ON HEREDITARY CLIQUE HELLY SELF-CLIQUE GRAPHS

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ABSTRACT. A graph is *clique-Helly* if any family of mutually intersecting (maximal) cliques has non-empty intersection, and it is *hereditary clique-Helly* (abbreviated HCH) if its induced subgraphs are clique-Helly. The *clique graph* of a graph G is the intersection graph of its cliques, and G is *self-clique* if it is connected and isomorphic to its clique graph. We show that every HCH graph is an induced subgraph of a self-clique HCH graph, and give a characterization of self-clique HCH graphs in terms of its constructibility starting from certain digraphs with some forbidden subdigraphs. We also specialize this results to *involutive* HCH graphs, i.e. self-clique HCH graphs whose vertex-clique bipartite graph admits a part-switching involution.

1. INTRODUCTION

By a *graph G* we will always mean a finite, simple and nonempty graph. For auxiliary purposes only, we will also consider digraphs D and graphs H that can have loops. A *clique* of a graph is a maximal complete subgraph, which we often identify with its vertex set. A graph is *clique-Helly* if the family of its cliques satisfies the *Helly property*: any family of mutually intersecting cliques has non-empty intersection. Ever since the rise of the study of clique graphs [8, 15] the Helly property and clique-Helly graphs have played a central role.

Hereditary clique-Helly graphs (or just *HCH* graphs) were defined by Prisner in [14] by the property that every induced subgraph is again clique-Helly. This important subclass of clique-Helly graphs admits various characterizations [14, 9] and contains several families of graphs which have been significant in the study of clique graphs: triangle-free graphs, diamond-free graphs, strongly chordal graphs and others. A polynomial time recognition algorithm for HCH graphs was given by Prisner in [14], and Szwarcfiter gave in [16] the general one for clique-Helly graphs. In terms of Szwarcfiter's characterization, HCH graphs are those for which not only every extended triangle has a universal vertex, but such a vertex exists even in the original triangle. See [5] for a recent work on HCH graphs.

The *clique graph* of a graph is the intersection graph of its cliques. A graph is *self-clique* if it is connected and isomorphic to its own clique graph. Escalante [7] discovered self-clique graphs in 1973; among other things he proved the existence of both clique-Helly and non-clique-Helly self-clique graphs, and also that every graph is an induced subgraph of a clique-Helly self-clique graph [7, Satz 7]. The rest of the century saw few results: in the 1980's Lim and Peng [13] and Balakrishnan

and Paulraja [1] gave some new families of examples, and in 2000 Chia gave in
[6] a characterization of the first family of self-clique graphs with triangles: those
having at most one clique that is not an edge. All those examples, save for one of
Escalante's families, were indeed clique-Helly self-clique graphs. They have been
generalized and obtained by a unified method in [12]. It was only recently that,
with the use of more general methods, larger families and the first characterizations
of clique-Helly self-clique graphs were found: Bondy, Durán, Lin and Szwarcfiter
showed that they are the graphs with a quasi-symmetric clique matrix [4, Thm.2.1],
or those admitting a vertex-clique duality [4, Thm.6], and Larrión, Neumann-Lara,
Pizaña and Porter showed them to be those with self-dual vertex-clique bipartite
graph [12, Thm.4.4].

An important subclass of clique-Helly self-clique graphs is formed by *involutive*
graphs: those for which the vertex-clique bipartite graph admits an involutive self-
duality, or, in the language of [4], those that admit a symmetric clique matrix. All
clique-Helly self-clique graphs known last century were in fact involutive, but this
is not always the case: both [4] and [12] contain counterexamples. Also, all those
old graphs were in fact HCH, but neither this is always the case: Any graph G that
inducedly contains one from the Hajós family is not HCH by [14, Thm.2.1] (see 2.2
below) but then G is an induced subgraph of some clique-Helly self-clique graph G'
by [7, Satz 7], and G' is not HCH once again by [14, Thm.2.1]; however, the first
explicit (minimal!) counterexample settling this question was given by Bonomo
(reported in [4, Fig.4]).

Owing to [17], the problem of recognizing HCH self-clique graphs is clearly in
NP. Furthermore, it follows from the proof of Theorem 2.1 in [11] that this problem
is polynomially equivalent to the graph isomorphism problem.

Our approach and techniques, as in [10] and [11], come mainly from [12], but we
will also reap much benefit from those of Balconi, Grieco and Zucchetti [2], which
we present, update and adapt to our purposes in §3.

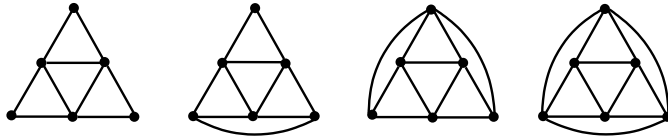
2. Vertex-clique bipartite graphs

If G is a graph, we denote by $BK(G)$ the *vertex-clique bipartite graph* of G.
This is the bipartite graph $B = BK(G)$ with vertex set $V(B) = V(G) \cup V(K(G))$
and edge set $E(B) = \{\{v, Q\} : v \in Q\}$. Thus $BK(G)$ is just the incidence graph
of the vertices and cliques of G, but in addition we will distinguish between the
left vertices $L = V(G)$ and the *right vertices* $R = V(K(G))$ of $BK(G)$. The
notation $B = (L, R)$ indicates this distinction. In fact, all our bipartite graphs
B will come with a fixed (and ordered) bipartition $B = (L, R)$. In the case in
which $B = BK(G)$ for some graph G, this will always be the *standard bipartition*
$BK(G) = (L, R)$ where $L = V(G)$ and $R = V(K(G))$. We say that a bipartite
graph $B = (L, R)$ is *self-dual* if there is a *part-switching automorphism* $\delta : B \to B$,
i.e. an automorphism δ with $\delta(L) = R$ and $\delta(R) = L$. See [12] for examples.

The following characterization from [12] will be basic to this work.

Theorem 2.1. [12, Thm.4.4] *A connected graph G is clique-Helly and self-clique
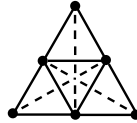if and only if $BK(G)$ is self-dual.*

In this section we shall characterize self-clique HCH graphs in terms of their vertex-clique bipartite graphs. We start with the characterization of HCH graphs by forbidden subgraphs due to Prisner [14]:

Theorem 2.2. [14, Thm.2.1] *A graph G is hereditary clique-Helly if and only if it contains none of the following four graphs as an induced subgraph.*

The Hajós family.

Another way of stating this result is saying that G is HCH if and only if it is *compatible* with the following diagram
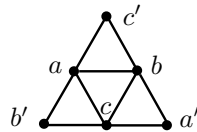

The Hajós diagram.

in the sense that any subgraph of G which is isomorphic to the solid part of the diagram induces also at least one of the dashed edges. This concept of diagram-compatibility will help us later to abbreviate the forbiddance of large families of induced subgraphs.
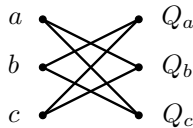
Prisner also showed in [14, Cor.2.3] that HCH graphs are characterized by the property that every triangle has a *good edge*: any common neighbour of the vertices of the edge is also adjacent or equal to the third vertex of the triangle. In terms of Szwarcfiter's characterization [16] of clique-Helly graphs, this means that not only every extended triangle has a universal vertex, but that such a universal vertex can be found even in the original triangle. Something similar will happen in our 2.6.

Theorem 2.3. *A graph G is hereditary clique-Helly if and only if $BK(G)$ does not have induced hexagons.*

Proof: Assume first that G is not HCH. By 2.2, G must have a subgraph of the following form

where x is not a neighbour of x' for $x = a, b, c$. Now consider cliques Q_a, Q_b and Q_c in G such that $a', b, c \in Q_a$, $a, b', c \in Q_b$ and $a, b, c' \in Q_c$. Then $x \notin Q_x$ for $x = a, b, c$ and

is an induced hexagon in $BK(G)$.

Reciprocally, if we have an induced hexagon as above in $BK(G)$, $T = \{a, b, c\}$ is a triangle of G. Since $a \notin Q_a$ there must be a vertex $a' \in Q_a$ which is not a neighbour of a, and in particular $a' \notin T$. Similarly we have $b' \in Q_b - T$ and $c' \in Q_c - T$ such that $b \notin N[b']$ and $c \notin N[c']$, but then G is not compatible with the Hajós diagram.   □

Using 2.3 and 2.1 we get our characterization of self-clique HCH graphs in terms of their vertex-clique bipartite graphs:

Theorem 2.4. *Let G be a connected graph. Then G is hereditary clique-Helly and self-clique if and only if $BK(G)$ is self-dual and does not have induced hexagons.*

We will need the following concepts and result from [12]. Given a graph G, denote by $\mathcal{N}(G)$ the (indexed) family of its neighbourhoods: $\mathcal{N}(G) = \{N(v) : v \in G\}$. Then G is said to be *N-Helly* if $\mathcal{N}(G)$ satisfies the Helly property, and G is called *N-Sperner* if $\mathcal{N}(G)$ is a *Sperner family*: $N(v) \subseteq N(w) \Rightarrow v = w$. The graph G is *good* if it is both N-Helly and N-Sperner. For a bipartite graph $B = (L, R)$ we have one-sided versions of the N-Helly and N-Sperner conditions: for instance B is *left-N-Helly* if $\{N(v) : v \in L\}$ satisfies the Helly property, and B is *right-N-Sperner* if $\{N(v) : v \in R\}$ is a Sperner family. By the way, Theorem 2.6 of [12] ensures that $B \cong BK(G)$ for some graph G if and only if B has no isolated vertices and is left-N-Helly and right-N-Sperner for some bipartition $B = (L, R)$.

Theorem 2.5. [12, Thm.4.3+Proof] *Let $B = (L, R)$ be a bipartite graph. Then $B \cong BK(G)$ for some clique-Helly graph G with $K(G) \cong G$ if and only if B is good and self-dual. If this is the case, G is self-clique if and only if B is connected.*

We will need later that any bipartite graph without induced hexagons is N-Helly. This holds in a more general setting:

Theorem 2.6. *Any graph G without triangles or induced hexagons is N-Helly.*

Proof: We will use the following characterization of the Helly property, which is due to Berge, Roberts and Spencer [15]: Let $\mathcal{F}$ be a family of subsets of a set $X \neq \varnothing$. Then $\mathcal{F}$ satisfies the Helly property if and only if $\cap \mathcal{F}(x, y, z) \neq \varnothing$ for all $x, y, z \in X$, where $\mathcal{F}(x, y, z) = \{A \in \mathcal{F} : |A \cap \{x, y, z\}| \geq 2\}$.

Notice that $\cap \mathcal{F}(x, y, z) \neq \varnothing$ is automatically satisfied if either $|\{x, y, z\}| \leq 2$, or for some pair of elements of $\{x, y, z\}$ there is no $A \in \mathcal{F}$ containing it. Accordingly, in order to prove that $\cap \mathcal{F}(x, y, z) \neq \varnothing$ one can assume that x, y and z are different and that any two of them are contained in some $A \in \mathcal{F}$.

We shall apply the above result to the family of neighbourhoods of G. We will denote by $N(a, b)$ the set of the common neighbours of the vertices $a, b \in G$, thus, $N(a, b) = N(a) \cap N(b)$. Observe that a neighbourhood $A = N(v)$ contains both a and b iff $v \in N(a, b)$.

Take three different vertices $a, b, c \in G$, and suppose that $N(x, y) \neq \varnothing$ for each 2-element set $\{x, y\} \subseteq \{a, b, c\}$. Note that, in our case, $\mathcal{F}(a, b, c) = \{N(x) : x \in N(a, b) \cup N(a, c) \cup N(b, c)\}$, and that we need to prove that there is some vertex in all these neighbourhoods $N(x)$. In fact, even more is true: one of a, b, c must be in $\cap \mathcal{F}(a, b, c)$. This means that either $a \in N(x)$ for all $x \in N(b, c)$, or $b \in N(x)$ for all $x \in N(a, c)$, or $c \in N(x)$ for all $x \in N(a, b)$. If this were not the case, we would have vertices $a' \in N(b, c)$, $b' \in N(a, c)$ and $c' \in N(a, b)$ such that $a \notin N(a')$,

$b \notin N(b')$ and $c \notin N(c')$. But then consider the hexagon $\mathcal{H} = (a, c', b, a', c, b', a)$: there are no edges in G between opposing vertices of $\mathcal{H}$, but any other chord of $\mathcal{H}$ would produce a triangle in G. This contradiction ends the proof. $\qquad\square$

3. Duality digraphs

The *duality digraph* associated to a duality of some incidence structure was introduced by Balconi, Grieco and Zucchetti in [2]. In the next section we shall transform our characterization 2.4 of self-clique HCH graphs G into another one in terms of the duality digraph associated to a self-duality of $BK(G)$. In this section we introduce the needed material from [2]. Owing to hindsight and the recent works [4, 12] we will be able to give stronger versions and correct a mistake. Our presentation is slightly different but equivalent: instead of incidence structures we will work with bipartite graphs, and instead of "stelle doppie" we will use "translations". An important fact is that there is an algorithm constructing, for each input n, all duality digraphs of order at most n [3].

Consider finite digraphs D with no parallel arrows. Thus, each vertex admits at most one loop and between two vertices there are at most two arrows, one in each direction. By $N^+(v)$ and $N^-(v)$ we denote the sets of out-neighbours and in-neighbours of a vertex $v \in V(D)$. In particular, denoting by $\mathcal{N}^+(D)$ the family $\{N^+(v) : v \in D\}$, we say that D is N^+-*Helly* if $\mathcal{N}^+(D)$ satisfies the Helly property, and that D is N^+-*Sperner* if $\mathcal{N}^+(D)$ is a Sperner family.

A *translation* in a digraph D is any bijection $\tau : V(D) \to V(D)$ such that $N^+(\tau(v)) = N^-(v)$ for all $v \in D$, see Figure 1. A *duality digraph* is any digraph D admitting some translation τ. More formally, a duality digraph should perhaps be defined as a pair (D, τ) where D is a digraph and τ is a translation in D, but in the cases that will be of interest to us the translation τ is unique as soon as it exists. Since there is an arrow $x \to x'$ in a duality digraph D iff there is an arrow $\tau(x') \to x$, any translation τ in D induces a bijection $N^+(x) \to N^-(x)$ for each $x \in D$. It is then very easy to see that duality digraphs are balanced and, if connected, Eulerian and strongly connected.
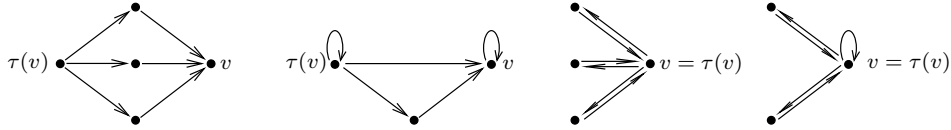

FIGURE 1. The possible *"stelle doppie"* of a vertex v of indegree 3. Only those arrows ending at v, or starting at $\tau(v)$, were depicted.

Let B be a self-dual bipartite graph with bipartition $B = (L, R)$. The *duality digraph associated* to a duality $\delta : B \to B$ is the digraph $D = D(\delta)$ with vertices $V(D) = L$ and arrows $A(D) = \{x \to x' : x \sim \delta(x') \text{ in } B\}$. This is indeed a duality digraph: let $\tau : L \to L$ be the restriction to L of the bijection $\delta^2 : B \to B$; then τ is bijective and we have an arrow $x \to x'$ in $D \Leftrightarrow x \sim \delta(x')$ in $B \Leftrightarrow \tau(x') \sim \delta(x)$ in $B \Leftrightarrow$ there is an arrow $\tau(x') \to x$ in D. More formally, we would say that $D(\delta) = (D, \tau)$ for this specific $\tau = (\delta^2)_{|L}$, to be called *the translation of* $D(\delta)$. It is easy to see that B connected implies D connected.

Reciprocally, given a duality digraph D with translation τ, we shall construct a self-dual bipartite graph $B = (V(D), R)$ and a duality $\delta : B \to B$ such that $D = D(\delta)$ and $\tau = (\delta^2)_{|V(D)}$. Indeed, put $L = V(D)$ and construct a new set $R = \{y_x : x \in L\}$. Now define B as having bipartition $B = (L, R)$ and an edge $x \sim y_{x'}$ for each arrow $x \to x'$ in D. Further, define $\delta : B \to B$ by $\delta(x) = y_x$ and $\delta(y_x) = \tau(x)$ for all $x \in L$. These definitions already ensure that $D = D(\delta)$ provided that δ is a self-duality of B, but this indeed holds: δ is clearly bijective and switches the parts of B, and we have $x \sim y_{x'}$ in $B \Leftrightarrow x \to x'$ in $D \Leftrightarrow \tau(x') \to x$ in $D \Leftrightarrow \tau(x') \sim y_x$ in $B \Leftrightarrow \delta(y_{x'}) \sim \delta(x)$ in B. Of course, one also has $\delta^2(x) = \delta(y_x) = \tau(x)$ for all $x \in V(D)$, so the translation of $D(\delta)$ is the original one of D.

Going the other way around, start with (B, δ) where $B = (L, R)$ is bipartite and $\delta : B \to B$ is a duality. Construct first $D(\delta)$ with its translation $\tau = (\delta^2)_{|L}$, and then consider in turn the above-constructed bipartite graph $B' = (L, R')$ with duality $\delta' : B' \to B'$ such that $D(\delta') = D(\delta)$. We claim that B and B' are isomorphic via an isomorphism which sends left part to left part, right part to right part, and transforms δ into δ'. Indeed, define $\theta : B \to B'$ by $\theta(x) = x$ for $x \in L$ and $\theta(x) = y_{\delta^{-1}(x)}$ for $x \in R$. We have that $\theta(L) = L$ and $\theta(R) = R'$, so we only need to see that $\theta \circ \delta = \delta' \circ \theta$. If $x \in L$, $\theta(\delta(x)) = y_x = \delta'(x) = \delta'(\theta(x))$. If $x \in R$ we have $\delta(x) \in L$, so $\theta(\delta(x)) = \delta(x) = \delta^2(\delta^{-1}(x)) = \tau(\delta^{-1}(x)) = \delta'(y_{\delta^{-1}(x)}) = \delta'(\theta(x))$.

We have thus an improved version of Teorema 1 of [2]:

Theorem 3.1. [2] *A digraph D is a duality digraph if and only if there is a duality $\delta : B \to B$ of some bipartite graph $B = (L, R)$ such that $D = D(\delta)$. Furthermore, the above two constructions set up a bijection between isomorphism types of bipartite graphs with fixed self-duality on the one hand, and isomorphism types of duality digraphs with fixed translation on the other hand.* □

As observed in [2], if the duality digraph D admits two different translations τ and τ', it is easy to see that the self-dual bipartite graphs B and B' constructed above are isomorphic, and only the dualities δ and δ' can differ from each other. However, under the condition that any two different vertices of B have different neighbourhoods, the translation τ is uniquely determined for the duality digraph D associated to any self-duality of B: Indeed, for any $v \in D$ there exists a *unique* $\tau(v) \in D$ such that $N^+(\tau(v)) = N^-(v)$. To see this, notice first that $N_D^+(w) = \delta^{-1}(N_B(w))$ for each vertex $w \in D$ and then observe that, since δ^{-1} is bijective, the condition implies that all the $N^+(w)$ are different. Even a stronger condition holds for the case of vertex-clique bipartite graphs, since B is N-Sperner in this case (see our proof of 3.3 below).

The *alternating square* of a duality digraph D is defined as the graph $D^{\wedge 2}$ with the same vertex set of D and in which two different vertices x, y are neighbours if and only if they are joined by an alternating path of length two in D, i.e. there are two arrows $x \to v \leftarrow y$ in D (note that v can be equal to x or y) or, equivalently, two arrows $x \leftarrow \tau(v) \to y$ in D. The following is implicit in [2]:

Lemma 3.2. *Let G be a graph, $B = BK(G)$, and $\delta : B \to B$ a self-duality. Then, if $D = D(\delta)$, we have $G = D^{\wedge 2}$.*

Proof: If $BK(G)$ has standard bipartition (L, R), then G, D and $D^{\wedge 2}$ have the same vertex set L. Take two different vertices $x, y \in L$. Then x and y are neighbours in G if and only if they are both contained in some clique $Q \in V(K(G))$. This holds if and only if x and y have a common successor $\delta^{-1}(Q)$ in D, but this means that they are neighbours in $D^{\wedge 2}$. $\qquad \square$

The following is an amended and strengthened version of Teorema 2 of [2]:

Theorem 3.3. [2] *Let D be a duality digraph with translation τ. Then there exists some graph G such that $D = D(\delta)$ for some duality δ of $BK(G)$ if an only if*
(1): D is N^+-Helly and (2): D is N^+-Sperner.
Moreover, if this is the case, $G = D^{\wedge 2}$ is clique-Helly and, if connected, self-clique.

Proof: Using 3.1, let $B = (L, R)$ be the bipartite graph with self-duality δ such that $D = D(\delta)$. By the already mentioned fact that $N_D^+(v) = \delta^{-1}(N_B(v))$ for each $v \in D$ and since δ^{-1} is bijective, D is N^+-Helly and N^+-Sperner if and only if B is left N-Helly and left N-Sperner. Since B is self-dual, this last is the case if and only if B is N-Helly and N-Sperner, i.e. B is good. But then, by 2.5, B is good and self-dual if and only if $B \cong BK(G)$ for some clique-Helly graph G with $K(G) \cong G$. By 3.2, $G = D^{\wedge 2}$. $\qquad \square$

The above is an amended version: the original one lacked our hypothesis (2), which is really needed: Indeed, in the language of [2], the incidence structure with points 1 and 2 and blocks $\{1, 2\}$ and $\{1\}$ is self-dual (it has a unique duality) and the duality digraph is N^+-Helly; however, the bipartite graph is the path on four vertices, so it is not a vertex-clique bipartite graph. The reinforcement of the original result (the application to clique-Helly self-clique graphs) only became possible after the characterizations in [4] or [12].

If the duality digraph D is disconnected, then clearly the alternating square $D^{\wedge 2}$ is disconnected, but $D^{\wedge 2}$ can be disconnected even for connected D. The possibilities for this were characterized in [2], and again our presentation is slightly different. A *morphism* of digraphs $f : D \to D'$ is just an arrow-preserving vertex map $f : V(D) \to V(D')$, and we say that f is *onto* if it is surjective both in vertices and arrows. Given an integer $n \geq 2$, a *cyclically n-partite digraph* (or a *cyclically multipartite digraph* with n parts) is a digraph D which admits an onto morphism to the loopless directed cycle $\overrightarrow{C_n}$ (with arrows $0 \to 1 \to ... \to n - 1 \to 0$). In other words, the vertex set of D can be partitioned into a cyclically ordered family of n independent sets (the *parts*, which must be non-empty) in such a way that any arrow starting at a vertex in some part ends in a vertex in the next part, and also some arrow starts at each part. If the duality digraph D is cyclically multipartite, notice that any connected component of $D^{\wedge 2}$ must be contained in one of the parts, so $D^{\wedge 2}$ is disconnected. The following is, essentially, Proposizione 5 of [2], but we need it for $D^{\wedge 2}$ rather than B:

Proposition 3.4. [2] *Let D be a connected duality digraph. Then $D^{\wedge 2}$ is connected if and only if D is not cyclically multipartite.*

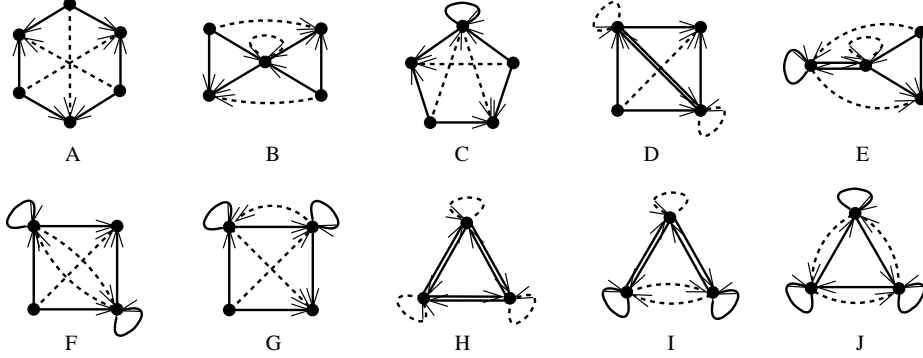**Proof:** Let C and C' be two distinct connected components of $D^{\wedge 2}$ such that there exist $u \in C$, $v \in C'$ and an arrow $u \to v$. Any other out-neighbour of u is

a neighbour of v in $D^{\wedge 2}$, so $N^+(u) \subseteq C'$. Any neighbour w of u in C has an out-neighbour in C' (even in $N^+(u)$) so $N^+(w) \subseteq C'$ for all $w \in C$. Dually, $N^-(w) \subseteq C$ for all $w \in C'$, and it follows easily that D is cyclically multipartite.   □
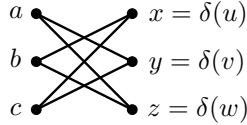
4. Self-clique HCH graphs and duality digraphs

In this section we transform our characterization 2.4 of self-clique HCH graphs G into another one in terms of the duality digraph associated to a self-duality of $BK(G)$. We start with the "hexagon-free" condition:

Theorem 4.1. *Let $B = (L, R)$ be a self-dual bipartite graph, and $D = D(\delta)$ the duality digraph of some duality δ of B. Then B is free of induced hexagons if and only if D is compatible with the following ten diagrams:*

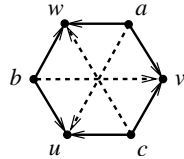A   B   C   D   E

F   G   H   I   J

Proof: Let us consider an hexagon $\mathcal{H}$ in B. Since B is bipartite, the vertices of $\mathcal{H}$ lie alternately in L (say a, b, c) and R (say x, y, z) as in the following picture, where we have also named the δ-preimages (in L) of the vertices in R:
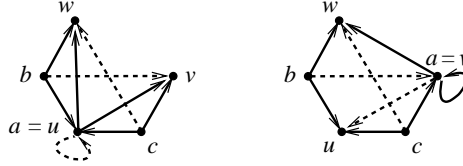

$$a \qquad\qquad x = \delta(u)$$
$$b \qquad\qquad y = \delta(v)$$
$$c \qquad\qquad z = \delta(w)$$

The vertices u, v and w are distinct, as well as a, b and c. We have then, in D, the six arrows $a \to v$, $a \to w$, $b \to u$, $b \to w$, $c \to u$ and $c \to v$. Now $\mathcal{H}$ is not induced in B if and only if either $a \sim x$ or $b \sim y$ or $c \sim y$ in B, and this is equivalent to the existence of an arrow $a \to u$, $b \to v$ or $c \to w$ in D. Therefore we only need to translate this last condition into the compatibility of D with our ten diagrams. There are four cases to consider, depending on the value of $m = |\{a, b, c\} \cap \{u, v, w\}|$.

If $m = 0$, the vertices a, b, c, u, v and w are all distinct, so we have in D the solid part of the diagram

and our condition is that we must also have at least one of the dotted arrows: this is precisely compatibility with diagram A.

If $m = 1$ (say, a in $\{u, v, w\}$ but not b or c) by symmetry there are essentially two possibilities: either $a = u$ or $a = v$. The two corresponding diagrams are obtained by identifying a with u or v in the previous diagram:

and this is clearly compatibility with diagrams B and C.

If $m = 2$ (say, a, b in $\{u, v, w\}$) there are six possibilities that give only four new diagrams. If $a = u$ and $b = v$ (let's denote this as case $[u, v]$) we obtain diagram D. In the cases $[u, w]$, $[v, u]$ and $[v, w]$, diagrams E, F and G are obtained. Cases $[w, u]$ and $[w, v]$ correspond again to diagrams G and E. For $m = 3$, the six possibilities give the remaining three diagrams: Cases $[u, v, w]$, $[u, w, v]$ and $[v, w, u]$ yield diagrams H, I and J. Two of the remaining cases give diagram I again, and the other one diagram J. □

Let us remark that in the above theorem the condition that D is compatible with the ten diagrams A-J is equivalent to the condition that D has no induced subdigraph isomorphic to one in a list of 18,512,100 non-isomorphic digraphs. Diagram A alone forbids 18,448,328 non-isomorphic subdigraphs.

Theorem 4.2. *Let G be a connected graph. Then G is a self-clique hereditary clique-Helly graph if and only if G is the alternating square $G = D^{\wedge 2}$ of some duality digraph D which is N^+-Sperner and compatible with the diagrams of 4.1.*

Proof: If G is self-clique and HCH, take $B = BK(G)$. By 2.4, B is self-dual and hexagon-free. Let $D = D(\delta)$, where δ is a self-duality of B. By 3.3, $G = D^{\wedge 2}$ and D is N^+-Sperner. By 4.1, D is compatible with the diagrams A-J.

Conversely, assume that $G = D^{\wedge 2}$ where D is an N^+-Sperner duality digraph compatible with the diagrams A-J. Using 3.1, let $B = (L, R)$ be a self-dual bipartite graph with self-duality $\delta : B \to B$ such that $D = D(\delta)$. We claim that B is N-Helly by 2.6: indeed, B is hexagon-free by 4.1, and bipartite implies triangleless. Using again that $N_D^+(v) = \delta^{-1}(N_B(v))$ for each $v \in D$, we get that D is N^+-Helly because B is N-Helly. Therefore D is N^+-Helly and N^+-Sperner, so by 3.3 it follows that $D = D(\delta')$ for some duality δ' of $BK(G)$ where G is our graph $G = D^{\wedge 2}$ and it is a clique-Helly self-clique graph. Since $BK(G) \cong B$ by 3.1 and B is hexagon-free, G is HCH by 2.4. □
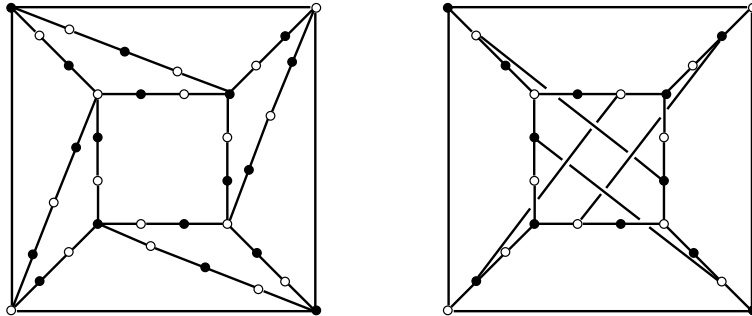
Taking 3.4 into account and recalling that a self-clique graph must by definition be connected, we can remove in 4.2 the hypothesis that G is connected by adding one condition:

Theorem 4.3. *Let G be a graph. Then G is a self-clique hereditary clique-Helly graph if and only if G is the alternating square $G = D^{\wedge 2}$ of some connected duality digraph D which is not cyclically multipartite and is N^+-Sperner and compatible with the diagrams A-J of 4.1.* □

Let $n \geq 2$ and recall the characterization (due to Balconi) of cyclically n-partite strongly connected digraphs that was announced in [2]: they are precisely those strongly connected digraphs for which the length of each directed circuit is a multiple of n. Indeed, let D be a strongly connected digraph such that the length of each directed circuit of D is a multiple of n. Fix $v \in V(D)$. If $w \in V(D)$, the lengths of any two directed vw-paths are congruent modulo n, since adding each of them to the length of a single directed wv-path we get 0 modulo n. Thus, each vertex of D gets a well defined label in $\{0, 1, ..., n-1\}$. Since any arrow starting at a vertex with label i clearly ends at one with label $i+1$, D is cyclically n-partite. Therefore, a duality digraph is not cyclically multipartite if and only if the greatest common divisor of the lengths of its directed circuits is 1.

5. Involutive HCH graphs and possibly loopy graphs

An *involutive graph* is a connected graph G such that $BK(G)$ has a *part-switching involution*, i.e. a self-duality $\sigma : BK(G) \to BK(G)$ such that σ^2 is the identity. Thus, any involutive graph is self-clique and clique-Helly by 2.1. We will characterize involutive HCH graphs in terms of their constructibility from certain graphs that are allowed to have loops. Let us notice that not every self-clique HCH graph is involutive: Indeed, consider the following bipartite graphs B_1 and B_2:

B_1 comes from [12, Fig.2] and B_2 is the vertex-clique bipartite graph of an example in [4, Fig.1]. They are N-Helly by 2.6 and, since they are N-Sperner, they are good. A quarter-turn is a self-duality for both of them, so they are the vertex-clique bipartite graphs of some self-clique HCH graphs G_1 and G_2 by 2.5 and 2.3. However, none of B_1 and B_2 has an involutive self-duality: indeed, the automorphism group in both cases is $\mathbb{Z}_4$ (look at the outer squares) and the only involution does not switch the parts.

A *possibly loopy graph* (or *pl-graph* for short) is a connected "graph" H which is allowed to have some loops (at most one at each vertex) i.e. H is finite, connected and without multiple edges. As in the case of simple graphs, we say that H is N-*Sperner* if $\mathcal{N}(G) = \{N(v) : v \in G\}$ is an antichain, that H is N-*Helly* if $\mathcal{N}(G)$ satisfies the Helly condition and that H is good if it is both N-*Helly* and N-*Sperner*. One only has to keep in mind that for a vertex v in a pl-graph H we have $v \in N(v)$ if and only if there is a loop in v. The *strict square* of the pl-graph H is the graph $G = H^{[2]}$ on the same vertex set as H and in which two vertices $u \neq v$ are adjacent if and only if there are two edges $\{u, x\}, \{x, v\}$ in H for some vertex x.

A forerunner of the result that we shall give for HCH involutive graphs was given in [12] for the subfamily of diamond-free graphs (the *diamond* is $K_4 - e$, and G is diamond-free if it has no induced diamonds):

Theorem 5.1. [12, Thm.9.3] *Let G be a non-trivial graph. Then G is diamond-free and involutive if and only if $G \cong H^{[2]}$ for some non-bipartite pl-graph H which satisfies the following conditions:*

(1) $\delta(H) \geq 2$.
(2) *There are no triangles, squares or hexagons in H.*
(3) *No pentagon of H has a loop.*
(4) *The distance between any two loops of H is at least 3.*

We shall specialize now the conditions in 4.3 to the case in which G is an involutive HCH graph. Let G be a graph and let $B = BK(G)$ with its standard bipartition $B = (L, R)$. By 2.4, G is an involutive HCH graph if and only if G is connected and B has no induced hexagons and has an involutive self-duality $\delta : B \to B$. Consider now some duality δ of B, and let $D = D(\delta)$.
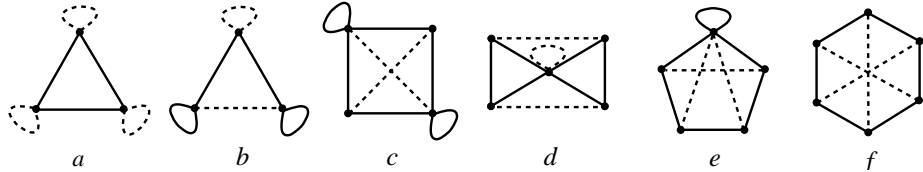
We already know by 4.1 that B has no induced hexagons if and only if D is compatible with our diagrams A-J, and since the translation of D is $\tau = (\delta^2)_{|L}$, it follows that δ is an involution iff τ is the identity (since all the neighbourhoods in B are different, if δ^2 is the identity in L it must also be so in R). But τ is the identity iff D has, together with any arrow $v \to w$ which is not a loop, the inverse arrow $w \to v$. (A loop is its own inverse, anyway.)

Therefore, if $\tau = \mathrm{id}$ and we call H the underlying pl-graph of D (take the heads out of the arrows and merge any resulting double edges) we have that $D^{\wedge 2} \cong H^{[2]}$.

Reciprocally, if H is a pl-graph and we replace each edge by two arrows in opposite directions if it is not a loop, or by an arrow if it is, we get a duality digraph D with $\tau = \mathrm{id}$ such that $D^{\wedge 2} \cong H^{[2]}$.
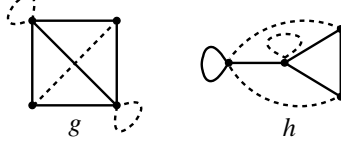
In other words, duality digraphs D with $\tau = \mathrm{id}$ are essentially the same that pl-graphs H, with alternating squares corresponding to strict squares. Since such a D is cyclically multipartite iff D is cyclically 2-partite ($v \to w \to v$ implies this) it is also clear that in our present setting a cyclically multipartite D corresponds to a bipartite H and viceversa. Again, D is N^+-Sperner iff H is N-Sperner.

Theorem 5.2. *Let G be a graph. Then G is an involutive HCH graph if and only if $G = H^{[2]}$ for some non-bipartite and N-Sperner pl-graph H that is compatible with the following six diagrams:*

Proof: By our previous remarks, it is enough to show that a duality digraph D with $\tau = \mathrm{id}$ is compatible with the diagrams A-J of 4.1 if and only if its underlying pl-graph H is compatible with the above diagrams *a-f*. To begin with, such a D is compatible with the diagrams A-J if and only if it is compatible with their underlying pl-graphs, and *a-f* are the underlying pl-graphs of H, I, F, B, C and A

respectively. Any D with $\tau = $ id is compatible with diagrams G and J, so we don't need them here. The underlying pl-graphs of diagrams D and E are the following:

but compatibility with a and b clearly implies compatibility with g and h.   □

6. Embeddings in self-clique HCH graphs

Escalante proved in [7, Satz 7] that any graph is an induced subgraph of a clique-Helly self-clique graph. This was strengthened in [12, Thm.7.2]: any graph is an induced subgraph of some involutive graph. Notice that not every graph is an induced subgraph of some involutive HCH graph, since for these any induced subgraph is necessarily HCH. Therefore, the following is the best possible result in this direction:

Theorem 6.1. *Any hereditary clique-Helly graph G is an induced subgraph of some involutive graph G' which is also hereditary clique-Helly.*

Before giving the proof we recall some needed concepts and results, and prove an auxiliary one. A graph G is said to be K-*periodic* (or just *periodic*) if $K^n(G) \cong G$ for some $n > 0$, and the smallest such n is the *period* of G. A vertex $v \in G$ is *dominated* if $N[v] \subseteq N[w]$ for some other vertex $w \in G$. Escalante [7, Satz 3] proved essentially that a clique-Helly graph G is periodic if and only if G does not have dominated vertices, and that in this case the period is 1 or 2. A reformulation of [7, Satz 3] can be found in [12, Thm.2.4].

Theorem 6.2. [12, Thm.2.7] *Let B be a bipartite graph. Then B is good if and only if there exists a periodic clique-Helly graph G with $B \cong BK(G)$.*

Theorem 6.3. [12, Thm.5.4] *A graph G is involutive if and only if $G \cong H^{[2]}$ for some pl-graph H which is good, connected and non-bipartite.*

Theorem 6.4. *Let G be a hereditary clique-Helly graph. Then G is an induced subgraph of some periodic graph G' which is also hereditary clique-Helly.*

Proof: We apply a method due to Escalante [7]. We can assume that G is non-trivial. By 2.2 we know that G is Hajós-free. If G has less than 4 vertices, replace it by the disjoint union of some copies of it. Now G' is obtained by attaching a pendant edge to each vertex of G and forming a cycle with the free vertices of these edges. Clearly G' is connected and also Hajós-free, so by 2.2 G' is HCH. Since G' is clique-Helly and has no dominated vertices, it is periodic.   □

Proof of 6.1: We first apply the method of 6.4, so we can assume that G is periodic and has some clique Q that is an edge. Since G is HCH, it follows from [14, Cor.4.1] that $K(G)$ is also HCH. The following is an instance of the *vertex-clique construction*, which works more generally (see [12, §7] for the details). Start with $B = BK(G)$, which is good by 6.2. Clearly $B^{[2]} = G \cup K(G)$ is its own clique graph and it is Hajós-free, but it is disconnected. Now construct the graph H by attaching to the vertex Q of B a new pendant edge and putting a loop at the free vertex of

this edge. Since H is still good and connected, but now it is non-bipartite, we have by 6.3 that $G' = H^{[2]}$ is involutive. We have that G' is obtained from $G \cup K(G)$ by adding a new vertex u and connecting it to $Q \in V(K(G))$ and to the two vertices of $Q = \{v, w\}$ in $V(G)$. Only the new triangle $\{u, v, w\}$ is formed, but it does not share an edge with any other triangle because $\{v, w\}$ was a clique in $G \cup K(G)$, so no new Hajós graph is formed and G' is HCH by 2.2. Finally, it is clear that both G and $K(G)$ are induced subgraphs of G'.   □

References

[1] R. Balakrishnan and P. Paulraja. *Self-clique graphs and diameters of iterated clique graphs.* Utilitas Math. **29** (1986) 263–268.

[2] G. Balconi, M. Grieco and B. Zucchetti. *Grafi di dualità di strutture d'incidenza.* Bolletino U. M. I. (5) **18**-B (1981) 799–810.

[3] G. Balconi, M. Grieco and B. Zucchetti. *Algoritmo per la construzione di grafi di dualitá.* Atti Sem. Mat. Fis. Univ. Modena **XXX** (1981) 131-137.

[4] A. Bondy, G. Durán, M.C. Lin and J.L. Szwarcfiter. *Self-clique graphs and matrix permutations.* Journal of Graph Theory **44** (2003) 178-192.

[5] F. Bonomo and G. Durán *Computational complexity of classical problems for hereditary clique-Helly graphs.* Pesquisa Operacional **24** (2004) 435-443.

[6] G.L. Chia. *On self-clique graphs with given clique sizes.* Discrete Math. **212** (2000) 185–189.

[7] F. Escalante. *Über Iterierte Clique-Graphen.* Abh. Math. Sem. Univ. Hamburg. **39** (1973) 58-68.

[8] R.C. Hamelink. *A partial characterization of clique graphs.* J. Combin. Theory **5** (1968) 192-197.

[9] T.A. McKee. *A new characterization of strongly chordal graphs.* Discrete Math. **205** (1999) 245-247.

[10] F. Larrión, V. Neumann-Lara, M.A. Pizaña and T.D. Porter. *Self-Clique Graphs with Prescribed Clique-Sizes.* Congressus Numerantium **157** (2002) 173-182.

[11] F. Larrión, V. Neumann-Lara, M.A. Pizaña and T.D. Porter. *Recognizing Self-Clique Graphs.* Matemática Contemporânea **25** (2003) 125-133.

[12] F. Larrión, V. Neumann-Lara, M.A. Pizaña and T.D. Porter. *A hierarchy of self-clique graphs.* Discrete Math. **282** (2004) 193-208.

[13] C.K. Lim and Y.H. Peng. *On graphs without multicliqual edges.* J. Graph Theory **5** (1981) 443–451.

[14] E. Prisner. *Hereditary clique-Helly graphs.* J. Combin. Math. Combin. Comput. **14** (1993), 216–220.

[15] F.S. Roberts and J.H. Spencer. *A characterization of clique graphs.* J. Combinatorial Theory Ser. B **10** (1971) 102–108.

[16] J.L. Swarcfiter. *Recognizing Clique-Helly Graphs.* Ars Combin. **45** (1997) 29-32.

[17] S. Tsukiyama, M. Ide, H. Ariyoshi and I. Shirakawa. *A new algorithm for generating all the maximal independent sets.* SIAM J. Comput. **6** (1977) 505–517.

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