SELF-CLIQUE GRAPHS WITH PRESCRIBED CLIQUE-SIZES

F. Larrión*, V. Neumann-Lara*, M.A. Pizaña†, and T.D. Porter‡

Abstract

The clique graph of a graph G is the intersection graph $K(G)$ of the (maximal) cliques of G. A graph G is called self-clique whenever $G \cong K(G)$. This paper gives various constructions of self-clique graphs. In particular, we employ (r, g)-cages to construct self-clique graphs whose set of clique-sizes is any given finite set of integers greater than 1.

Keywords: Cage, clique Helly graphs, self-clique graphs, vertex-clique bipartite graph.

The clique graph $K(G)$ of a graph G is the intersection graph of the (maximal) cliques of G. A connected graph G is called *self-clique* if $G \cong K(G)$. In [9] the authors presented a hierarchy and classification of self-clique graphs. It can be shown that the previous self-clique constructions [1, 3, 4] are contained in one of these classes, the so-called *involutive* self-clique graphs.

A family $\mathcal{F}$ of subsets of a set $X \neq \emptyset$ has the *Helly property* (or is Helly) if $\cap S \neq \emptyset$ for any $S \subseteq \mathcal{F}$ such that $p, p' \in S \Rightarrow p \cap p' \neq \emptyset$. In other words, if the members of S are *pairwise* intersecting, then there exists an element common to all. A graph G is said to be *clique-Helly* if the family of all cliques of G has the Helly property. Wallis and Wu introduced the *vertex-clique bipartite graph* [12] of a graph G. It is the bipartite graph $BK(G) = (X, Y)$, where X denotes the vertices of G, Y denotes the cliques of G and xy is an edge iff x is in clique y. The tensor product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set consisting of those pairs of vertices (g, h), (g', h') where g is adjacent to

*Instituto de Matemáticas, U.N.A.M., México. {paco, neumann} @math.unam.mx

†Depto. de Ingeniería Eléctrica, Universidad Autónoma Metropolitana, México. map@xanum.uam.mx.

‡Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4408, USA. tporter@math.siu.edu.

g' and h is adjacent to h'. In [7], George, Porter, and Wallis characterized balanced bipartite graphs (X, Y) that admit an involution that maps the vertices of one part X to vertices of the other part Y. We include that result here. Notice that $K_2 \times H$ is a balanced bipartite graph for any H.

Theorem 1 ([7]). *The following conditions on a bipartite graph (X, Y) are equivalent:*

(1) (X, Y) has a symmetric bipartite adjacency matrix;

(2) (X, Y) may be written as $K_2 \times H$ for some H, (H may have loops);

(3) (X, Y) has an involution that maps the vertices from X to Y. □

We now describe the relationship between a self-clique graph G and the corresponding graph H, where $(X, Y) = BK(G) \cong K_2 \times H$. Such self-clique graphs G are called *involutive*. All involutive graphs are clique-Helly [9].

First, notice that if $(X, Y) = BK(G)$ admits a part-switching involution, then G is a self-clique graph. ¿From Theorem 1, we also have $(X, Y) = BK(G) \cong K_2 \times H$, for some H. Let the vertex set X of G be labelled $1, \ldots, n$ and let Y be labelled $1, \ldots, n$, where the involution from X to Y takes i to i. Then H is the graph with vertex set $1, \ldots, n$, where the open neighborhood, $N(i)$, of vertex i in $V(H)$ is the set of vertices in the corresponding clique i in Y. H is allowed one loop at each vertex. More useful though, is the other way, i.e., reconstructing G, where $BK(G) = K_2 \times H$, from H. For this, define the *strict-square*, $H^{[2]}$ of H, to be the graph with $V(H^{[2]}) = V(H)$, and uv is an edge in $H^{[2]}$ iff there is a $u - v$ walk in H that uses exactly two different edges. Then, we have $G \cong H^{[2]}$.

Also, if one has H, where $(X, Y) = BK(G) \cong K_2 \times H$, then to obtain the isomorphism ϕ between G and its clique graph $K(G)$, you may take the function that maps each u in $V(H) = V(G)$ to the open neighborhood $N(u)$ in H, i.e., the open neighborhoods of H are the maximal cliques in G. (Please see figures 1(a,b,c) for an example.) Note the loops present at vertices 5, 6, 7 in Fig. 1(a). We use the convention that a loop at a vertex adds only one to the degree of the vertex.

We can classify these involutive self-clique graphs. We call a graph H *good* if $(X, Y) = K_2 \times H \cong BK(G)$ for some involutive self-clique graph G. We say that H is neighborhood-Sperner (N-Sperner) if no open neighborhood of H is contained in another. We say that H is neighborhood-Helly (N-Helly) if the open neighborhoods of H satisfy the Helly property.

Lemma 2. *If H is good, then it is N-Sperner.*

Proof. Contrarily, suppose $N(i) \subseteq N(j)$ for some $i \neq j$. Then with $(X,Y) = BK(G) \cong K_2 \times H$, we would have the corresponding cliques C_i, C_j with vertex sets $N(i)$, resp., $N(j)$ as vertices of Y. But then $C_i \subseteq C_j$ contradicts the definition of $(X,Y) = BK(G)$, i.e., that Y contains only the maximal cliques of G. □

Lemma 3. *If H is good, then it is N-Helly.*

Proof. Contrarily, suppose H is not N-Helly. Then let F be a maximal family of vertices in H whose open neighborhoods are pairwise intersecting but do not possess a common vertex. By the strict-square transformation $H^{[2]}$, the corresponding vertices F in G form a maximal clique in G, but without this intersecting common vertex, there is thus no vertex j in H with $N(j) = F$, hence this maximal clique is not represented at any vertex in Y, contradicting the definition of Y. □

Notice that if H is N-Helly and N-Sperner, then by our defined construction $(X,Y) = BK(G) \cong K_2 \times H$, we have $G = H^{[2]}$ is an involutive self-clique graph. In conjunction with Lemmas 2, 3 we have:

Theorem 4. *A graph H is good iff it is N-Helly and N-Sperner.*

□

We remark that if H is good, the involutive self-clique graph $G = H^{[2]}$ is clique-Helly. However, not all self-clique graphs are clique-Helly. This was first shown by Escalante [6]. In [9], it is also shown that not every Helly self-clique graph is involutive. In [11], Szwarcfiter gives a polynomial time algorithm that determines whether G is clique-Helly. This is remarkable, since a graph of order n may have exponentially many cliques.

Given a good graph H and via the transformation $G = H^{[2]}$, we see that beginning with H gives a very clean and powerful method to generate self-clique graphs. First, notice that the degree sequence $(d_1, d_2, \ldots, d_m)$ of H, with $d_1 \leq \cdots \leq d_n$, also represents the sizes of the m cliques in G.

We now construct self-clique graphs which contain given lists of clique sizes. This idea has been studied by Escalante [6] and Chia [4]. Escalante's result gives:

Theorem 5 ([6]). *Let G be a connected graph whose cliques are all of size 2. Then G is self-clique if and only if G is a cycle of length at least 4.*

Chia [4] characterized the self-clique graphs whose cliques all have size 2, except precisely for one clique. For a graph G, whose m cliques have sizes $a_1 \leq a_2 \leq \cdots \leq a_m$, call $(a_1, \ldots, a_m)$ the *clique size sequence* of G. We then have by definition of H and $H^{[2]}$:

Theorem 6. *If H is good with degree sequence $(d_1, \ldots, d_m)$ and $G = H^{[2]}$ has clique size sequence $(a_1, \ldots, a_m)$, then $(d_1, \ldots, d_m) = (a_1, \ldots, a_m)$.* □

Chia characterized the self-clique graphs with clique size sequence $(2, 2, \dots, p)$ where $p \geq 3$. Notice that Fig. 1(b) is a Chia graph with clique size sequence $(2, 2, 2, 2, 2, 2, 3)$, and Fig. 1(a) is its associated good graph H.

A convenient way to generate a Chia graph through H is as follows: Let H be the graph with a central vertex designated c, attach to c, p paths each of length at least two, and then attach a loop to the p ending leafs of each path. (See Fig. 2.) It is easy to check this H is *good*, and $H^{[2]}$ then gives a self-clique Chia graph with clique size sequence $(2, 2, \dots, p)$.

We now generalize this concept. Define the clique-size set, $\mathcal{C}(G)$, of a graph G, to be the set of clique sizes in G, disregarding multiplicities. For example, a Chia graph has clique-size set $\{2, p\}$, for some $p \geq 3$. We first give an interpolation type result; which gives a self-clique graph with clique-size set $\{2, 3, 4, \dots, p\}$, for any $p \geq 3$. We then show that for any finite set of distinct integers $X = \{2, l_1, l_2, \dots, l_k\}$ with $l_i \geq 3$ for $1 \leq i \leq k$, there is a self-clique graph G, with $X = \mathcal{C}(G)$.

We use Gilmore's [2] theorem on conformal hypergraphs to check if a graph H is N-Helly. First we give a restatement of the Helly property due to Roberts and Spencer [10].

Theorem 7 ([10]). *Let $\mathcal{F}$ be a family of subsets of X. Put $\widehat{A} = \{S \in \mathcal{F} : |S \cap A| \geq 2\}$ for $A \subseteq X$. Then $\mathcal{F}$ is Helly if and only if $\cap \widehat{A} \neq \emptyset$ for any $A = \{x, y, z\} \subseteq X$.* □

Now we apply Gilmore's theorem and Theorem 7.

Theorem 8. *Let H be a graph that may contain loops. Then H is N-Helly if and only if Gilmore's condition holds: For any three different vertices with pairwise intersecting neighborhoods there exists a vertex whose neighborhood contains the union of the pairwise intersections of the neighborhoods of those vertices.*

Proof. This follows from Theorem 7: Put $\mathcal{F} = \{N(v) : v \in V(H)\}$ and fix $A = \{v_1, v_2, v_3\}$. Put $I_{ij} = N(v_i) \cap N(v_j)$ for $i \neq j$, and $I = I_{12} \cup I_{13} \cup I_{23}$. Assume that $|A| = 3$ and all $I_{ij} \neq \emptyset$, for otherwise $\cap \widehat{A} \neq \emptyset$. Since $|N(v) \cap A| \geq 2 \Leftrightarrow v \in I$, if $w \in V(H)$: $w \in \cap \widehat{A} \Leftrightarrow w \in N(v) \, \forall v$ s.t. $|N(v) \cap A| \geq 2 \Leftrightarrow v \in N(w) \, \forall v \in I \Leftrightarrow I \subseteq N(w)$. □

We now give an interpolation type construction. Consider the graph $K_{1,p}$ and 'identify' (glue-together) each leaf i in $K_{1,p}$ with a vertex in a given graph H_i, call this resulting graph $star(H_1, \dots, H_p)$. Now define $S(3) = star(C_5, C_5, C_5)$ (see Fig. 3), and recursively generate a sequence of star graphs as follows:

$$S(3) = star(C_5, C_5, C_5)$$
$$S(n+1) = star(\underbrace{S(n), S(n), \dots, S(n)}_{n+1\text{-times}}), \quad n \geq 3.$$

Here, in $S(n+1)$, each leaf vertex in the underlying graph $K_{1,n+1}$ is identified with the center vertex in $S(n)$. First, $S(3)$ is N-Helly and N-Sperner, and it is straightforward to check by Theorem 8 that if $S(n)$ is N-Helly then so is $S(n+1)$. Also, by symmetry, a case-by-case check that $S(n)$ is N-Sperner is easily verified. So we have that the star graphs $S(p)$, $p \geq 3$ are N-Helly and N-Sperner. Hence by Theorem 4, the strict square graph $S(p)^{[2]}$ is an involutive self-clique graph. Notice that since the degree set of $S(p)$ is $\{2, \dots, p\}$, the clique-size set of $S(p)^{[2]}$ is also $\{2, \dots, p\}$, i.e., $\mathcal{C}(S(p)^{[2]}) = \{2, \dots, p\}$. Here as usual, the degree set of a graph is the set of vertex degrees in G.

We remark that if a connected graph H with at least three vertices contains a vertex u of degree 1, then H is not N-Sperner. Since, with uv being an edge of H, and $z \in N(v)$, then $N(u) \subseteq N(z)$. So we require $\delta(H) \geq 2$. Also, if $H_1, \dots, H_p$ is a family of loopless good graphs, with $\delta(H_i) \geq 2$, then $star(H_1, \dots, H_p)$ is also good. This is readily checked, the only considerations are the leaf vertices in the underlying $K_{1,p}$. (See Fig. 4.) So we have:

Theorem 9. *If $H_1, \dots, H_p$ is a family of loopless good graphs, with $\delta(H_i) \geq 2$, then $star(H_1, \dots, H_p)$ is good.* □

Let $X = \{2, l_1, \dots, l_p\}$ denote any finite set of distinct integers. We construct an involutive self-clique graph with clique-size set X. First, notice the 7-cycle. C_7 is N-Helly and N-Sperner. It is straightforward to show that C_n, $n \geq 7$ is good. The construction from X is as follows:

Begin with a cycle C_m, where $m = \max\{7, p\}$.

Case 1. $m = p$.

To the cycle C_m with the vertex set $\{1, \dots, m\}$, attach to vertex i, $l_i - 2$ paths each of length two, then attach a loop at the end of each of these paths. Notice the degree set of this resulting graph is $\{2, l_1, \dots, l_p\}$. A quick check gives that this graph is N-Helly, N-Sperner, and hence the strict-square transformation gives an involutive self-clique graph with clique set X.

Case 2. $m = 7$.

Here with $X = \{2, l_1, \dots, l_p\}$, $p < 7$. We use the same construction as Case 1, but here leave the vertices $p+1, \dots, 7$ on C_7 as original. So we have:

Theorem 10. *For any finite set $X = \{2, x_1, \dots, x_p\}$ of distinct integers there exists a self-clique graph with clique-size set X.* $\square$

Notice our above constructions contain vertices of degree 2, and hence, after the strict-square transformation, the resulting self-clique graph contains cliques of size two. So we now ask; for any Y, does there exist self-clique graphs with clique-size set $Y = \{l_1, \dots, l_p\}$ with $l_i \geq 2$? We can answer this in the affirmative. Before we consider this we state a theorem from [9].

Theorem 11 ([9]). *Any graph with girth ≥ 7 and minimum degree ≥ 2 is good.* $\square$

Consider the case $Y = \{r\}$, $r \geq 2$. If one can find an r-regular good graph H, then by Theorem 6, $H^{[2]}$ is an involutive self-clique graph with clique-size set $Y = \{r\}$. It can be shown that amongst the five platonic solids, only the dodecahedron is N-Helly and N-Sperner. Hence, the strict-square of the dodecahedron is an involutive self-clique graph with clique-size set $\{3\}$. For the general case $Y = \{r\}$, $r \geq 2$, we need the result of Erdös and Sachs [5] concerning cages.

For any $r \geq 2$, $g \geq 2$, there exists an r-regular graph with girth g. Such a graph with minimum order is an *(r, g)-cage*. So, for the case $Y = \{r\}$, let H be an $(r, 7)$-cage, then by Theorem 11, we have H is good, hence $H^{[2]}$ is an involutive self-clique graph with clique-size set $\{r\}$.

For the general case, let $Y = \{l_1, \dots, l_p\}$, $l_i \geq 2$, $p \geq 2$ be given. We construct a chain of p cages. Take any $g \geq 7$, we form a chain of (l_i, g)-cages for $1 \leq i \leq p$ from left-to-right as follows:

The first cage C_1, is a (l_1, g)-cage; the second cage C_2 is a (l_2, g)-cage. Let ab be any edge from C_1, and cd any edge from C_2. We attach C_1, C_2 as follows; remove edge ab from C_1, and edge cd from C_2, and then add two 'new' edges ac and bd. Notice this new graph has degree set $\{l_1, l_2\}$ and girth $\geq g$. We continue this process; to connect C_2 to C_3, we take any other edge ef in C_2, where ef is not incident with cd; and then take gh in C_3, again remove edges ef and gh, and then connect C_2 to C_3 by adding the two 'new' edges eg and fh. (See Fig. 5.) We continue this process, forming a chain $C_1, C_2, \dots, C_p$ of p cages. Call this resulting graph H. Notice, by the construction that H has degree set $Y = \{l_1, \dots, l_p\}$ and girth $\geq g$. Hence, by Theorem 11, the strict-square graph $H^{[2]}$ is an involutive self-clique graph with clique set Y. Notice we can form this construction using any $g \in \{7, 8, \dots\}$, hence:

Theorem 12. *For any set $Y = \{l_1, \dots, l_p\}$, with $l_i \geq 2$, there exists infinitely many self-clique graphs with clique-size set Y.* $\square$

A final construction using the girth 7 property is as follows. Consider any simple connected graph H; replace each edge in H with two paths,

P_1, P_2, where P_1 has length 3, P_2 has length 4. Notice, this new graph, $\widehat{H}$, now has girth 7 and by Theorem 11 $\widehat{H}$ is good. If the degree set of H is $\{x_1, \dots, x_p\}$, then the degree set of $\widehat{H}$ is $\{2, 2x_1, \dots, 2x_p\}$. Recall that Kapoor, Polimeni, and Wall [8] showed that for any set $\{a_1, \dots, a_p\}$, there exists a simple graph whose degree set is $\{a_1, \dots, a_p\}$, Hence, if $Y = \{l_1, \dots, l_p\}$ is a set with each l_i even, then let H be the graph with degree set $\{l_1/2,\ l_2/2,\ \dots,\ l_p/2\}$, then $\widehat{H}$ has degree set $\{2, l_1, \dots, l_p\}$. Since $\widehat{H}$ is good, $\widehat{H}^{[2]}$ is an involutive self-clique graph with clique-size set $\{2, l_1, \dots, l_p\}$.

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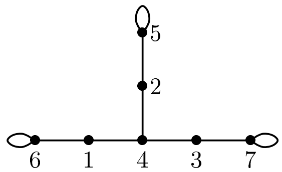

Fig. 1(a): A good graph H

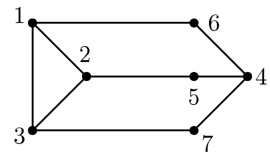

Fig. 1(b): The self-clique graph $G = H^{[2]}$

$$\phi: \begin{array}{ccc} 1 & \to & \{4,6\} \\ 2 & \to & \{4,5\} \\ 3 & \to & \{4,7\} \\ 4 & \to & \{1,2,3\} \\ 5 & \to & \{2,5\} \\ 6 & \to & \{1,6\} \\ 7 & \to & \{3,7\} \end{array}$$

Fig. 1(c): The isomorphism ϕ between G and $K(G)$

Figure 1

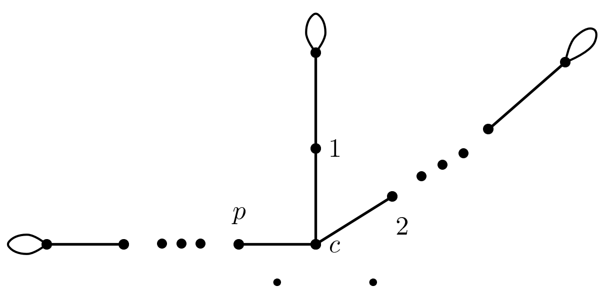

Figure 2: A good graph H where $H^{[2]}$ is a Chia Graph

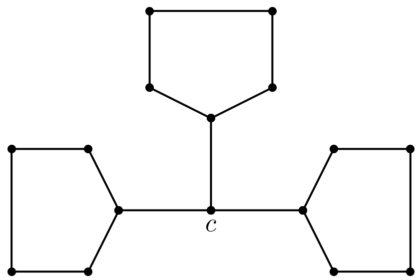

Figure 3(a): The star graph $S(3)$

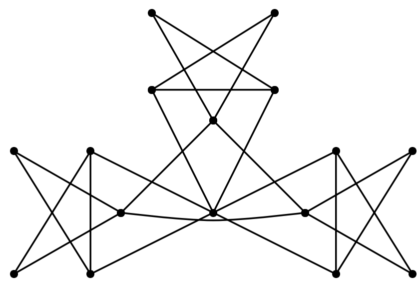

Figure 3(b): The self-clique graph $H^{[2]}$
with $H = S(3)$ and clique set $= \{2, 3\}$

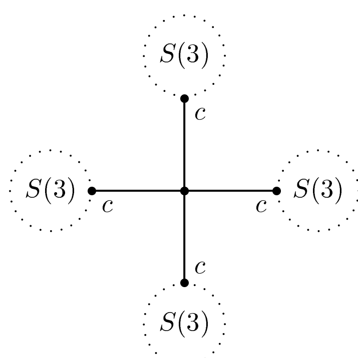

Figure 3(c): The star graph $S(4)$

Figure 3

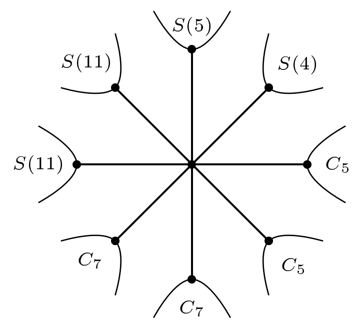

Figure 4: A $star(H_1, \dots, H_8)$ graph

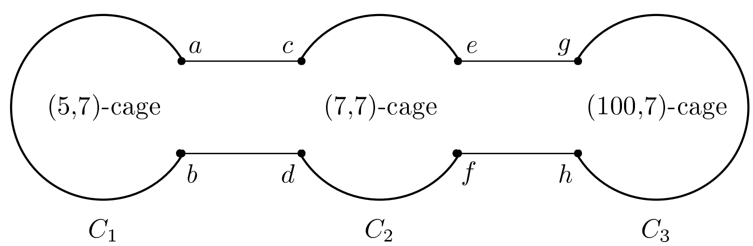

Figure 5: An example of a chain of cages with degree set $\{5, 7, 100\}$ and girth ≥ 7