THE CLIQUE OPERATOR ON MATCHING AND CHESSBOARD GRAPHS

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Abstract. Given positive integers m, n, we consider the graphs G_n and $G_{m,n}$ whose simplicial complexes of complete subgraphs are the well-known matching complex M_n and chessboard complex $M_{m,n}$. Those are the *matching* and *chessboard* graphs. We determine which matching and chessboard graphs are clique-Helly. If the parameters are small enough, we show that these graphs (even if not clique-Helly) are homotopy equivalent to their clique graphs. We determine the clique behavior of the chessboard graph $G_{m,n}$ in terms of m and n, and show that $G_{m,n}$ is clique-divergent if and only if it is not clique-Helly. We give partial results for the clique behavior of the matching graph G_n.

1. Introduction

Our graphs are finite, simple and nonempty. A *matching* of a graph G is a set of disjoint edges of G. The *matching complex* of G is the simplicial complex $M(G)$ whose simplexes are the matchings of G. When $G = K_n$ (complete graph) the complex $M_n = M(K_n)$ is known just as the *matching complex*. For $G = K_{m,n}$ (complete bipartite graph) $M_{m,n} = M(K_{m,n})$ is known as the *chessboard complex*.

Chessboard and matching complexes have received a lot of attention in the literature, given that they occur in several, seemingly unrelated contexts. Chessboard complexes appeared first as certain "coset complexes" in the thesis of P. Garst ([9]), and then they appeared in work of Vrećica and Živaljević as "complexes of injective functions" ([27]). The first unifying survey of the combinatorial properties of matching and chessboard complexes was done by Björner, Lovász, Vrećica and Živaljević ([2]). From a different perspective, Bouc found matching complexes in his study of the topology of Quillen complexes ([3]). He observed that the fundamental group of M_7 is cyclic of order 3. This was explained later by group-theoretic methods in [1], but in a combinatorial context, already Hall ([11]) had proved a result that implies that the universal cover of M_7 is 3-to-1. Our treatment in this paper is closer to that of Hall, in the sense that we consider graph-theoretical properties of the 1-skeleton of matchings and chessboard complexes. Recent papers on these complexes and generalizations include [22, 23, 26, 6].

Given a graph G, the collection of complete subgraphs of G also forms a simplicial complex, which we denote by $\Delta(G)$. By means of the geometric realization of $\Delta(G)$ one usually attaches topological concepts to G. For instance, we say that the graphs G and H are *homotopy equivalent*, and denote it by $G \simeq H$, if the geometric realizations of $\Delta(G)$ and $\Delta(H)$ are so.

A simplicial complex of the form $\Delta = \Delta(G)$ for some graph G is called a *Whitney complex* (also known as a *clique complex*). Necessarily, G is the 1-skeleton of Δ. The matching complex of any graph is Whitney, since $M(G) = \Delta(\overline{L(G)})$ where $L(G)$ is the *line graph* of G (the intersection graph of the edges of G). It is therefore natural to call the complement $\overline{L(G)}$ the *matching graph of* G. The matching graph of K_n will be denoted by G_n and called the *matching graph*. Similarly, the *chessboard graph* is the matching graph $G_{m,n}$ of $K_{m,n}$. Note that G_n is also the Kneser graph $K_{n:2}$ of [10, Chap. 7]. All these graphs are clearly vertex-transitive, and this will be implicitly used in several arguments in this work.

A *clique* of a graph is a maximal complete subgraph. The *clique graph* of G is the intersection graph $K(G)$ of the cliques of G (see [24] for a survey). *Iterated clique graphs* are defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. We study the dinamics of the clique operator K and distinguish several kinds of K-*behavior*: The graph G is called *clique-divergent* or K-*divergent* if the order of $K^n(G)$ tends to infinity with n. If this is not the case, it is easy to see that G is *eventually* K-*periodic* (also called K-*convergent*): $K^t(G) \cong K^{t+p}(G)$ for some integers $t \geq 0$ and $p \geq 1$; when $t = 0$ we say that G is K-*periodic*, and if p is minimal we call it the *period* of G. In case that G is connected and K-periodic of period one, we say that G is *self-clique*.

A graph G is *clique-Helly* if any collection of pairwise intersecting cliques has a nonempty intersection. In Section 3 we determine which matching and chessboard graphs are clique-Helly. By a pioneering result of Prisner [21], each clique-Helly graph is homotopy equivalent to its clique graph. However, many non-clique-Helly graphs G still satisfy $K(G) \simeq G$. We show in Section 4 that some non-clique-Helly matching and chessboard graphs have this property. In order to do this we shall use a generalization of Prisner's result due to Larrión, Neumann-Lara and Pizaña [16] and a further similar result (4.2) of our own.

Escalante [7] proved that all clique-Helly graphs are eventually K-periodic, so they are not K-divergent. By a result of Szwarcfiter [25], clique-Hellyness is recognizable in polynomial time. On the other hand, there is no known algorithm to recognize K-divergence. Indeed, even to determine whether such an algorithm exists is an open problem [18]. However, it is known that for some restricted classes of graphs one can in fact detect K-divergence algorithmically: extended P_4-sparse graphs [5], regular locally cyclic graphs [17], cographs [14], complements of cycles [20, 19] and powers of cycles [19]. Furthermore, in the last three of these classes, the notions of K-divergence and non-clique-Hellyness coincide. We shall prove in Section 5 that chessboard graphs also exhibit this property.

2. Preliminaries

All our graphs are finite, simple and non-empty. We usually identify induced subgraphs with their vertex sets; for instance, we write $v \in G$ rather than $v \in V(G)$. A complete subgraph will be called just a *complete*, so here a *clique* is a maximal complete. We denote the cyclic graph on n vertices by C_n and the disjoint union of three copies of K_2 as $3K_2$. The complement of $3K_2$ will play an important role in this work. It can also be described as the complete multipartite graph $K_{2,2,2}$ with three parts of size two, or as the (1-skeleton of the) octahedron $\mathcal{O}_3$.

The *open neighborhood* of a vertex $a \in G$ is the set $N_G(a)$ of all neighbors of a in G, and the *closed neighborhood* is $N_G[a] = N_G(a) \cup \{a\}$. We say that a vertex a is *dominated* if $N_G[a] \subseteq N_G[x]$ for some vertex $x \neq a$ in G.

Theorem 2.1. (Escalante, [7]) *Let G be a clique-Helly graph. Then G is eventually K-periodic of period at most two. Furthermore, G is K-periodic (i.e. $K^2(G) \cong G$) if and only if G has no dominated vertices.*

Let us denote $N_G[a] \cap N_G[b]$ by $N_G[a, b]$. A *triangle* is a complete with three vertices. If $T = \{a, b, c\}$ is a triangle of G, its *extended triangle* is the subgraph $\hat{T}$ of G induced by the vertex set $N_G[a, b] \cup N_G[a, c] \cup N_G[b, c]$. A *cone* is a graph containing a *universal vertex*, i.e. a vertex $v \in G$ such that $N_G[v] = G$.

The following result, due to Szwarcfiter, gives a very useful criterion for clique-Hellyness of a graph. It clearly leads to a polynomial time recognition algorithm.

Theorem 2.2. (Szwarcfiter, [25]) *A graph G is clique-Helly if and only if every extended triangle of G is a cone.*

As mentioned in the Introduction, besides the K-behavior and clique-Hellyness of matching and chessboard graphs, we will study the relation between the homotopy type of these graphs and that of their clique graphs. The first result of this kind was proved for clique-Helly graphs by Prisner in 1992 and we record it here for future reference:

Theorem 2.3. (Prisner, [21]) *If G is clique-Helly, then $G \simeq K(G)$.*

Larrión, Neumann-Lara and Pizaña [16] gave a condition that shows that many graphs G which are not clique-Helly still satisfy $G \simeq K(G)$. A clique of cliques $Q = \{q_1, \ldots, q_n\} \in K^2(G)$ is a *necktie* if $\cap Q = \cap_{i=1}^{n} q_i = \varnothing$. Clique-Helly graphs do not have neckties. If X is a complete of $K(G)$ such that $\cap X = \varnothing$, then $q_0 \in K(G)$ is a *center* of X if $\cap(Y \cup \{q_0\}) \neq \varnothing$ whenever $Y \subseteq X$ and $\cap Y \neq \varnothing$.

Theorem 2.4. [16] *Let G be such that each complete X of $K(G)$ with $\cap X = \varnothing$ has a center that is contained in every necktie that contains X. Then $G \simeq K(G)$.*

The *product* of two graphs G and H (also called *times*, or *tensor product*) is the graph $G \times H$ on the cartesian product of the vertex sets of G and H in which the ordered pairs (g, h), (g', h') are neighbors if g, g' are so in G and h, h' are so in H.

We will use an alternative description of the chessboard graph $G_{m,n} = \overline{L(K_{m,n})}$:

Proposition 2.5. *The chessboard graph $G_{m,n}$ is isomorphic to $K_m \times K_n$.*

Proof: The edges of the bipartite graph $K_{m,n}$ can be thought of as ordered pairs (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$ joining the i-th vertex of the first part of the bipartition to the j-th vertex of the second part. Two edges (i, j), (i', j') are declared adjacent in $G_{m,n}$ whenever they are disjoint, that is, if $i \neq i'$ and $j \neq j'$. But this is exactly the situation in which (i, j) and (i', j') are adjacent in the product $K_m \times K_n$. □

In other words, 2.5 says that we can label the vertices of the chessboard graph $G_{m,n}$ as (i, j) for $1 \le i \le m$, $1 \le j \le n$, and that two of these pairs are adjacent whenever they differ in both coordinates. This result also makes apparent the reason for the name "chessboard graph": the vertices can be taken to be the squares of an $m \times n$ chessboard, and two vertices are neighbors if two rooks put at the corresponding squares cannot take each other.

In order to avoid trivially settled cases that would only clutter the statements of our results, we will only consider matching graphs G_n with $n \ge 3$ and chessboard graphs $G_{m,n}$ with $2 \le m \le n$.

3. Connectedness and clique-Hellyness

We will always label the vertices of the complete graph K_n as $1, 2, \ldots, n$.

Since we are mainly interested in connected graphs, we note:

Theorem 3.1. *The matching graph G_n is connected if and only if $n \ge 5$. The chessboard graph $G_{m,n}$ is connected if and only if $m + n \ge 5$.*

Proof: It is immediate that G_n is disconnected if $n = 3, 4$, and that $G_{m,n}$ is disconnected if $m = n = 2$. So, assume $n \ge 5$ and consider two distinct non-adjacent vertices of G_n, that is, two distinct but intersecting edges of the complete graph K_n. Without loss of generality, we can assume those edges to be 12, 13. Then $12, 45, 13$ form a path in G_n. In the case of $G_{m,n}$, we assume $2 \le m < n$. Two distinct nonadjacent vertices in $G_{m,n}$ can be assumed to be $(1, 1)$ and $(1, 2)$, and we have a path $(1, 1), (2, 3), (1, 2)$. □

Now we determine, in terms of the parameters, which matching and chessboard graphs are clique-Helly.

Theorem 3.2. *The matching graph G_n is clique-Helly if and only if $n \le 6$. The chessboard graph $G_{m,n}$ is clique-Helly if and only if $m = 2$ or $m + n \le 6$.*

Proof: We apply 2.2. It is clear that G_n for $n \le 5$, and $G_{m,n}$ for $m = 2$ (hence for $m+n \le 5$) are clique-Helly since they have no triangles. Now, for $n = 6$, a triangle in G_6 can be assumed to be $T = \{12, 34, 56\}$. But then $\hat{T} = T$, which is a cone. We consider $G_{3,3}$. A triangle T can be assumed to be $T = \{(1, 1), (2, 2), (3, 3)\}$, and in this case we have again $\hat{T} = T$. Thus, the conditions on the parameters are sufficient in both statements.

Let us assume that $n \ge 7$, and let $T = \{12, 34, 56\}$ be a triangle in G_n. For each $v \in \{7, \ldots, n\}$, consider the edge $1v$ of K_n, which lies in $N_{G_n}[34, 56]$. Let $X = T \cup \{\, 1v : 7 \le v \le n \,\} \subseteq \hat{T}$. No edge of K_n which is outside of X can be disjoint to all edges in X. Thus only 34 and 56 are candidates to be universal in $\hat{T}$. But 34 is not disjoint to $37 \in N_{G_n}[12, 56] \subseteq \hat{T}$, and 56 is not disjoint to $57 \in \hat{T}$. Hence the extended triangle of T is not a cone, so G_n is not clique-Helly.

For the chessboard graph, we consider now $m \ge 3$, $n \ge 4$, and the triangle $T = \{(1, 1), (2, 2), (3, 3)\}$. For each $i \in \{4, \ldots, n\}$ the edge $(1, i)$ of $K_{m,n}$ lies in

$N_{G_{m,n}}[(2,2),(3,3)]$. Again, if $X = T \cup \{\,(1,i) : 4 \le i \le n\,\}$, we get that $X \subseteq \hat{T}$ and that no edge of $K_{m,n}$ which is outside of X can be disjoint to all edges in X. Then only $(2,2)$ and $(3,3)$ could be universal in $\hat{T}$, but $(2,2)$ is not disjoint to $(2,4) \in N_{G_{m,n}}[(1,1),(3,3)]$ and $(3,3)$ is not disjoint to $(3,4) \in N_{G_{m,n}}[(1,1),(2,2)]$. Therefore the extended triangle of T is not a cone, and $G_{m,n}$ is not clique-Helly in this case. □

Lemma 3.3. *The matching graph G_n has no dominated vertices for $n \ge 5$.*
The chessboard graph $G_{m,n}$ has no dominated vertices for $m+n \ge 5$.

Proof: Consider G_n for $n \ge 5$. Since G_n is vertex-transitive, it will suffice to show that its vertex 12 is not dominated. A neighbor of 12 has the form ij, with $i,j \ge 3$. Let $k \in \{1,\dots,n\}\setminus\{1,2,i,j\}$. Then ik is a neighbor of 12 which is not adjacent to ij. Hence 12 is not dominated in G_n in this case.

Consider now $G_{m,n}$ with $m+n \ge 5$. Then $m \ge 2$ and $n \ge 3$. A neighbor of $(1,1)$ has the form (i,j) with $i,j \ge 2$. Let $k \in \{1,\dots,n\}\setminus\{1,j\}$. Therefore (i,k) is a neighbor of $(1,1)$ which is not a neighbor of (i,j). Hence $(1,1)$ is not a dominated vertex in $G_{m,n}$. □

If G is a matching or chessboard graph, and G is connected, clique-Helly and without dominated vertices, then G is either self-clique or 2-periodic by 2.1. By 3.1, 3.2 and 3.3, this covers the cases G_5, G_6, $G_{2,n}$ with $n \ge 3$, and $G_{3,3}$.

The graph G_5 is the well known Petersen graph. It has 10 vertices, 15 edges, and no triangles. The cliques of G_5 are its edges, so $K(G_5) \not\cong G_5$ and G_5 is 2-periodic.

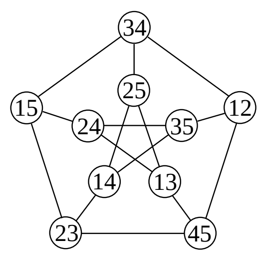

FIGURE 1. The Petersen graph as G_5

The graph $G_{2,3}$ is isomorphic to the cycle C_6, and it is clearly self-clique.

More generally, $G_{2,n}$ has $2n$ vertices, $n(n-1)$ edges and it has no triangles. In consequence, $G_{2,n}$ is 2-periodic whenever $n \ge 4$.

A clique of $G_{3,3}$ has the form $\{(1,i),(2,j),(3,k)\}$ with $\{i,j,k\} = \{1,2,3\}$. We can identify such a clique with the permutation of $\{1,2,3\}$ given by the assignations $1 \mapsto i$, $2 \mapsto j$, $3 \mapsto k$. Hence $K(G_{3,3})$ has six vertices, and $G_{3,3}$ is 2-periodic.

Finally, we will prove that G_6 is self-clique after two results of independent interest. Given graphs G and H, we say that G is *locally* H if $N_G(x) \cong H$ for all $x \in G$. For instance, it is easy to see that G_{n+2} is locally G_n for $n \ge 2$ and that $G_{m+1,n+1}$ is locally $G_{m,n}$ for $n \ge m \ge 1$.

Proposition 3.4. *If a graph G is locally $3K_2$, then $K(G)$ is locally $3K_2$ too.*

Proof: If G is locally $3K_2$, then for any $v \in G$ we have that $N_G[v]$ is composed of three triangles whose pairwise intersections consist only of the vertex v. It follows that the cliques of G are exactly its triangles and that every edge of G is contained in exactly one triangle. Let $T = \{0, 1, 2\}$ be a triangle in G. For $i = 0, 1, 2$, write $N_G[i]$ as the union of three triangles $N_G[i] = T \cup T_i \cup T'_i$, where $T \cap T_i = T \cap T'_i = T_i \cap T'_i = \{i\}$. See Figure 2(a). Now, if we had some $w \in T_0 \cap T_1$, then $N_G[w]$ would not be isomorphic to $3K_2$. It then follows that $N_{K(G)}(T) = \{T_0, T'_0, T_1, T'_1, T_2, T'_2\}$ is isomorphic to $3K_2$. Hence $K(G)$ is locally $3K_2$, as we wanted to prove. □

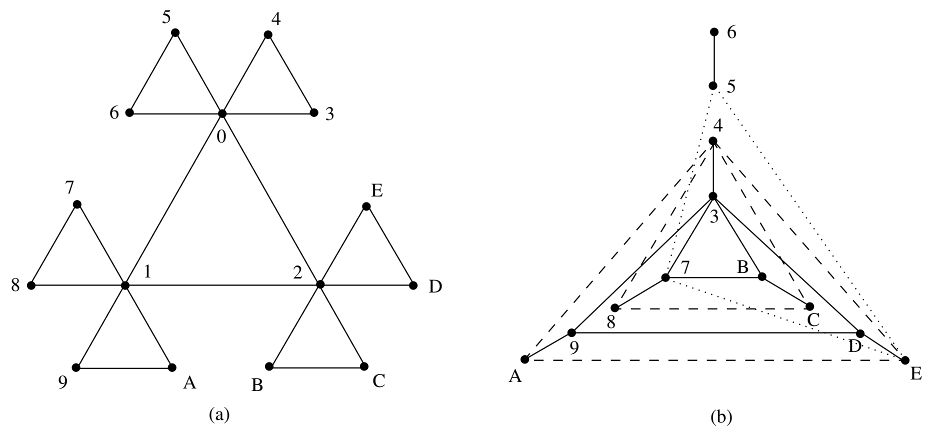

FIGURE 2. Partial drawings of a locally $3K_2$ graph.

Proposition 3.5. *The only locally $3K_2$ graph with 15 vertices is G_6.*
In fact, G_6 is the unique smallest locally $3K_2$ graph.

Proof: $G_4 \cong 3K_2$, so G_6 is locally $3K_2$. If $M_0 = \{0, 1, 2\} = 012$ is a triangle of a locally $3K_2$ graph H, we know already that H has at least the 15 distinct vertices of Figure 2(a). But there is essentially one way to complete Figure 2(a) to a locally $3K_2$-graph of order 15, as we shall see now.

Note that no two triangles of H can share an edge. Let $M_1 = 034, M_2 = 056, \ldots, M_6 = 2DE$ and $P_0 = \{3, 4, 5, 6\}$, $P_1 = \{7, 8, 9, A\}$ and $P_2 = \{B, C, D, E\}$. Besides the seven triangles already drawn in Figure 2(a), each vertex in $\{3, 4, \ldots E\}$ still needs two incident triangles and hence H must have another 8 triangles, say $T_1, T_2 \ldots, T_8$. Note that each of these triangles T_i uses exactly one vertex of each P_j, for otherwise there would be unwanted triangles which would share and edge with some other. Note also that each vertex in $\{3, \ldots, E\}$ still needs exactly four edges and hence:

Observation. If $M_i \cap M_j = \varnothing$, then the edges between M_i and M_j form a perfect matching for $i, j = 1, \ldots, 6$.

It will be convenient to refer to Figure 2(b) for the rest of the proof. Now, relabelling if necesary we may assume $\boldsymbol{T_1 = 37B}$. There is another triangle at vertex 3, say T_2, but then $8, C \notin T_2$ (otherwise unwanted triangles would form) and $7, B \notin T_2$

(otherwise $|T_2 \cap T_1| \geq 2$). Relabeling if necesary we may assume without loss that $\boldsymbol{T_2 = 39D}$. Now the observation above yields the adjacencies: $4 \sim 8, A, C, E$ and $A \sim E$, $8 \sim C$ and thus, two new triangles are necessarily formed: $\boldsymbol{T_3 = 48C}$ and $\boldsymbol{T_4 = 4AE}$.

Edges between 056 and 178 form a perfect matching, hence, without loss, we have $5 \sim 7$ and $5, 7 \in T_5$. Then $B, C \notin T_5$, but also $D \notin T_5$ (otherwise it would form a triangle $37D$ sharing an edge with T_1). Hence $\boldsymbol{T_5 = 57E}$. Let T_6 be the other triangle meeting vertex 5. Then it follows that $7, 8, D, E \notin T_6$. Now observe that AC can not be an edge (triangle $4AC$ would share an edge with T_3). Hence the matching between $19A$ and $2BC$ must use edges AB and $9C$. But $5B$ can not be and edge since then the triangle $57B$ would share an edge with T_1. Hence $\boldsymbol{T_6 = 59C}$. Now the observation says that $\boldsymbol{T_7 = 68D}$ and $\boldsymbol{T_8 = 6AB}$. A direct verification shows that the graph constructed is indeed a locally $3K_2$ graph. □

Theorem 3.6. *The graph G_6 is self-clique.*

Proof: We have seen that the cliques of G_6 are its triangles. Since each vertex of G_6 is contained in three triangles and each triangle contains three vertices, G_6 has as many triangles as vertices, and so, by 3.4, $K(G_6)$ is a locally $3K_2$ graph with 15 vertices. We have then $K(G_6) \cong G_6$ by 3.5. □

We point out that there are other results involving matching and chessboard graphs which are analogous to 3.5. For example, a similar proof shows that $G_{3,4}$ is the unique smallest locally $G_{2,3} = C_6$ graph. Another similar proof, but much easier, shows that also $G_{3,3}$ is the smallest locally $G_{2,2} = 2K_2$ graph. Buset proves in [4] that there are only two locally $G_{2,4}$ graphs, namely, $G_{3,5}$ and H_{24}, the 1-skeleton of the 24-cell. A stronger unpublished result of Brouwer is mentioned without proof in [4] and [13], namely, that if $m + n \geq 6$ and G is a locally $G_{m,n}$ graph, then $G \cong G_{m+1,n+1}$ save for Buset's exception at $m = 2$, $n = 4$, where we can have $G \cong H_{24}$, or for $m = 3$, $n = 3$ and $G \cong J(6, 3)$, a Johnson graph. From all this it follows that for all the cases considered, i.e. when $n \geq m \geq 2$, $G_{m+1,n+1}$ is the smallest locally $G_{m,n}$ graph. On the other hand, since G_5 is the Petersen graph, G_7 is locally Petersen. By work of Hall [11], there are only three connected locally Petersen graphs, of which G_7 is the smallest.

4. Homotopy equivalences

If the graph G has no induced subgraph isomorphic to a given graph H, we say that G is *H-free.*

Lemma 4.1. *If a graph G has at most eight vertices, then its matching graph $\overline{L(G)}$ is $\mathcal{O}_3$-free.*

Proof: The graph $\overline{L(G)}$ is $\mathcal{O}_3$-free if, and only if, $L(G)$ does not have $\overline{\mathcal{O}_3} = 3K_2$ as an induced subgraph. A copy of $3K_2$ in $L(G)$ involves nine vertices in G. □

On the other hand, note that for $n \geq 9$, $\{12, 13, 45, 46, 78, 79\}$ induces a copy of $\mathcal{O}_3$ inside G_n. If $m \geq 3$ and $m + n \geq 9$, then the dots marked in Figure 3 represent

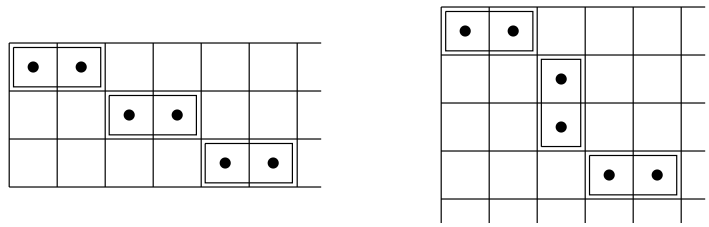

FIGURE 3. Copies of $\mathcal{O}_3$ in $G_{m,n}$

vertices of $G_{m,n}$ that induce a copy of $\mathcal{O}_3$; the figure on the left gives the case $m = 3$, and that on the right the case where $m \geq 4$.

Theorem 4.2. *Let G be an $\mathcal{O}_3$-free graph such that every triangle in G is contained in a unique clique. Then $K(G) \simeq G$.*

Proof: Clearly, a clique containing a face of a tetrahedron contains it all. We will prove the hypothesis of 2.4.

Let X be any complete of $K(G)$ with $\cap X = \varnothing$. Let Z be a minimal subset of X satisfying $\cap Z = \varnothing$. Clearly $Z = \{q_1, q_2, \ldots, q_s\}$ with $s \geq 3$. Since Z is minimal we can choose, for each i, some $x_i \in \cap(Z \setminus \{q_i\})$. Therefore $x_i \notin q_i$ but $x_i \in q_j$ for every $j \neq i$. Now take the triangle $T = \{x_1, x_2, x_3\}$ and define $\hat{T} = \{q \in K(G) : |q \cap T| \geq 2\}$ which is clearly a complete of $K(G)$.

We claim that if $q \in K(G)$ intersects each of q_1, q_2 and q_3, then $q \in \hat{T}$: If $|q \cap T| = 1$ we would have (say) $q \cap T = \{x_1\}$. Choose $z \in q \cap q_1$. Now $T \cup \{z\}$ is a tetrahedron and q_1 contains one of its faces, hence q_1 contains all of the tetrahedron, but then $x_1 \in T \cup \{z\} \subseteq q_1$ which is a contradiction. On the other hand if $|q \cap T| = 0$ we could choose $z_i \in q \cap q_i$ for $i = 1, 2, 3$ which together with the vertices in T would necesarily induce an $\mathcal{O}_3$, another contradiction.

From the previous claim it follows that $X \subseteq \hat{T}$ and that $\hat{T}$ is the unique clique (hence the unique necktie) of $K(G)$ containing X. It only remains to be shown that there is a center q_0 of X which is contained in $\hat{T}$. Indeed, let q_0 be the unique clique containing T and let $Y \subseteq X$ with $\cap Y \neq \varnothing$. If $(\cap Y) \cap T \neq \varnothing$ we are done. Otherwise, take $z \in \cap Y$. It follows that $T \cup \{z\}$ is a tetrahedron which is therefore contained in q_0, hence $(\cap Y) \cap q_0 \neq \varnothing$. Hence q_0 is a center of X and clearly $q_0 \in \hat{T}$. Thus we have verified the hypothesis of 2.4, and therefore $G \simeq K(G)$. □

Corollary 4.3. *If the graph G has at most eight vertices, then its matching graph $\overline{L(G)}$ satisfies $K(\overline{L(G)}) \simeq \overline{L(G)}$.*

Proof: We have already observed in 4.1 that $\overline{L(G)}$ is $\mathcal{O}_3$-free in this case. Now, if there is a triangle T in $\overline{L(G)}$, then there are three disjoint edges in G. A vertex v in $\overline{L(G)}$ such that $v \notin T$ but with $T \cup \{v\}$ complete corresponds to a fourth edge in G disjoint from the first three, and by our assumption on G, the vertex v can be chosen in at most one way. Hence the hypothesis of 4.2 are satisfied. □

Theorem 4.4. *If $n \leq 8$, then $G_n \simeq K(G_n)$.*
If $m = 2$ or $m + n \leq 8$, then $G_{m,n} \simeq K(G_{m,n})$.

Proof: If $m = 2$, then $G_{m,n}$ is clique-Helly by 3.2 and the claim follows from 2.3. In the other cases the result follows from 4.3. □

Let G be a matching or chessboard graph. We believe that the homotopy equivalence $K(G) \simeq G$ only holds for the values indicated in 4.4, i.e. it would not hold for the matching graph G_n if $n \geq 9$, nor for $G_{m,n}$ if $m \geq 3$, $m + n \geq 9$:

Björner, Lovász, Vrećica, and Živaljević proved in [2] that the matching complex $\Delta(G_n)$ is $(\nu_n - 1)$-connected and the chessboard complex $\Delta(G_{m,n})$ is $(\nu_{m,n} - 1)$-connected, where $\nu_n = \lfloor \frac{n+1}{3} \rfloor - 1, \nu_{m,n} = \min\{m, n, \lfloor \frac{m+n+1}{3} \rfloor\} - 1$, and they conjectured that these connectivity bounds are sharp. The conjecture was finally settled by Shareshian and Wachs in [23], and this means that $H_{\nu_n}(\Delta(G_n), \mathbb{Z}) \neq 0$ and $H_{\nu_{m,n}}(\Delta(G_{m,n}), \mathbb{Z}) \neq 0$. Using the software system GAP [8], together with the Simplicial Homology package [12], we have obtained $H_{\nu_{m,n}}(\Delta(K(G_{m,n})), \mathbb{Z}) = 0$ for $(m, n) = (3, 6), (3, 7), (3, 8), (4, 5), (5, 5)$, and so $G_{m,n} \not\simeq K(G_{m,n})$ in those cases. But the general case remains to be done. As for matching graphs, the computer is unable even to calculate $H_2(\Delta(K(G_9)), \mathbb{Z})$: the complex $\Delta(K(G_9))$ has 945 vertices, dimension 104 and 55,476 facets.

5. Clique divergence

In this section we prove that if a chessboard graph $G = G_{m,n}$ is not clique-Helly, then it must be clique-divergent. We will use two results on expansive graphs from [19]. First, a few definitions.

A *coaffination* of a graph G is an automorphism $\gamma : G \to G$ with $d_G(v, \gamma(v)) \geq 2$ for all $v \in G$. A *coaffine graph* is a graph G together with a fixed coaffination of G. If G is coaffine and its subgraph G' is invariant under the coaffination, we say that G' is a *coaffine subgraph* of G. In this case, G' is coaffine on its own with the restriction of the coaffination of G. We will not need the definition of an expansive graph, so we merely record here that every expansive graph is K-divergent [19]. If G and H are graphs, their *Zykov sum* $G + H$ is the disjoint union $G \cup H$ plus all the edges between G and H; if G and H are coaffine, then $G + H$ is coaffine with the union of the coaffinations of G and H.

Theorem 5.1. (Neumann-Lara's Connected Summand Theorem [19])
If G and H are coaffine graphs and H is connected, then $G + H$ is expansive.

Theorem 5.2. [19] *If G' is a coaffine subgraph of G and G' is expansive, then G is expansive.*

Theorem 5.3. *The chessboard graph $G_{m,n}$ is K-divergent whenever both $m, n \geq 3$ and $(m, n) \neq (3, 3)$.*

Proof: As usual, we assume $m \leq n$. The smallest case is when $(m, n) = (3, 4)$ and, since $G_{2,3} \cong C_6$, $G_{3,4}$ is a locally C_6 graph, hence K-divergent by [15]. Otherwise, $G_{m,n}$ is expansive since $G_{m,n}$ contains a coaffine subgraph which is isomorphic

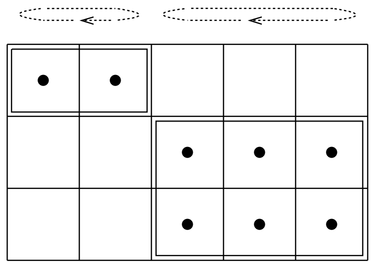

FIGURE 4. $G_{1,2} + G_{2,3}$ coaffinely embedded in $G_{3,5}$

to $G_{1,2} + G_{m-1,n-2}$ and $G_{m-1,n-2}$ is connected by 3.1. Here, the coaffination considered for $G_{m,n}$ is the permutation of columns given by $(1,2)(3,4,5,\ldots,n)$. □

The only chessboard graphs that we have ignored are those of the form $G_{1,n} \cong nK_1$, and these are certainly clique-Helly and not K-divergent. Therefore, from 3.2, 2.1 and 5.3 we have, in general, that:

Theorem 5.4. *The chessboard graph $G_{m,n}$ is K-divergent if and only if $G_{m,n}$ is not clique-Helly, if and only if $m, n \geq 3$ and $(m,n) \neq (3,3)$.* □

The determination of the K-behavior of matching graphs seems to be quite a harder problem. It follows from our results in Section 3 that G_n is K-convergent for $n = 2, 3, \ldots, 6$, and it could well be that only for these values of n. It is easy to see that matching graphs do not have coaffine automorphisms, so our technique in 5.3 will not work for $n \geq 7$. The computer quickly loses track of the iterated clique graphs of matching graphs:

G	$\lvert G\rvert$	$\lvert K(G)\rvert$	$\lvert K^2(G)\rvert$	$\lvert K^3(G)\rvert$	$\lvert K^4(G)\rvert$
G_7	21	105	126	4,893	168,756
G_8	28	105	448	401,928,849	-
G_9	36	945	55,476	-	-
G_{10}	45	945	7,482,240	-	-
G_{11}	55	10,395	-	-	-
G_{12}	66	10,395	-	-	-
G_{13}	78	135,135	-	-	-
G_{14}	91	135,135	-	-	-
G_{15}	105	2,027,025	-	-	-

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