On Defensive Alliances and Strong Global Offensive Alliances

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Abstract

We consider complexity issues and upper bounds for defensive alliances and strong global offensive alliances in graphs. We prove that it is NP-complete to decide for a given 6-regular graph G and a given integer a, whether G contains a defensive alliance of order at most a. Furthermore, we prove that determining the strong global offensive alliance number $\gamma_{\hat{o}}(G)$ of a graph G is APX-hard for cubic graphs and NP-hard for chordal graphs. For a graph G of minimum degree at least 2, we prove $\gamma_{\hat{o}}(G) \leq 3n(G)/4$, which improves previous results by Favaron et al. and Sigarreta and Rodríguez. Finally, we prove $\gamma_{\hat{o}}(G) \leq \left(\frac{1}{2} + (1 + o(\delta(G)))\frac{\ln(\delta(G)+1)}{\delta(G)+1}\right) n(G)$.

Keywords: defensive alliance; strong global offensive alliance

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1 Introduction

The everyday phenomenon of defensive and offensive alliances is pervasive and occurs as a natural concept in numerous situations. In [12] Kristiansen et al. initiated the systematic study of alliances as a graph theoretic notion. They defined several versions of alliances corresponding to different interpretations and scenarios. In the present paper we contribute results on so-called defensive alliances and strong global offensive alliances. Defensive alliances were proposed by Flake et al. [8] as an abstract model of web communities in the internet. The study of offensive alliances was initiated by Favaron et al. [6] and the global version was studied by Sigarreta et al. [16].

We consider finite, simple, and undirected graphs. For a graph G, the *vertex set* and the *edge set* are denoted by $V(G)$ and $E(G)$, respectively. The *order* and *size* of G are denoted by $n(G)$ and $m(G)$, respectively. For a vertex u of a graph G, the *neighborhood* and the *degree* of u in G are denoted by $N_G(u)$ and $d_G(u)$, respectively. Furthermore, the *closed neighborhood* $N_G(u) \cup \{u\}$ of u in G is denoted by $N_G[u]$. For a set U of vertices of a graph G, the *boundary* $\partial_G(U)$ of U in G is the set $\{u \in V(G) \setminus U : N_G(u) \cap U \neq \emptyset\}$. The *minimum degree* and *maximum degree* of vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph is *r-regular* if all its vertices have degree r. A set of pairwise non-adjacent vertices of a graph G is *independent*. An independent set is *maximal* if it is not contained in a larger independent set. The maximum cardinality of an independent set of G is the *independence number* $\alpha(G)$ of G. A set D of vertices of a graph G is *k-dominating* for some positive integer k, if every vertex in $V(G) \setminus D$ has at least k neighbors in D. A 1-dominating set is *dominating*.

A non-empty set A of vertices of a graph G is

- a *defensive alliance* if $|N_G[u] \cap A| \geq |N_G[u] \setminus A|$ for every $u \in A$ and
- an *offensive alliance* if $|N_G[u] \setminus A| \geq |N_G[u] \cap A|$ for every $u \in \partial_G(A)$.

If a defensive/offensive alliance is dominating it is called *global*; note that in this case $\partial_G(A) = V(G) \setminus A$. Furthermore, if the inequalities in the above definitions are required to be strict, then the defensive/offensive alliance is called *strong*. The minimum cardinalities of a defensive alliance, offensive alliance, strong offensive alliance, and strong global offensive alliance of a graph G are denoted by $a(G)$, $a_o(G)$, $a_{\hat{o}}(G)$, and $\gamma_{\hat{o}}(G)$, respectively. Clearly, $a_o(G) \leq a_{\hat{o}}(G) \leq \gamma_{\hat{o}}(G)$ for every graph G.

Our contributions concern the complexity of defensive alliances and strong global offensive alliances. Furthermore, we present new bounds for $\gamma_{\hat{o}}(G)$. We are now going to explain our contributions and closely related known results in detail.

The first problem we consider is the following.

DEFENSIVE ALLIANCE
Instance: A graph G and an integer a.
Question: Is there a defensive alliance A in G with $|A| \leq a$, that is, is $a(G) \leq a$?

In [10, 11] Jamieson et al. prove that DEFENSIVE ALLIANCE is NP-complete even when restricted to either split graphs, or chordal graphs, or bipartite graphs. Their constructions necessarily lead to vertices of high degree. Since no defensive alliance of order at most a can contain a vertex of degree at least $2a+1$, high degree vertices are trivially excluded from small alliances, which facilitates the proof of the hardness result. It is therefore interesting to study the complexity of DEFENSIVE ALLIANCE restricted to regular graphs. In [3] Araujo-Pardo and Barrière observe that DEFENSIVE ALLIANCE is easy for r-regular graphs with $r \leq 5$.

Our first result is that DEFENSIVE ALLIANCE is NP-complete for 6-regular graphs (Theorem 2), which completely settles the complexity for regular graphs.

The next problem we consider is the following.

STRONG GLOBAL OFFENSIVE ALLIANCE

Instance: A graph G and an integer a.

Question: Is there a strong global offensive alliance A in G with $|A| \leq a$, that is, is $\gamma_{\hat{o}}(G) \leq a$?

The complexity of global offensive alliances was considered by Cami et al. in [5] where they prove NP-completeness of deciding for a given graph G and a given integer a, the existence of a global offensive alliance of G with at most a vertices. In [15] Rodríguez and Sigarreta observe that STRONG GLOBAL OFFENSIVE ALLIANCE is NP-complete.

Our results are that STRONG GLOBAL OFFENSIVE ALLIANCE is NP-complete for chordal graphs (Theorem 3) and that the minimization version of STRONG GLOBAL OFFENSIVE ALLIANCE is APX-hard for cubic graphs (Theorem 5).

Finally, we consider bounds.

In [6] Favaron et al. prove

$$a_{\hat{o}}(G) \leq \frac{5}{6} n(G) \tag{1}$$

for every graph G of order at least 3 and

$$a_{\hat{o}}(G) \leq \frac{3}{4} n(G) \tag{2}$$

for every graph G of minimum degree at least 2.

In [16] Sigarreta and Rodríguez prove

$$\gamma_{\hat{o}}(G) \leq \frac{5}{6} n(G) \tag{3}$$

for every graph G of minimum degree at least 2. Furthermore, they prove [15, 16] that

$$\gamma_{\hat{o}}(G) \leq \frac{3}{4} n(G) \tag{4}$$

for every cubic graph G.

We prove that

$$\gamma_{\hat{o}}(G) \leq \frac{3}{4} n(G)$$

for every graph G of minimum degree at least 2 (Theorem 6), which implies (2), (3), and (4). Note that (1) cannot be extended to $\gamma_{\hat{o}}(G)$ in view of the edge-less graph.

Furthermore, we prove (Theorem 7)

$$\gamma_{\hat{o}}(G) \leq \left(\frac{1}{2} + (1 + o(\delta(G))) \frac{\ln(\delta(G) + 1)}{\delta(G) + 1} \right) n(G),$$

which is better than any of the above estimates for large minimum degree. Finally we discuss algorithmic consequences of our bounds.

2 Results

For the sake of completeness, we give a short argument for the observation of Araujo-Pardo and Barrière [3] mentioned above.

Observation 1 (Araujo-Pardo and Barrière [3]) Defensive Alliance *can be solved in polynomial time for r-regular graphs with $r \leq 5$.*

Proof: If $r \leq 1$, then a single vertex is a defensive alliance. If $2 \leq r \leq 3$, then two adjacent vertices form a defensive alliance. Finally, if $4 \leq r \leq 5$, then the vertex set of a cycle forms a defensive alliance and every minimal defensive alliance must be a cycle. Since a shortest cycle in a graph can be determined in polynomial time, the desired statement follows. □

We proceed to our first main result.

Theorem 2 Defensive Alliance *is NP-complete for 6-regular graphs.*

Proof: Defensive Alliance is clearly in NP. In order to show NP-completeness, we describe a reduction from the NP-complete problem One-In-Three 3Sat having only positive literals (cf. problem [L04] in [9]). Let $\mathcal{C}$ be an instance of One-In-Three 3Sat that uses the boolean variables $x_1, \ldots, x_n$ and consists of the clauses $C^1, \ldots, C^m$, where each clause consists of exactly three positive literals. Without affecting the NP-completeness, we assume additionally that for every two variables x and x', there is a sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ of variables such that $x = x_{i_1}$, $x' = x_{i_k}$, and for every $\ell \in [k-1]$, $\mathcal{C}$ contains a clause that contains both x_{i_ℓ} and $x_{i_{\ell+1}}$, that is, the bipartite variable-clause incidence graph is connected. If this condition would fail, then $\mathcal{C}$ could be partitioned into independent smaller instances, which could be solved separately.

We construct a 6-regular graph G and some integer a such that the encoding length of (G, a) is polynomially bounded in n and m and there exists a truth assignment for the x_i satisfying exactly one literal of each clause C^j of $\mathcal{C}$ if and only if there is a defensive alliance A in G with $|A| \leq a$.

First, we construct a graph H starting with the empty graph.

- For every $j \in [m]$, we add to H a clause gadget G^j as in Figure 1.

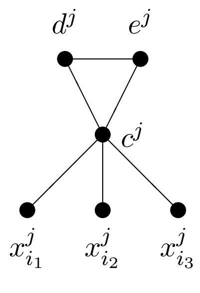

Figure 1: The clause gadget G^j for the clause C^j consisting of the three literals x_{i_1}, x_{i_2}, and x_{i_3}.

For every $i \in [n]$, let k_i denote the number of clauses of $\mathcal{C}$ that contain the literal x_i. Note that $k_1 + \cdots + k_n = 3m$.

- For every $i \in [n]$, we add to H a variable gadget G_i that arises from a path $P_i = x_{i,1} \dots x_{i,4k_i}$ by adding the $2k_i$ edges $x_{i,\ell}x_{i,2k_i+\ell}$ for $\ell \in [2k_i]$.

- We add m further edges among the vertices in $\{d^j : j \in [m]\} \cup \{e^j : j \in [m]\}$ such that $d^1e^1d^2e^2 \dots d^me^md^1$ forms a cycle.

- For every $i \in [n]$, if $C^{j_1}, \dots, C^{j_{k_i}}$ denote the k_i clauses of $\mathcal{C}$ that contain x_i, then we add $k_i + 1$ further edges among the vertices in $\{x_{i,1}, x_{i,4k_i}\} \cup \{x_i^{j_\ell} : \ell \in [k_i]\}$ such that $x_{i,1}x_i^{j_1}x_i^{j_2} \dots x_i^{j_{k_i}}x_{i,4k_i}$ forms a path.

This completes the construction of H; see Figure 2 for an example. Note that H has order $6m + \sum\limits_{i=1}^{n} 4k_i = 18m$ and size $6m + \sum\limits_{i=1}^{n}(4k_i - 1 + 2k_i) + m + \sum\limits_{i=1}^{n}(k_i + 1) = 28m$. In fact, the vertices $c^1, \dots, c^m$ have degree 5 in H and all remaining vertices have degree 3 in H. Let $a = 8m$.

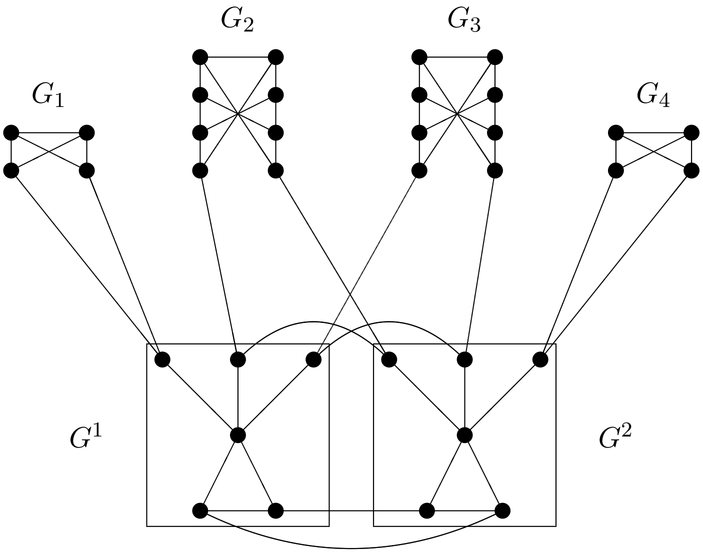

Figure 2: The graph H for the instance $\mathcal{C}$ consisting of the two clauses $C_1 : x_1 \vee x_2 \vee x_3$ and $C_2 : x_2 \vee x_3 \vee x_4$.

Claim A *There is a truth assignment for the x_i satisfying exactly one literal of each clause C^j of $\mathcal{C}$ if and only if there is a non-empty set A of vertices of H that induces a subgraph of minimum degree at least 3 in H and is of order at most a.*

Proof of claim A: First, we assume that there is a truth assignment as in the statement. Since in each clause exactly one literal is true, we have $\sum\limits_{i:x_i \text{ true}} k_i = m$ and the set

$$A = \bigcup_{j=1}^{m} \Big(V(G^j) \setminus \{x_i^j : i \in [n] \text{ and } x_i \text{ is a false literal in } C^j\} \Big) \cup \bigcup_{i:x_i \text{ true}} V(G_i)$$

induces a 3-regular subgraph of H of order exactly $4m + \sum\limits_{i:x_i \text{ true}} 4k_i = a$.

Conversely, let A be a non-empty set of vertices of H that induces a subgraph of minimum degree at least 3 in H and is of order at most a. Note that if A contains a vertex u of degree 3 in H, then A contains $N_H(u)$ completely.

If A does not contain a vertex from $\{d^j : j \in [m]\} \cup \{e^j : j \in [m]\}$, then the connectivity of the bipartite variable-clause incidence graph implies

$$A = \bigcup_{j=1}^{m} \left(V(G^j) \setminus \{d^j, e^j\}\right) \cup \bigcup_{i=1}^{m} V(G_i),$$

that is, A is of order $4m + \sum_{i=1}^{n} 4k_i = 16m$, which is a contradiction. Hence, A intersects $\{d^j : j \in [m]\} \cup \{e^j : j \in [m]\}$, which, by the above remark, implies that A contains $\{c^j : j \in [m]\} \cup \{d^j : j \in [m]\} \cup \{e^j : j \in [m]\}$ and for every $j \in [m]$, A contains at least one of the three neighbors of c^j distinct from d^j and e^j.

Clearly, for every $i \in [n]$, either A and $V(G_i)$ are disjoint or A contains $V(G_i)$.

Let $T = \{i \in [n] : V(G_i) \subseteq A\}$. Note that $\sum_{i \in T} k_i \geq m$.

If there is some $j \in [m]$ such that A contains two of the three neighbors of c^j distinct from d^j and e^j, then A is of order at least $(4m + 1) + \sum_{i \in T} 4k_i > 8m$, which is a contradiction. This implies that for every $j \in [m]$, A contains exactly one of the three neighbors of c^j distinct from d^j and e^j. This implies that setting x_i to true exactly if $i \in T$ defines a truth assignment for the x_i satisfying exactly one literal of each clause C^j of $\mathcal{C}$, which completes the proof. $\square$

The construction of G relies on a regularity gadget R. We construct R starting with the empty graph as follows. Let $h = \lceil \log_2(a) \rceil + 1$.

- We add four complete 5-ary trees T_1, T_2, T_3, and T_4 of depth h rooted in the vertices r_1, r_2, r_3, and r_4, respectively.
- We add the two edges r_1r_2 and r_3r_4.
- Let L denote the set of leaves of the forest constructed so far. We add further edges such that L induces a 5-regular graph.

This completes the construction of R. Clearly, R is 6-regular and the order of R is polynomial in a.

Claim B *If A is a set of vertices of R that contains r_1 and induces a subgraph $R[A]$ of R such that $d_{R[A]}(r_1) \geq 2$ and $d_{R[A]}(v) \geq 3$ for every $v \in A \setminus \{r_1\}$, then $|A| > a$.*

Proof of claim B: It follows by an inductive argument that for every $1 \leq i \leq h$, A contains at least 2^{i-1} vertices in $V(T_1)$ that are of depth i. Since $1 + \sum_{i=1}^{h} 2^{i-1} > a$, this completes the proof of the claim. $\square$

We proceed to the iterative construction of G starting with H. While there is a vertex v of degree at most 4, we add a copy of R, delete the edge r_1r_2, and add the two edges vr_1 and vr_2. Furthermore, while there are two vertices v_1 and v_2 of degree 5, we add a copy of R, delete the edge r_1r_2, and add the two edges v_1r_1 and v_2r_2. Since the number of vertices of odd degree is always even, this results in a 6-regular graph. Furthermore, the number of disjoint copies of R that we added to H is polynomially bounded in n and m. Therefore, the order of G is polynomially bounded in n and m.

Claim B implies that H has a non-empty induced subgraph of minimum degree at least 3 and order at most a if and only if G has a non-empty induced subgraph of minimum degree at least 3 and order at most a. Since G is 6-regular, this happens if and only if there is an alliance A in G with $|A| \leq a$. By Claim A, the proof is complete. $\Box$

We proceed to our results concerning Strong Global Offensive Alliance.

Theorem 3 Strong Global Offensive Alliance *is NP-complete for chordal graphs.*

Proof: Clearly, Strong Global Offensive Alliance is in NP. It is known [4] that the problem

Minimum Dominating Set

Instance:	A graph G and an integer k.
Question:	Does G have a dominating set D of order k?

is NP-complete even restricted to chordal graphs. We describe a reduction from Minimum Dominating Set to Strong Global Offensive Alliance.

Let G be a chordal graph. Let G' arise from G by appending to every vertex v of G exactly $d_G(v)$ new vertices of degree 1. Note that the number of added vertices equals $2m(G)$ and that every strong global offensive alliance of G' necessarily contains all new vertices.

If D is a dominating set of G, then let $A = D \cup (V(G') \setminus V(G))$. Clearly, every vertex u in $V(G') \setminus A$ has at least $d_G(u) + 1$ many neighbors in A, that is, A is a strong global offensive alliance of G'.

Conversely, if A is a strong global offensive alliance of G', then $V(G') \setminus V(G) \subseteq A$. Let $D = A \setminus (V(G') \setminus V(G))$. Since every vertex u in $V(G) \setminus D$ has at least $d_G(u) + 1$ neighbors in A, every such vertex has a neighbor in D, that is, D is a dominating set of G.

Since $|A| - |D| = 2m(G)$ in both cases, the desired result follows. $\Box$

In view of Theorem 3 and the fact that Minimum Dominating Set can be solved in polynomial time for strongly chordal graphs [6], we pose the following problem.

Problem 4 *Is* Strong Global Offensive Alliance *NP-complete for strongly chordal graphs?*

Our next result concerns the minimization version of Strong Global Offensive Alliance.

Minimum Strong Global Offensive Alliance

Instance:	A graph G.
Task:	Find a strong global offensive alliance in G of minimum cardinality.

Theorem 5 *There is some $\epsilon > 0$ such that approximating* Minimum Strong Global Offensive Alliance *within a factor $(1 + \epsilon)$ is NP-hard for cubic graphs.*

Proof: We use the technique of L-reductions [13].

It is known [1] that there is some $\epsilon > 0$ such that approximating MAXIMUM INDEPENDENT SET within a factor $(1+\epsilon)$ is NP-hard for cubic graphs, that is, given a cubic graph G, it is NP-hard to determine an independent set I with $|I| \geq \frac{1}{1+\epsilon}\alpha(G)$.

It is obvious that a set of vertices in a cubic graph is a strong global offensive alliance if and only if its complement is an independent set, that is, $\alpha(G) = n(G) - \gamma_{\hat{o}}(G)$ for a cubic graph G. Therefore, given a cubic graph G, it is NP-hard to determine a strong global offensive alliance A with $n(G) - |A| \geq \frac{1}{1+\epsilon}(n(G) - \gamma_{\hat{o}}(G))$ or equivalently $|A| \leq \frac{1}{1+\epsilon}\gamma_{\hat{o}}(G) + \frac{\epsilon}{1+\epsilon}n(G)$. Since G is cubic, the independence number of G is at least $\frac{n(G)}{4}$ and hence $n(G) \geq \frac{4}{3}\gamma_{\hat{o}}(G)$. This implies that, given a cubic graph G, it is NP-hard to determine a strong global offensive alliance A with

$$|A| \leq \frac{1}{1+\epsilon}\gamma_{\hat{o}}(G) + \frac{\epsilon}{1+\epsilon}\left(\frac{4}{3}\gamma_{\hat{o}}(G)\right) = \left(1 + \frac{\epsilon/3}{1+\epsilon}\right)\gamma_{\hat{o}}(G).$$

Altogether, setting $\epsilon' = \frac{\epsilon/3}{1+\epsilon}$, it is NP-hard to approximate $\gamma_{\hat{o}}(G)$ for a cubic graph G within a factor of $1+\epsilon'$. $\Box$

We proceed to bounds on $\gamma_{\hat{o}}(G)$.

Theorem 6 *If G is a graph of minimum degree at least 2, then*

$$\gamma_{\hat{o}}(G) \leq \frac{3}{4}n(G).$$

Furthermore, a strong global offensive alliance of order at most $\frac{3}{4}n(G)$ can be found in polynomial time.

Proof: Our proof relies on a variation of the method of [16].

Let I_1 be a maximal independent set of G. Let I_2 be a maximal independent set of the subgraph of G induced by $\{u \in V(G) \setminus I_1 : |N_G(u) \cap I_1| = 1\}$. As long as $|I_2| > |I_1|$, we iteratively replace I_1 by a maximal independent set of G containing I_2 and redefine I_2 as above. After polynomially many steps this results in sets I_1 and I_2 with $|I_2| \leq |I_1|$.

Since G is of minimum degree at least 2, $V(G) \setminus I_1$ is a strong global offensive alliance, which implies

$$\gamma_{\hat{o}}(G) \leq n(G) - |I_1|.$$

Let $X \cup Y$ be a partition of $V(G) \setminus (I_1 \cup I_2)$ such that $|N_G(u) \cap Y| \geq |N_G(u) \cap X|$ for every vertex u in X and $|N_G(u) \cap X| \geq |N_G(u) \cap Y|$ for every vertex u in Y. Such a partition can be found by starting with an arbitrary partition and exchanging vertices that do not satisfy the given conditions. Since the number of edges between X and Y increases with every exchange, the process ends after polynomially many steps. We may assume that $|X| \leq |Y|$.

Let $A = V(G) \setminus Y$. Let $u \in Y$. Since I_1 is a maximal independent set of G, u has at least one neighbor in I_1. If u has two neighbors in I_1, then $|N_G[u] \cap A| > |N_G[u] \setminus A|$. If u has exactly one neighbor in I_1, then the choice of I_2 implies that u has a neighor in I_2. Hence also in this case

$|N_G[u] \cap A| > |N_G[u] \setminus A|$. It follows that A is a strong global offensive alliance, which implies

$$\gamma_{\hat{o}}(G) \leq |I_1| + |I_2| + |X| \leq |I_1| + |I_2| + \frac{n(G) - |I_1| - |I_2|}{2} = \frac{n(G) + |I_1| + |I_2|}{2} \leq \frac{n(G) + 2|I_1|}{2}.$$

Altogether, we obtain

$$\gamma_{\hat{o}}(G) \leq \min\left\{n(G) - |I_1|, \frac{n(G) + 2|I_1|}{2}\right\} \leq \frac{3}{4}n(G),$$

which completes the proof. $\Box$

Theorem 7 *If G is a graph with $\delta(G) + 1 \geq 4\ln(\delta(G) + 1)$, then*

$$\gamma_{\hat{o}}(G) \leq \left(\frac{1}{2} + \frac{1}{2(\delta(G) + 1)} + \frac{1}{2(\delta(G) + 1)^2} + \frac{\ln(\delta(G) + 1)}{\delta(G) + 1}\right) n(G). \tag{5}$$

Furthermore, a strong global offensive alliance of order at most the right hand side of (5) can be found in polynomial time.

Proof: The proof relies on results of [14] and [16].

Let D be a 2-dominating set of G. As a generalization of an observation made by Favaron et al. [7], Sigarreta and Rodríguez [16] show that $\gamma_{\hat{o}}(G) \leq \frac{n(G)+|D|}{2}$. In fact, if $X \cup Y$ is a partition of $V(G) \setminus D$ as in the proof of Theorem 6, then $D \cup X$ is a strong global offensive alliance of order at most $\frac{n(G)+|D|}{2}$.

In [14] it is shown that G has a 2-dominating set D of order at most

$$\frac{n(G)}{\delta(G) + 1}\left(\frac{2\ln(\delta(G) + 1)}{\delta(G) + 1} + \frac{1}{\delta(G) + 1} + 1\right).$$

Since the proof in [14] relies on a simple application of the first moment method, it can be easily derandomized using the method of conditional expectation [2]. Since the partition $X \cup Y$ can be found in polynomial time, the bound (5) and the algorithmic statement follow. $\Box$

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