CONTRACTIBILITY AND THE CLIQUE GRAPH OPERATOR

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Abstract. To any graph G we can associate a simplicial complex $\Delta(G)$ whose simplices are the complete subgraphs of G, and thus we say that G is *contractible* whenever $\Delta(G)$ is so. We study the relationship between contractibility and K-nullity of G, where G is called K-*null* if some iterated clique graph of G is trivial. We show that there are contractible graphs which are not K-null, and that any graph whose clique graph is a cone is contractible.

1. Introduction

Our graphs are simple, finite, connected and non-empty. Making a noun out of an adjective, we often refer to complete subgraphs just as *completes*. We identify induced subgraphs (hence completes) with their vertex sets. A *clique* of a graph is a maximal complete. The *clique graph* of a graph G is the intersection graph $K(G)$ of the set of all cliques of G. *Iterated clique graphs* $K^n(G)$ are defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$.

We say that G is K-*null* if $K^n(G)$ is the *trivial* (i.e. one-vertex) graph for some $n \geq 0$; if n is minimal, it is called the (*nullity*) *index* of G. More generally, if there are m, n with $m \neq n$ such that $K^m(G) \cong K^n(G)$, we say that G is K-*convergent*. It is easy to see that if G is not K-convergent, then the sequence of orders $|K^n(G)|$ tends to infinity, and in this case we say that G is K-*divergent*. The first examples of K-divergent graphs were given by Neumann-Lara [17]: defining the n-*th octahedron* O_n as the complement of the disjoint union of n copies of K_2, then one has $K(O_n) \cong O_{2^{n-1}}$, and so O_n is K-divergent for $n \geq 3$.

Given a graph G, the *complex of completes* of G is the simplicial complex $\Delta(G)$ whose simplices (or faces) are the complete subgraphs of G. On the other hand, we say that a simplicial complex Δ is *Whitney* if $\Delta = \Delta(G)$ for some graph G (the only candidate for G is the 1-skeleton of Δ). We remark that Whitney complexes have also been called *flag complexes*. We can thus attach topological concepts to G via the geometric realization $|\Delta(G)|$ of its associated complex. For instance, we say that a graph G is a *disk* (or a *sphere*) whenever $|\Delta(G)|$ is so, in which case we can also say that G is a *Whitney triangulation* of the disk (or a sphere). Again, G is *contractible* when $|\Delta(G)|$ is so and, more generally, we refer to the *homotopy type* of G as that of $|\Delta(G)|$. For example, the homotopy type of O_n is that of the sphere S^{n-1}.

The study of the clique operator under the topological viewpoint of the complex of completes was initiated by Prisner in [18] and has been further pursued in [10, 11, 12, 13, 15]. In this

Key words and phrases. clique graphs, homotopy type, contractibility.

paper, we explore the relation between K-nullity and contractibility of graphs. For a long time, we thought that several examples, results, problems and conjectures in the literature hinted at the equivalence of these concepts. Let us just mention three of them, others will be recalled later. Trees, which are the easiest examples of contractible graphs, are known to be K-null since the earliest result on iterated clique graphs: Hedetniemi and Slater proved in [9] that if G is connected, triangleless and with at least three vertices, then $K^2(G)$ is obtained from G by removing the vertices of degree one. Or take the K-null graphs F_n, H_n^i which were studied by Bornstein and Szwarcfiter in [4]: they are Whitney triangulations of the disk, thus contractible. In fact, it is conjectured in [14] that every Whitney triangulation of the disk is K-null, and this has been proved in [13] for the particular case in which each interior vertex has degree at least six.

We shall prove in Section 3 that K-nullity and contractibility are not equivalent, as there are contractible graphs which are K-divergent. In fact, the example given in Theorem 3.2 also disproves a related conjecture we upheld for some time: indeed, that graph is contractible but its clique graph is not. By adapting the example in Section 3, we obtain in Section 4 a comparability graph which is contractible and K-divergent. This gives a new answer to a problem in [8] and [21], and also settles a question that remained unanswered in [14]. In our last two sections we consider the remaining question of whether K-null graphs are always contractible. The first non-trivial case (index 2) is posed by clique-complete graphs, which were previously studied by Lucchesi, de Mello and Szwarcfiter [16]. In Section 5 we show that, as well as cones, critical clique complete graphs (which were classified in [16]) are always contractible, but that not all clique-complete graphs can be shown to be contractible by the simple argument used in that section, which goes back to Prisner [18]. In Section 6 we shall prove that all clique-cone graphs, a family including all clique-complete graphs, are contractible. We will accomplish this by first proving a result on simplicial quotients (6.1) which holds only for Whitney complexes. This result is related to the Contractible Subcomplex Lemma [3, Lemma 10.2], and may be of independent interest.

2. Preliminaries

In this work we denote by G a graph and by Δ a finite simplicial complex. If $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G[X]$. The *closed neighborhood* of $x \in G$ is $N_G[x] = \{ y \in G \mid xy \in E(G) \} \cup \{x\}$. If $N_G[x] = N_G[y]$, we say that x and y are *twins.* Any two twin vertices of G are interchangeable: there is an automorphism of G sending each one into the other and fixing everything else.

The *simplicial cone* C_Δ is obtained from Δ by adding a new vertex v and all the simplices $\sigma \cup \{v\}$ where $\sigma \in \Delta$ or $\sigma = \varnothing$. Clearly $|C_\Delta|$ is a topological cone, therefore contractible. We say that Δ is a *cone* if $\Delta \cong C_{\Delta_0}$ for some complex Δ_0. Similarly, given a graph H, define the *cone of* H as the graph C_H obtained from H by adding a new vertex and making it a neighbor of all vertices in H. The graph G is *a cone* if $G \cong C_H$ for some graph H. Note that this holds precisely when G has a *universal vertex*, i.e. a vertex a with $N_G[a] = G$. We also have that $\Delta(C_H)$ is the simplicial cone over $\Delta(H)$, hence each graph which is a cone is contractible. If G is a cone, the universal vertex (or *apex*) a needs not to be unique, but any two apices of G are twins, and so the *base* $G - a$ is determined by G up to an automorphism.

Since the cliques of a cone are clearly obtained adding the apex to the cliques of the base, $K(G)$ is complete and $K^2(G)$ is trivial for each cone G.

We say that the vertex v is *dominated by* $w \in G$ if $N_G[v] \subseteq N_G[w]$. We say that the vertex v is *dominated* if it is dominated by some $w \neq v$. The useful fact that removing a dominated vertex does not change the homotopy type was proved by Prisner as a part of his proof of Proposition 3.2 in [18]:

Theorem 2.1. [18] *Let v be a dominated vertex of the graph G. Then $G \simeq G - v$.*

The graph G is *dismantlable to the graph H* if there are vertices $x_1, \ldots, x_r$ in G such that x_{i+1} is a dominated vertex of $G - \{x_1, \ldots, x_i\}$ for $i = 0, \ldots, r-1$ and $G - \{x_1, \ldots, x_r\} \cong H$. In this case, using Theorem 2.1 r times we obtain that $G \simeq H$. A graph G is called just *dismantlable* if it is dismantlable to the trivial graph. Hence dismantlable graphs are a special class of contractible graphs, and Prisner proved that they are all K-null:

Theorem 2.2. [18, Cor. 4.2] *Every dismantlable graph is K-null.*

The converse is not true, as there are K-null graphs without dominated vertices. The example given in [18] is a 19-vertex K-null graph of index 7. It is a Whitney triangulation of the disk, so it is contractible. We present a simpler example in the next Proposition. In the class of Whitney triangulations of the disk without dominated vertices, the graph shown has the least possible number of vertices. This can be proved using the fact that every interior vertex in a disc has a neighborhood isomorphic to a cycle while every vertex in the border has a neighborhood isomorphic to a path.

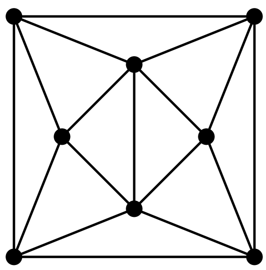

FIGURE 1. A K-null graph G without dominated vertices

Proposition 2.3. *The graph G in Fig.1 is K-null of index 4 and has no dominated vertices.*

Proof: The cliques of G are its 10 triangles, so $K(G)$ has order 10. The vertices of $K^2(G)$ are the cliques of $K(G)$ (i.e. the *cliques of cliques*). They are ten in number and still easy to see in Figure 1. First, for each of the four interior vertices v of G, the set of all triangles containing v is a clique of cliques, called the "star" of the vertex v and denoted by v^*. Second, for each of the six interior triangles T of G, the triangle T and the three triangles sharing some edge with T form a clique of cliques called the "necktie" of T. For more on stars and neckties see [12]. Now let v be one of the two interior vertices of degree five. Since v^* shares a triangle with any other clique of cliques, v^* is a neighbor of any other vertex of $K^2(G)$. Since $K^2(G)$ is a cone, $K^4(G)$ is the trivial graph. It is clear that $K^2(G)$ is not complete, so G is K-null of index four. □

Graphs very similar to G will play a role in our sections 3 and 4.

A graph G is *clique-Helly* if the collection of all cliques of G has the *Helly property*: any family of mutually intersecting cliques has a vertex in common. With respect to the clique graph operator, clique-Helly graphs are the best understood. Prisner proved in [18, Prop.2.2] that each clique-Helly graph has the same homotopy type as its clique graph, and then observed that K-null clique-Helly graphs are contractible. Furthermore, the converse of Theorem 2.2 holds for clique-Helly graphs: Bandelt and Prisner's Thm. 2.2 in [2] ensures that a clique-Helly graph G is K-null if and only if G is dismantlable. In the last paragraph of page 205 in [18], Prisner conjectures implicitly that a clique-Helly graph is K-null if and only if it is contractible.

A *map of graphs* f from G to H is a map $f\colon V(G) \to V(H)$ such that for every $xy \in E(G)$ we have either $f(x) = f(y)$ or $f(x)f(y) \in E(H)$. A map of a graph onto a subgraph $\rho\colon G \to H$ is a *retraction* if $\rho(x) = x$ for all $x \in H$. The following result is due to Neumann-Lara [17]:

Theorem 2.4. [17] *Any retraction $\rho\colon G \to H$ induces a retraction $\rho_K\colon K(G) \to K(H)$. In particular, if H is K-divergent, then G is also K-divergent.*

For example, if there is a retraction from G to an octahedron O_n for $n \geq 3$, then G is K-divergent. Furthermore, since a retraction of graphs induces a simplicial retraction, and this in turn a retraction of geometric realizations, in this case we would also have that G is not contractible, and in fact the index of the top nontrivial Betti number of $K^n(G)$ increases without bound as n increases.

For a finite poset P, the chains (i.e. linearly ordered subsets) of P form a simplicial complex $\Delta(P)$. In fact, $\Delta(P) = \Delta(\operatorname{Comp} P)$, where $\operatorname{Comp} P$ is the *comparability graph* of P: $V(\operatorname{Comp} P) = P$, $E(\operatorname{Comp} P) = \{\, xy \mid x < y \text{ or } x > y \,\}$. The complex $\Delta(P)$ allows us to attach topological concepts to P. For instance, we say that two posets P and P' are *homotopy equivalent* (and write $P \simeq P'$) if the geometric realizations of $\Delta(P)$ and $\Delta(P')$ are so. The following is Quillen's Fiber Theorem, which is a powerful tool for proving homotopy equivalences of posets:

Theorem 2.5. [19, 1.6] *Let P and P' be posets, and $f\colon P \to P'$ an order-preserving map. If $f^{-1}(P'_{\leq x}) = \{\, a \in P \mid f(a) \leq x \,\}$ is contractible for all $x \in P'$, then $P \simeq P'$.*

We will use an easy consequence of Theorem 2.5 for simplicial complexes:

Corollary 2.6. *Let $f : \Delta \to \Delta'$ be a simplicial map such that for each $\sigma' \in \Delta'$ the vertex set $f^{-1}(\sigma')$ induces a contractible subcomplex $\Phi_{\sigma'}$ of Δ . Then $\Delta \simeq \Delta'$.*

Proof: Let P and P' be the posets of simplices of Δ and Δ', ordered by inclusion. Then $\Delta(P)$ and $\Delta(P')$ are the barycentric subdivisions of Δ and Δ', so it suffices to show that $P \simeq P'$. Note that $f : \Delta \to \Delta'$ induces an order-preserving map $f\colon P \to P'$ and that, for $\sigma' \in P'$, the "fiber" $f^{-1}(P'_{\leq\sigma'}) = \{\, \sigma \in P \mid f(\sigma) \subseteq \sigma' \,\} = \big\{\, \sigma \in P \mid \sigma \subseteq f^{-1}(\sigma') \,\big\}$ is just the poset of simplices of $\Phi_{\sigma'}$ ordered by inclusion. The result follows from Theorem 2.5. □

3. Contractible K-divergent graphs

We shall show in this section that not every contractible graph is K-null, as even K-divergent contractible graphs do exist.

The *suspension* (or *double cone*) $\operatorname{Susp} H$ of a graph H is obtained adding two new vertices A, B to H and making them neighbors of all vertices of H. Then $\operatorname{Susp} H$ has twice the number of cliques as H: $V(K(\operatorname{Susp} H)) = \{\, q \cup \{A\} \mid q \in V(K(H))\,\} \cup \{\, q \cup \{B\} \mid q \in V(K(H))\,\}$.

Theorem 3.1. *Let H be a graph such that $K(H)$ has an induced four-cycle $\mathcal{C} \cong C_4$. Assume that no vertex of $K(H)$ outside of $\mathcal{C}$ is a neighbor of two opposite vertices of $\mathcal{C}$. Then there is a retraction from $K(\operatorname{Susp} H)$ to the octahedron O_4.*

Proof: Let $V(\mathcal{C}) = \{q_1, q_2, q_3, q_4\}$ and $E(\mathcal{C}) = \{q_1q_2, q_2q_3, q_3q_4, q_4q_1\}$. Note that the eight vertices of $K(\operatorname{Susp} H)$ of the form $q_i \cup \{A\}$ or $q_i \cup \{B\}$ $(i = 1, \ldots, 4)$ induce an octahedron O_4 in $K(\operatorname{Susp} H)$: for example, the only one not adjacent to $q_2 \cup \{A\}$ is $q_4 \cup \{B\}$. For a vertex $q \cup \{X\}$ of $K(\operatorname{Susp} H)$, define $\overline{N}(q \cup \{X\})$ as the set of vertices in $K(\operatorname{Susp} H)$ which are neither adjacent nor equal to $q \cup \{X\}$. We define a map $\rho\colon V(K(\operatorname{Susp} H)) \to V(O_4)$ by first leaving the vertices in O_4 fixed and then by the following rules:

$$\overline{N}(q_1 \cup \{A\}) \to q_3 \cup \{B\} \tag{1}$$
$$\overline{N}(q_3 \cup \{A\}) \to q_1 \cup \{B\} \tag{2}$$
$$\overline{N}(q_2 \cup \{B\}) \to q_4 \cup \{A\} \tag{3}$$
$$\overline{N}(q_4 \cup \{B\}) \to q_2 \cup \{A\} \tag{4}$$

where, for example, rule (1) means that all vertices in $\overline{N}(q_1 \cup \{A\})$, are to be sent by ρ to $q_3 \cup \{B\}$. If a vertex of $K(\operatorname{Susp} H)$ satisfies (i.e. lies in the left hand side of) two or more rules, we obey the first of these. As any vertex of $K(\operatorname{Susp} H)$ contains either A or B, and no $q \in K(H) \setminus \mathcal{C}$ is a neighbor of two opposite vertices in $\mathcal{C}$, each element in $V(K(\operatorname{Susp} H)) \setminus O_4$ satisfies at least one rule.

We claim now that ρ is a map of graphs, and in order to see this it will be enough to prove that if xy is an edge in the complement of O_4, and $a \in \rho^{-1}(x)$, $b \in \rho^{-1}(y)$, then ab is an edge in the complement of $K(\operatorname{Susp} H)$. Consider the case $x = q_2 \cup \{A\}$, $y = q_4 \cup \{B\}$. Then $\rho^{-1}(x) \subseteq \overline{N}(y)$ and $\rho^{-1}(y) = \{y\}$, from where our condition follows. The cases of the other three edges in $\overline{O_4} = 4K_2$ are similar. □

We give in Figure 2 an example of a graph H satisfying the hypotheses of Theorem 3.1, where $\mathcal{C}$ is induced in $K(H)$ by those triangles that have some edge in the border.

The graph H is contractible, since it is a Whitney triangulation of the disk. It seems to be the smallest contractible graph that satisfies the hypotheses of Theorem 3.1. Since the geometric realization of $\Delta(\operatorname{Susp} H)$ is homeomorphic to the topological suspension of $|\Delta(H)|$, we have that $\operatorname{Susp} H$ (in this case a 3-ball) is also contractible.

Hence $G = \operatorname{Susp} H$ is contractible but $K(G)$ is not, by Theorem 3.1. Since there is a retraction of G onto an octahedron, using Theorem 2.4 we obtain that G is K-divergent and $K^n(G)$ is not contractible for any $n \geq 1$. Furthermore, the index of the top nontrivial Betti number of $K^n(G)$ increases without bound as n increases. In particular we have:

Theorem 3.2. *There are contractible graphs which are K-divergent. Such graphs can even be found with non-contractible clique graphs.* □

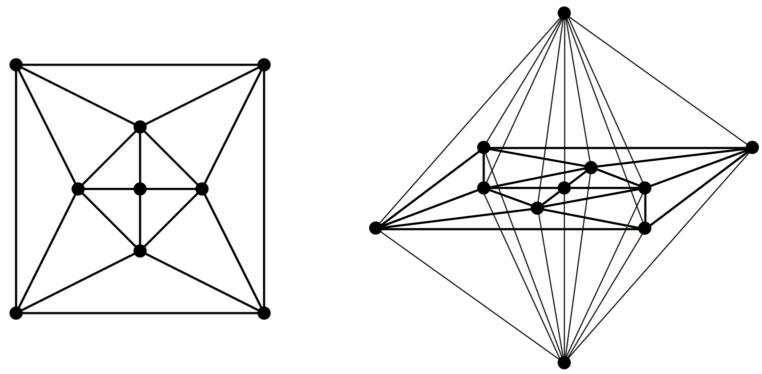

Figure 2. The graph H at the left satisfies the hypotheses of 3.1; the graph at the right is its suspension $G = \operatorname{Susp} H$.

4. Partially ordered sets

A poset P has the *fixed point property* (FPP) if any order-preserving endomorphism of P has a fixed point. Rival proved in [20] that if P is dismantlable then P has the FPP. Hazan and Neumann-Lara proved in [8] that if the comparability graph $\operatorname{Comp} P$ is K-null, then P has the FPP. Since by [6, Cor. 2.5] P is dismantlable if and only if $\operatorname{Comp} P$ is dismantlable, and dismantlable graphs are K-null by Theorem 2.2, the result in [8] generalizes that in [20]. On the other hand, Baclawski and Björner proved in [1, p. 271] that any contractible poset has the FPP, and as dismantlable graphs are contractible by Theorem 2.1, this is also a generalization of Rival's result; it could even be a generalization of Hazan and Neumann-Lara's result, but we still do not know whether K-null graphs are always contractible, or if at least this holds for comparability graphs.

In [8] and Schröder's book [21, p. 160], the question was raised as to whether posets with the FPP do exist with non K-null comparability graph. Two such posets were given in [14], and for both of them the comparability graph is K-divergent and has the same Betti numbers as the circle. In fact, it was remarked in [14] that "we do not know an example of a non K-null graph with the same Betti numbers as the disk". By adapting our graph G in Fig. 2, we show here that such examples do exist even within the realm of comparability graphs of posets with the FPP; thus, the new examples also solve in the affirmative the above question of [8] and [21], but with the additional property of contractibility.

Theorem 4.1. *Finite posets P with the fixed point property do exist, where* $\operatorname{Comp} P$ *is K-divergent and contractible.*

Proof: Let P' be the poset whose comparability graph H' is depicted in Figure 3. We show the transitive orientation of the edges of H' given by $x \to y$ if, and only if, $x > y$ in P'. As in the proof of Theorem 3.2, H' is a contractible graph satisfying the hypotheses of Theorem 3.1.

Since H' is contractible, so is P' (recall that $\Delta(P') = \Delta(\operatorname{Comp} P') = \Delta(H')$). Let P be the *suspension poset* $P = \operatorname{Susp} P'$, which is obtained by adding two new non-comparable points A, B to P' and making them greater than any point in P'. Since $\operatorname{Comp} P = \operatorname{Susp} H'$ is a

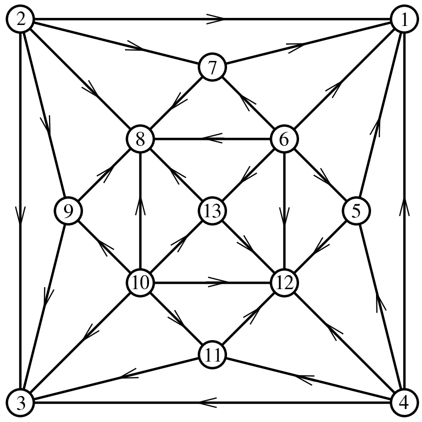

FIGURE 3. A transitive orientation of a comparability graph $H' = \text{Comp}\, P'$ that satisfies the hypotheses of 3.1.

3-ball, it is contractible and so is P. Thus, P has the fixed point property by the above-mentioned result of Baclawski and Björner [1]. By Theorems 3.1 and 2.4, $\text{Comp}\, P = \text{Susp}\, H'$ is K-divergent. The Hasse diagram of P is shown in Figure 4. □

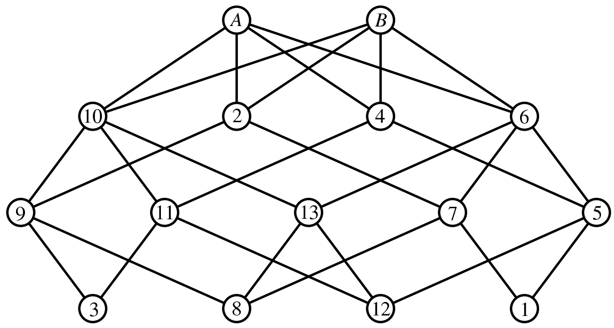

FIGURE 4. Hasse diagram of a poset P as in 4.1.

5. Clique-complete graphs

Now that we know that contractibility does not imply K-nullity, the problem remains whether this could hold the other way around. As far as we have been able to ascertain it, K-null graphs seem to be always contractible.

We said before that K-null clique-Helly graphs are contractible, and that dismantlable graphs are always both K-null and contractible. We now turn to look at the nullity index of graphs. Besides the trivial graph (which is the only graph with index 0), the simplest examples of K-null graphs are those of index one, i.e. the graphs with trivial clique graph. These are just the complete graphs, which topologically are balls and hence contractible.

For the index two case, note that $K^2(G)$ is trivial if and only if $K(G)$ is complete. These are just the *clique-complete* graphs G that were studied by Lucchesi, de Mello and Szwarcfiter in [16], where they showed that the problem of recognizing clique-complete graphs is Co-NP-complete.

The easiest examples of clique-complete graphs are cones, and they are contractible (see Section 2). A non-conical clique-complete graph is *critical* if each proper induced subgraph is either a cone or not clique-complete. A complete classification of critical non-conical clique-complete graphs was given in [16]: they are just the graphs G_m with m odd and $m \geq 3$, where the complement of G_m is obtained by attaching a pendant edge to each vertex of the m-cycle C_m. In Figure 5 we borrow the pictures of the complements of G_3 and G_5 in [16].

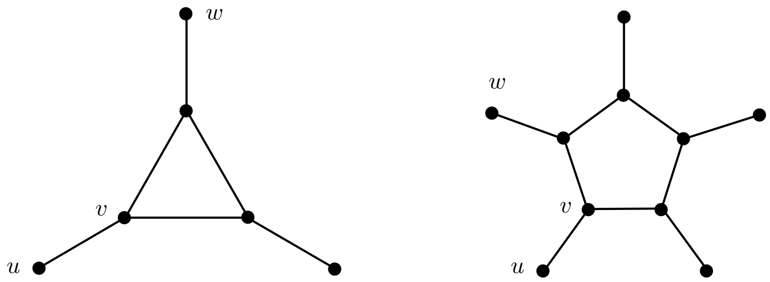

FIGURE 5. The graphs $\overline{G}_3$ and $\overline{G}_5$.

Proposition 5.1. *Every critical clique-complete graph is dismantlable, hence contractible.*

Proof: As just noted, by [16] it is enough to show that the graphs G_m are dismantlable for all $m \geq 3$. With the labels in Figure 5, observe first that the vertex v is always dominated by the vertex w. By Theorem 2.1 we can remove the vertex v from G_m without altering the homotopy type, but in the resulting graph $G_m - v$ the vertex u is universal. Thus $G_m - v$, being a cone, is dismantlable (since in a cone every vertex is dominated by the apex) and therefore so is G_m. □

However, not all clique-complete graphs are dismantlable.

Theorem 5.2. *There are clique-complete graphs without dominated vertices.*

Proof: In Figure 6 we depict the complement H of one such graph G.

The cliques of G are the maximal independent sets of H. There are 26 of these (21 with 5 vertices and 5 with 6) and each of them contains at least three of the five vertices of degree two in H. Therefore, G is clique-complete.

In order to see directly in H that G has no dominated vertices just observe that, for any pair of different and non-neighboring vertices, each one has a neighbor that is not a neighbor of the other. □

In our next section we shall prove (see Corollary 6.3) that all clique-complete graphs are contractible.

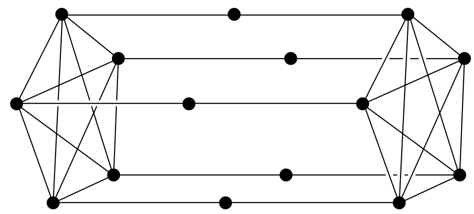

FIGURE 6. The complement $H = \overline{G}$ of a clique-complete graph G which does not have any dominated vertex.

6. Clique-cone graphs

If Δ is a simplicial complex and $\sim$ is an equivalence relation in $V(\Delta)$, the *simplicial quotient* $\Delta/\!\sim$ has vertex set $V(\Delta)/\!\sim$ and faces $\{\, \pi(\sigma) : \sigma \in \Delta \,\}$, where $\pi : V(\Delta) \to V(\Delta/\!\sim)$ is the natural projection. It follows immediately that $\Delta/\!\sim$ is indeed a simplicial complex and that each of its maximal faces is the image under π of a maximal face of Δ. We shall only use simplicial quotients $\Delta/\!\sim$ where $\sim$ has just one non-singular equivalence class and this class is a face C of Δ; in this case, we denote the quotient as $\Delta \,/\, C$ and say that it is obtained from Δ by *shrinking* the face C. In general, $\Delta \,/\, C$ does not need to have the same homotopy type as Δ: take Δ with just three vertices and three edges, and let C be an edge. However, for Whitney complexes, shrinking a face does not alter the homotopy type. Indeed, we have the following result, which is the strongest possible simplicial analogue of the Contractible Subcomplex Lemma (see Lemma 10.2 of [3]):

Theorem 6.1. *Let G be a graph and $\Delta = \Delta(G)$ its associated simplicial complex. Let C be a complete subgraph of G and let $\Delta' = \Delta \,/\, C$. Then $\Delta' \simeq \Delta$.*

Proof: We can think that $V(\Delta') = (G - C) \cup \{*\}$ and that the natural projection $\pi \colon \Delta \to \Delta'$ is given by $\pi(v) = v$ if $v \notin C$ and $\pi(v) = *$ if $v \in C$.

Let $\sigma' \in \Delta'$ be any face. Then there is a $\sigma \in \Delta$ such that $\pi(\sigma) = \sigma'$. By Corollary 2.6, it will be enough to show that $\pi^{-1}(\sigma')$ induces a contractible subcomplex of Δ. If $* \notin \sigma'$, then $\pi^{-1}(\sigma') = \sigma' = \sigma$ is a face of Δ and hence contractible. The same holds if $\sigma' = \{*\}$, since in this case we have $\sigma \subseteq C$ and $\pi^{-1}(\sigma') = C$.

Assume then that $\{*\} \subsetneq \sigma'$. Now $\pi^{-1}(\sigma') = \sigma \cup C$, which is not necessarily a face of Δ. Call H the subgraph of G induced by $\sigma \cup C$ and let $v \in \sigma \cap C$. Since both σ and C are faces of Δ, then v is a neighbor of every other vertex in H, so H is a cone. Since $\Delta = \Delta(G)$, the subcomplex of Δ induced by $\pi^{-1}(\sigma')$ is $\Delta(H)$, which is contractible. □

Note, however, that the complex $\Delta \,/\, C$ is not always Whitney.

Theorem 6.2. *Let G be a graph such that $K(G)$ is a cone. Then G is contractible.*

Proof: If $K(G)$ is a cone, there is a clique Q in G that intersects all cliques. Shrinking Q, form the simplicial quotient $\Delta' = \Delta(G) \,/\, Q$. By 6.1, $\Delta(G) \simeq \Delta'$, so we need to show that Δ' is contractible. The maximal faces of Δ' are some sets of the form $\pi(Q')$ for $Q' \in K(G)$, $Q' \neq Q$, but all these contain the vertex $* = \pi(Q)$, so Δ' is a cone and thus contractible. □

Since any complete graph is a cone, we get at once:

Corollary 6.3. *Let G be a clique-complete graph. Then G is contractible.* $\square$

Thus we have that K-null graphs of index at most 2 are always contractible.

Proposition 6.4. *There is an infinite number of K-null graphs of index 3 whose contractibility is ensured by Theorem 6.2.*

Proof: Indeed, starting with any non-complete graph H take its cone C_H. Since all cones are clearly clique-Helly (in fact, cones are just those graphs for which the intersection of all cliques is non-empty) we know by [7] that there is a graph G_H such that $K(G_H) = C_H$. Note that G_H is K-null of index 3 because $K^2(C_H)$ is trivial but $K(C_H)$ is not. As $C_H \cong C_{H'}$ only for $H \cong H'$, we also have that $G_H \cong G_{H'}$ only for $H \cong H'$ and the family of all G_H is indeed very large. $\square$

However, not all K-null graphs of index 3 are clique-cone, as shown by the contractible graph in Figure 7.

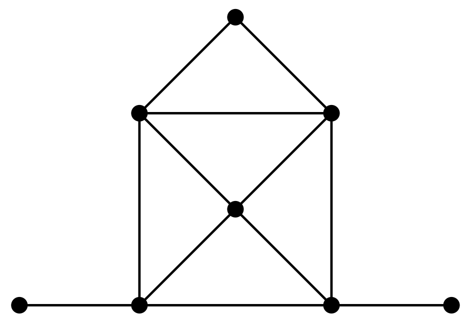

FIGURE 7. A graph G with $K^3(G)$ trivial but $K(G)$ not a cone.

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