THE FUNDAMENTAL GROUP OF THE CLIQUE GRAPH

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Abstract. Given a finite connected bipartite graph $B = (X, Y)$ we consider the simplicial complexes of complete subgraphs of the square B^2 of B and of its induced subgraphs $B^2[X]$ and $B^2[Y]$. We prove that these three complexes have isomorphic fundamental groups. Among other applications, we conclude that the fundamental group of the complex of complete subgraphs of a graph G is isomorphic to that of the clique graph $K(G)$, the line graph $L(G)$ and the total graph $T(G)$.

1. Introduction

All our graphs are simple, finite, connected and non-empty. Our simplicial complexes (or just *complexes*) are also finite and connected, and we identify them with their sets of simplexes. For a bipartite graph B we write $B = (X, Y)$ if $V(B)$ is the disjoint union of the independent sets X and Y, which we call the *parts* of B. Making a noun out of an adjective, we often refer to complete subgraphs just as *completes*. Given a graph G, the *complex of completes* of G is the simplicial complex $\Delta(G)$ whose simplices (or faces) are the completes of G. This is also called the clique complex or the flag complex of G in the literature. We can use the geometric realization $|\Delta(G)|$ of $\Delta(G)$ to attach topological concepts and constructions not only to the complex $\Delta(G)$, but also to the graph G itself. For instance, we say that the graphs G and H are *homotopic* if $|\Delta(G)|$ and $|\Delta(G)|$ are homotopic (i.e. homotopically equivalent), and that the *fundamental group* of G is just $\pi_1(G) = \pi_1(|\Delta(G)|)$.

Our main result is related to an old theorem due to Dowker [2] and Mather [10] which is called the Bipartite Relation Theorem in [1]: Let $B = (X_0, X_1)$ be a bipartite graph and, for $i = 0, 1$, let K_i be the complex with $V(K_i) = X_i$ and $\sigma \in K_i$ if $\sigma \subseteq N_B(v)$ for some $v \in X_{1-i}$. Then K_0 and K_1 are homotopic (see [1, Thm.10.9]). Prisner used this to prove in [12] that any clique-Helly graph G and its clique graph $K(G)$ are homotopic. Here, a *clique* is a maximal complete, the *clique graph* $K(G)$ is the intersection graph of the cliques of G, and G is *clique-Helly* if its cliques satisfy the *Helly property*: any family of pairwise intersecting cliques has a vertex in common. In the case in which G is not clique-Helly, G and $K(G)$ share at least the one-dimensional modulo 2 Betti numbers: this was also proved by Prisner in [12] using the Bipartite Relation Theorem.

Starting also from a bipartite graph $B = (X_0, X_1)$ we shall define two new complexes Δ_0 and Δ_1 with $V(\Delta_i) = X_i$ such that the Dowker-Mather complexes K_0 and K_1 are subcomplexes of Δ_0 and Δ_1. Our complexes Δ_0 and Δ_1 are not necessarily homotopic because they have, in general, more simplexes than K_0 and K_1; however, our main result (Theorem 3.1) ensures that they have always the same fundamental group. We will prove this by showing that the fundamental groups of Δ_0 and Δ_1 are isomorphic to that of a third complex Δ with $V(\Delta) = V(B)$.

We are mainly interested in clique graphs. The study of the clique operator under the topological viewpoint of the complex of completes was initiated by Prisner in [12, 11] and has been further pursued in [3, 4, 5, 6, 8, 9]. Thus, from our point of view, the main application of Theorem 3.1 is to the case in which $B = BK(G)$ is the *vertex-clique bipartite graph* of a graph G, that is, $X_0 = V(G)$, $X_1 = V(K(G))$ and $E(B) = \{vQ : v \in Q\}$. In this case our complexes Δ_0 and Δ_1 are precisely

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the complexes of completes $\Delta(G)$ and $\Delta(K(G))$ associated to G and $K(G)$, and they coincide with the Dowker-Mather complexes K_0 and K_1 of B precisely when G is clique-Helly (see §4). Thus, by Theorem 3.1, we could obtain directly Corollary 5.2: $\pi_1(K(G)) \cong \pi_1(G)$, thereby strengthening the above mentioned Prisner's result on one-dimensional modulo 2 Betti numbers. However, our Corollary 5.2 will depend on another consequence of Theorem 3.1, namely Proposition 5.1: If $\mathcal{F}$ is a complete edge cover of the graph G, and H is the intersection graph of $\mathcal{F}$, then $\pi_1(G) \cong \pi_1(H)$. This proposition will enable us to also prove the invariance of the fundamental group under several graph operators including, besides the clique graph $K(G)$, the line graph $L(G)$, the n-simplex graph $\nabla_n(G)$ for $n \geq 2$, and any finite composition of these operators, among others.

Some of the just-mentioned invariance results were proved, but in the weaker setting of one-dimensional modulo 2 homology, by Prisner in [11]. Besides working for the more fundamental setting of π_1, our proofs using Proposition 5.1 give a more unified approach. Using directly our Theorem 3.1, we will also prove that for a graph G the line graph $L(G)$ and the total graph $T(G)$ have the same fundamental group as G, a result also stated in [11] for one-dimensional modulo 2 Betti numbers, but with an incorrect proof. We will also prove that the graphs $\Omega(P)$ and $\mho(P)$, which were defined for any finite poset P in [9] and are not necessarily homotopic, have always the same fundamental group.

2. Preliminaries

If G is a graph and $v \in G$ (that is, $v \in V(G)$), we denote by $N_G(v)$ the set of neighbours of v. The distance in G between two vertices $u, v \in G$ will be denoted by $d_G(u,v)$. For a set of vertices $S \subseteq V(G)$ we denote by $G[S]$ the subgraph of G induced by S.

The *morphisms* between graphs $\varphi : G \to H$ that we shall use are vertex-maps $\varphi : V(G) \to V(H)$ such that the images of any two adjacent vertices of G are either adjacent or equal in H. These are sometimes called *reflexive morphisms* in the literature, and are akin to simplicial maps; they are better suited to the study of clique graphs than the other more usual morphisms, used for instance in graph colorings, where the images of adjacent vertices are required to be adjacent.

If Δ is a simplicial complex, the fundamental group of $|\Delta|$ can be described in combinatorial terms, and only the 2-skeleton $\Delta^{(2)}$ of Δ appears in this description, as indeed $\pi_1(|\Delta|) \cong \pi_1(|\Delta^{(2)}|)$. (See [14], mainly §3.6.) The following is essentially the construction in [14].

Let us call a *walk* in Δ any finite sequence $\gamma = v_0, v_1, \ldots, v_r$ of vertices of Δ where $r \geq 1$ and $\{ v_i, v_{i+1} \} \in \Delta$ for $i = 0, \ldots, r-1$. Thus, v_i and v_{i+1} are either equal or adjacent in the 1-skeleton of Δ for $i < r$. If some three consecutive vertices in γ form a simplex, say $\{ v_i, v_{i+1}, v_{i+2} \} \in \Delta$, removing the middle one we obtain a new walk $\gamma' = v_0, \ldots, v_i, v_{i+2}, \ldots, v_r$. We say that each of γ and γ' is obtained from the other by an *elementary transition.* Two walks γ and γ' are *homotopic* (denoted by $\gamma \simeq \gamma'$) if one can be obtained from the other by a finite sequence of elementary transitions. Fixing a base vertex x_0, the group $\pi_1(|\Delta|, x_0)$ is naturally equivalent to (i.e. can be safely thought of as) the group $\pi_1(\Delta, x_0)$ whose elements are the homotopy classes of the *closed* walks in x_0 (or *walks in* (Δ, x_0), i.e. $v_0 = v_r = x_0$) and in which the product is defined by juxtaposition: $[\gamma][\gamma'] = [\gamma\gamma']$. Since $\pi_1(\Delta, x_0) \cong \pi_1(\Delta, y_0)$ for any x_0, y_0, one usually writes just $\pi_1(\Delta)$.

If Δ' is another complex with base vertex x_0' and the map $\eta : \Delta \to \Delta'$ is simplicial and sends x_0 to x_0', there is a group morphism $\pi_1(\eta) : \pi_1(\Delta, x_0) \to \pi_1(\Delta', x_0')$ given by $\pi_1(\eta)([\gamma]) = [\eta(\gamma)]$, where $\eta(\gamma) = \eta(v_0), \eta(v_1), \ldots, \eta(v_r)$ if γ is as above. When acting on simplicial maps, π_1 preserves identities and compositions, that is $\pi_1(1_\Delta) = 1_{\pi_1(\Delta)}$ and $\pi_1(\eta \circ \zeta) = \pi_1(\eta) \circ \pi_1(\zeta)$.

3. Main Result

Let $B = (X, Y)$ be a connected bipartite graph. We think of X and Y as the *left* and *right* parts of B. Denote by B^2 be the square of B, so $V(B^2) = V(B)$ and $vw \in E(B^2)$ if and only if $d_B(v, w) \leq 2$. There are two kinds of edges in B^2: the original edges of B, which will be called *horizontal*, have one vertex in X and one in Y; the B^2-edges not in B, which have both vertices either in X or in Y, will be called *vertical.* If u and v are vertices in the same part of B, note that we have a vertical edge uv in B^2 if and only if $N_B(u) \cap N_B(v) \neq \varnothing$. Notice also that $B^2[X]$ and $B^2[Y]$ are connected and disjoint induced subgraphs of B^2.

Theorem 3.1. *Let $B = (X, Y)$ be a connected bipartite graph. Then B^2, $B^2[X]$ and $B^2[Y]$ have isomorphic fundamental groups.*

Proof: Denote B^2 as just G, and $G_X = G[X]$. It is clearly enough to show that $\pi_1(G) \cong \pi_1(G_X)$. As already noted, it suffices to work with the 2-skeleta $\Delta_X = \Delta(G_X)^{(2)}$ and $\Delta = \Delta(G)^{(2)}$. Rather than Δ, we will work with a subdivision Δ' which we define now. Let us begin by constructing a graph G' from G. First we subdivide each vertical edge $e = y_i y_j$ with vertices in Y by removing e from G and adding a *new* vertex y_{ij} which is to be adjacent to both y_i and y_j and to all their common neighbors in X. We say that e has been subdivided into two *semiedges*, and that the *new vertex* y_{ij} is incident in G' to these semiedges and the *new horizontal edges* xy_{ij} for $x \in N_B(y_i) \cap N_B(y_j)$. Now, for any triangle $\{ y_i, y_j, y_k \}$ in G with vertices in Y, we add the *little edges* $y_{ij}y_{jk}$, $y_{jk}y_{ki}$, and $y_{ki}y_{ij}$. We have now the graph G' which is the 1-skeleton of Δ'. Not all triangles of G' will be faces of Δ', but only those that are or subdivide triangles of Δ. Namely, the 2-dimensional faces of Δ' are the triangles of G' of the following types: (1) with no new vertices (hence with at least two vertices in X), (2) with two semiedges and a little edge, (3) with three little edges, (4) with a semiedge and a new horizontal edge. Thus, Δ' is obtained from Δ by subdividing each vertical edge in Y into two semiedges, each vertical triangle in Y into four triangles, and each horizontal triangle with two vertices in Y into two triangles. Then it is quite clear that $|\Delta'|$ is homeomorphic to $|\Delta|$ and that Δ_X is a subcomplex of Δ' as well as of Δ.

Let us call σ the inclusion $\sigma : G_X \to G'$. We claim that σ has a left inverse $\rho : G' \to G_X$. First define $\rho(x) = x$ for each $x \in X$. Now, for each old vertex $y \in Y$, choose a fixed $\rho(y) \in N_B(y)$ (there is at least one: B is connected). Finally, if y_{ij} is a new vertex, $N_B(y_i) \cap N_B(y_j) \neq \varnothing$, so we can pick a vertex $\rho(y_{ij})$ in this intersection. In order to show that ρ is a graph morphism we will check that for any edge uv of G' the vertices $\rho(u)$ and $\rho(v)$ lie in some complete subgraph of G_X, so they are either adjacent or equal. Any vertical edge in X is mapped to itself, so we need only to consider edges that have at most one vertex in X. If xy is an old horizontal edge, then $x, \rho(y) \in N_B(y)$, which is a complete of G_X. If xy_{ij} is a new horizontal edge, then $x, \rho(y_{ij}) \in N_B(y_i) \cap N_B(y_j)$ which is also complete. As for vertical edges with vertices not in X, for any semiedge $y_i y_{ij}$ we have $\rho(y_i), \rho(y_{ij}) \in N_B(y_i)$, and for each little edge $y_{ij} y_{jk}$ we have that $\rho(y_{ij}), \rho(y_{jk}) \in N_B(y_j)$. Therefore ρ is a graph morphism, and clearly $\rho \circ \sigma = 1_{G_X}$.

Since Δ_X is a subcomplex of Δ' we have an inclusion simplicial map which we also call σ. Indeed, $\sigma : \Delta_X \to \Delta'$ is, as a vertex function, the same as $\sigma : G_X \to G'$. On the other hand, the graph morphism $\rho : G' \to G_X$ gives us a simplicial map $\rho : \Delta' \to \Delta_X$. Indeed, it is enough to note that any 2-dimensional face of Δ', being a triangle of G', is mapped to a complete of G_X with at most three vertices, and all such completes are faces of Δ_X. Let us fix $x_0 \in X$ and use it as a base vertex both for Δ_X and Δ'. We will show that $\pi_1(\sigma) : \pi_1(\Delta_X, x_0) \to \pi_1(\Delta, x_0)$ is an isomorphism. Since $\rho \circ \sigma = 1_{\Delta_X}$, we have that $\pi_1(\rho) \circ \pi_1(\sigma) = \pi_1(\rho \circ \sigma) = \pi_1(1_{\Delta_X}) = 1_{\pi_1(\Delta_X)}$ and then $\pi_1(\sigma)$ is injective.

In order to see that $\pi_1(\sigma)$ is surjective, let $\gamma = v_0, v_1, \ldots, v_r$ be a walk in Δ' with $v_0 = v_r = x_0$. We need to show that $\gamma \simeq \gamma'$ for some walk γ' in (Δ_X, x_0), so $[\gamma] = \pi_1(\sigma)[\gamma']$. We will use elementary transitions until a walk in Δ_X is obtained, but after each step we will still denote the resulting walk by

$\gamma = v_0, v_1, \ldots, v_r$. If $v_i v_{i+1}$ is a new horizontal edge, it can be avoided using a triangle of type (4), and we keep doing this until γ uses no new horizontal edge. Likewise, little edges can be eliminated from γ using triangles of type (2). If $v_i = v_{i+1} \in Y$ we can delete v_{i+1} from γ, since in this case $i+1 < r$ and certainly $\{ v_i, v_{i+1}, v_{i+2} \} \in \Delta'$. Also, any subwalk of the form v, w, v in γ can be shortened to just v. We apply repeatedly operations of these last two kinds until it is no longer possible. Any occurrence of a new vertex in the resulting γ, say $v_t = y_{ij}$, must satisfy $0 < t < r$ and $\{ v_{t-1}, v_{t+1} \} = \{ y_i, y_j \}$. But then v_{t-1}, v_t, v_{t+1} can be replaced in γ by v_{t-1}, x, v_{t+1}, where $x \in N_B(y_i) \cap N_B(y_j)$: this is achieved by two elementary transitions using triangles of type (4). Repeating this as needed, we end up with a walk γ such that, if $v_i \notin X$, then $v_i \in Y$ and both v_{i-1} and v_{i+1} lie in X, in fact in $N_B(v_i)$. But then such vertices v_i can be deleted from γ either using a triangle of type (1) or because $v_{i-1} = v_{i+1}$. □

4. Relation to the Dowker-Mather Complexes

Recall from the introduction that the Dowker-Mather complexes (DM complexes) associated to a bipartite graph $B = (X, Y)$ are the complexes K_X and K_Y with $V(K_X) = X$ and $\sigma \in K_X$ if $\sigma \subseteq N_B(y)$ for some $y \in Y$, and similarly for K_Y upon interchange of X and Y. Thus, the maximal faces of, say, K_X are all of the form $N_B(y)$ for some $y \in Y$ and, since these are completes of $B^2[X]$, we see that K_X is a subcomplex of $\Delta(B^2[X])$. On the other hand, a complete σ of $B^2[X]$ is not necessarily contained in a neighbourhood $N_B(y)$ for some $y \in Y$: here we have only that each pair $x, x' \in \sigma$ is contained in one such $N_B(y)$. Looking at this the other way around, a complete of $B^2[X]$ is a set $\sigma \subseteq X$ such that for any pair $x, x' \in \sigma$ we have that $N_B(x) \cap N_B(x') \neq \varnothing$. This σ will be a face of K_X as soon as there is a $y \in Y$ with $\sigma \subseteq N_B(y)$ or, equivalently, $y \in N_B(x)$ for all $x \in \sigma$. In [7] a bipartite graph $B = (X, Y)$ was called *left N-Helly* (respectively, *right N-Helly*) if the family $\{N_B(x) : x \in X\}$ (respectively, $\{N_B(y) : y \in Y\}$) satisfies the Helly property. Also, B is *N-Helly* if B is left and right N-Helly, i.e. the family $\{N_B(v) : x \in B\}$ satisfies the Helly property. From our remarks in this paragraph we get at once that:

Proposition 4.1. *Let $B = (X, Y)$ a bipartite graph and let K_X and K_Y be its DM complexes. Then*
1. *$K_X = \Delta(B^2[X])$ if and only if B is left N-Helly.*
2. *$K_Y = \Delta(B^2[Y])$ if and only if B is right N-Helly.*
3. *$K_X = \Delta(B^2[X])$ and $K_Y = \Delta(B^2[Y])$ if and only if B is N-Helly.* □

The bipartite graphs $B = (X, Y)$ which are isomorphic to the vertex-clique bipartite graph $BK(G)$ of some graph G were characterized in [7]: B is called *right N-Sperner* if the family $\{N_B(y) : y \in Y\}$ is an antichain, that is $N_B(y) \subseteq N_B(y')$ implies $y = y'$. Then one has that there exists a graph G such that $B \cong BK(G)$ (with $B^2[X] \cong G$ and $B^2[Y] \cong K(G)$) if and only if B is both left N-Helly and right N-Sperner [7, Thm.2.6].

In particular, the vertex-clique bipartite graph $B = BK(G)$ of a graph G is left N-Helly and therefore $\Delta(B^2[X]) = \Delta(G)$ is always equal to the DM complex K_X of B. However, $\Delta(K(G)) = \Delta(B^2[Y])$ coincides with the other DM complex K_Y of B only in case that B is right N-Helly, and this means that G is clique-Helly, since the neighbourhoods in B of the vertices of $Y = V(K(G))$ are precisely the cliques of G.

5. Applications

Let G be a graph. By a *complete edge cover* of G we mean a family $\mathcal{F} = (G_i : i \in I)$ of complete subgraphs of G such that any vertex and any edge of G lie in some G_i. In other words, G is the union of its complete subgraphs G_i, $i \in I$. We are not excluding the possibility that $G_i = G_j$ for some $i \neq j$ in I, so $\mathcal{F}$ can have repeated members. However, the most interesting case is when there are no repetitions, and then the family $\mathcal{F}$ can be thought of as just a set $\mathcal{F} = \{G_i : i \in I\}$. Complete edge covers were important since the beginning of the study of clique graphs: a graph G is isomorphic to the clique graph of some graph if and only if G has a complete edge cover that satisfies the Helly property. This characterization, which we won't use here, is due to Roberts and Spencer [13].

Now take any complete edge cover $\mathcal{F} = (G_i : i \in I)$ of G and consider its intersection graph $H = \Omega(\mathcal{F})$, so the vertices of H are the elements of I (or just the members G_i of $\mathcal{F}$ if there are no repetitions), and there is an edge $ij \in E(H)$ iff G_i and G_j share some vertex. The incidence graph of vertices of G and members of $\mathcal{F}$ is the bipartite graph $B = B(\mathcal{F})$ with vertex set $V(B) = X \cup Y$, where $X = V(G)$ and $Y = V(H) = I$, and edge set $E(B) = \{vi : v \in G_i\}$. We have that $B^2[X] = G$ because $\mathcal{F}$ is a complete edge cover of G, and clearly $B^2[Y] = H$ as $H = \Omega(\mathcal{F})$. By Theorem 3.1 we get:

Proposition 5.1. *Let $\mathcal{F}$ be a complete edge cover of the graph G, and let H be the intersection graph of $\mathcal{F}$. Therefore, $\pi_1(H) \cong \pi_1(G)$.* □

In case that the complete edge cover is the set of cliques of G, one obtains:

Corollary 5.2. *If G is a graph, then $\pi_1(K(G)) \cong \pi_1(G)$.* □

In particular, as the first homology group $H_1(G, \mathbb{Z})$ of G with integer coefficients is just the abelianized group $\pi_1(G)/\pi_1(G)'$, this implies that $H_1(G, \mathbb{Z}) \cong H_1(K(G), \mathbb{Z})$ for any graph G. Similarly, $H_1(G, \mathbb{Z}_2)$ is just the modulo 2 reduction of $H_1(G, \mathbb{Z})$, so we get that $H_1(G, \mathbb{Z}_2) \cong H_1(K(G), \mathbb{Z}_2)$. The dimensions of these latter groups as vector spaces over the two-element field $\mathbb{Z}_2$ are the one-dimensional modulo 2 Betti numbers (denoted by β_1 in [12, 11]) of G and $K(G)$, so we get also that $\beta_1(K(G)) = \beta_1(G)$ for each graph G. Similar remarks apply whenever we have the invariance of the fundamental group under some graph operator, as in our following results, for which we will recall the definitions first.

Let G be a graph with more than one vertex. The *line graph* $L(G)$ is the intersection graph of the edges of G. The *graph of completes* $C(G)$ is the intersection graph of the completes of G. For $n \geq 2$, the *n-simplex graph* $\nabla_n(G)$ is the intersection graph of the subset of all inclusion-maximal elements in the set of all completes of cardinality at most n of G (see [11]). By Proposition 5.1 we have:

Corollary 5.3. *Let the graph operator $\mathcal{O}$ be a finite composition $\mathcal{O} = \mathcal{O}_1 \circ \mathcal{O}_2 \circ \cdots \circ \mathcal{O}_n$ where each $\mathcal{O}_i$ is one of L, C, K, or ∇_n ($n \geq 2$). Then $\pi_1(\mathcal{O}(G)) \cong \pi_1(G)$ for each graph G.* □

The *total graph* $T(G)$ has $V(G) \cup E(G)$ as vertex set and, in addition of all edges of G and $L(G)$, $T(G)$ has also all edges of the form ve where $v \in G$, $e \in L(G)$ and $v \in e$. Then $T(G) = B^2$, where the bipartite graph $B = (X, Y)$ is the *vertex-edge incidence graph* of G, that is $X = V(G)$, $Y = E(G)$ and $E(B) = \{ve : v \in G, e \in L(G), v \in e\}$. We clearly also have that $B^2[X] = G$ and $B^2[Y] = L(G)$, so the following is immediate from Theorem 3.1:

Corollary 5.4. *Any non-trivial graph G shares the fundamental group with $L(G)$ and $T(G)$.* □

Our last application will be to the graphs $\Omega(P)$ and $\mho(P)$ which were associated to any finite poset P in [9]. For such a (connected) poset P we denote by $\min(P)$ and $\max(P)$ respectively the sets of minimal and maximal elements of P. Then $\Omega(P)$ is the graph with vertex set $\min(P)$ in which two distinct vertices x, x' are adjacent if and only if there is $z \in P$ such that $x \leq z$ and $x' \leq z$. Dually we define the graph $\mho(P)$ with $V(\mho(P)) = \max(P)$ where $y \sim y'$ iff they have a common lower bound in P. In general $\Omega(P)$ and $\mho(P)$ are not homotopic (some sufficient conditions were given in [9]) but we always have that:

Corollary 5.5. *Let P be a finite connected poset. Then $\pi_1(\Omega(P)) \cong \pi_1(\mho(P))$.*

Proof: Define $X = \min(P)$, $Y = \max(P)$ and $B = (X, Y)$ with $E(B) = \{xy : x \in X, y \in Y, x \leq y\}$. Let $x, x' \in X = \min(P)$. Then $x \sim x'$ in $\Omega(P)$ iff they have a common upper bound in P, but this holds iff they have a common upper bound in $\max(P)$, and this is equivalent to $x \sim x'$ in B^2, so $\Omega(P) = B^2[X]$. Similarly, $\mho(P) = B^2[Y]$, and the result follows from Theorem 3.1. □

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