Whitney Triangulations, Local Girth and Iterated Clique Graphs

F. Larrión [a,1,*] V. Neumann-Lara [a,1] M. A. Pizaña [b]

[a] *Instituto de Matemáticas, U.N.A.M. Circuito Exterior, C.U. México 04510 D.F. MEXICO.*

[b] *Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica. Av. Michoacán y Purísima s/n. Col Vicentina. México 09340 D.F. MEXICO.*

Abstract

We study the dynamical behaviour of surface triangulations under the iterated application of the clique graph operator k, which transforms each graph G into the intersection graph kG of its (maximal) cliques. A graph G is said to be k-*divergent* if the sequence of the orders of its iterated clique graphs $|V(k^nG)|$ tends to infinity with n. If this is not the case, then G is *eventually* k-*periodic,* or k-*bounded*: $k^nG \cong k^mG$ for some $m > n$. The case in which G is the underlying graph of a regular triangulation of some closed surface has been previously studied under the additional (Whitney) hypothesis that every triangle of G is a face of the triangulation: if G is regular of degree d, it is known that G is k-bounded for $d = 3$ and k-divergent for $d = 4$, 5 and 6. We will show that G is k-bounded for all $d \geq 7$, thus completing the study of the regular case. Our proof works in the more general setting of graphs with local girth at least 7. As a consequence we obtain also the k-boundedness of the underlying graph G of any triangulation of a compact surface (with or without border) provided that any triangle of G is a face of the triangulation and that the minimum degree of the interior vertices of G is at least 7.

Key words: clique convergence, clique-Helly graphs, iterated clique graphs, local girth, surface triangulations

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* Contact author.

Email addresses: `paco@math.unam.mx` (F. Larrión), `neumann@math.unam.mx` (V. Neumann-Lara), `map@xanum.uam.mx` (M. A. Pizaña).

URLs: `http://calli.matem.unam.mx/personal/paco` (F. Larrión), `http://calli.matem.unam.mx/personal/neumann` (V. Neumann-Lara), `http://xamanek.uam.mx/map` (M. A. Pizaña).

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1 Introduction

Our graphs are simple, non-empty, finite and connected. We often identify induced subgraphs with their vertex-sets.

The *clique graph* kG of a graph G is the intersection graph of the family of all the *cliques* (maximal complete subgraphs) of G: the vertices of kG are the cliques of G and two different cliques of G are adjacent in kG if and only if they share at least one vertex. The *iterated clique graphs* k^nG are defined by $k^0G = G$ and $k^{n+1}G = kk^nG$. The study of this challenging subject was initiated by S. T. Hedetniemi and P. J. Slater in [5]. We refer to [8], [12] and [13] for the literature on iterated clique graphs. The problem of determining the *k-behaviour* of a given graph G, i.e. the dynamical behaviour of G under the iteration of the clique operator k, can be quite difficult. The behaviour can be k-bounded or k-divergent:

The graph G is said to be *k-null* if k^nG is the trivial graph K_1 for some n. For instance the tetrahedron K_4, being a complete graph, is k-null. More generally, G is said to be *k-bounded* if $k^nG \cong k^mG$ for some pair of non-negative integers $n < m$. If this is the case G is, in the obvious sense, *eventually k-periodic.* Hedetniemi and Slater [5] proved that for G connected, triangleless and with at least three vertices, $k^2G \cong G - \{v \mid \deg(v) = 1\}$. From this it follows that all such graphs are eventually k-periodic of period one or two. F. Escalante [2] generalized this last to the family of clique-Helly graphs (see §2) and he also constructed examples of k-periodic graphs G of any period $p \geq 1$. Escalante's generalization will be useful in this work. We call G *k-divergent* if the sequence of the orders $|V(k^nG)|$ tends to infinity with n. This last holds if and only if the sequence of the orders is unbounded, if and only if G is not eventually k-periodic.

A graph G is said to be a *locally C_t graph* if $N(v) \cong C_t$ for every $v \in V(G)$, where the *open neighbourhood* $N(v)$ is the subgraph of G induced by the neighbours of v, and C_t is the cycle of length t. For each $t \in \{3, 4, 5\}$ there exists a unique connected locally C_t graph G: For $t = 3$, G is the tetrahedron K_4 and we have already remarked that it is k-null. For $t = 4$, G is the octahedron and V. Neumann-Lara proved that it is k-divergent (see [2], [10]). For $t = 5$, G is the icosahedron, and M. A. Pizaña proved in [11] that it is k-divergent. Notice that these graphs are (1-skeletons of) triangulations of the sphere. There is an infinite number of locally C_6 graphs, all defining triangulations of the torus or the Klein bottle; they were all proved to be k-divergent by F. Larrión and V. Neumann-Lara in [8] and [9].

We will describe in this paper the k-behaviour of locally C_t graphs for $t \geq 7$. For any such t, there exists an infinite number of locally C_t graphs (see

Proposition 17). Contrary to the cases $t = 4, 5$, and 6, all the locally C_t graphs for $t \geq 7$ are k-bounded: for such a graph G one has $kG \cong k^3G$ (Theorem 13).

Our proof works for a more general case: A *Whitney triangulation* $\mathcal{T}$ of some compact surface is one such that any triangle of its underlying graph $G = \mathcal{T}_1$ is a face of $\mathcal{T}$. Such a triangulation $\mathcal{T}$ is fully determined by its 1-skeleton G, and these graphs admit a nice characterization in combinatorial terms: each open neighbourhood in such a graph G is either a cycle or a path and $\mathcal{T}$ is closed if and only if G is *locally cyclic,* i.e. each open neighbourhood in G is a cycle (see §4.1). Notice that a locally C_t graph is just a locally cyclic graph which is regular of degree t, and thus Theorem 13 will be a consequence of Theorem 12: $kG \cong k^3G$ for every locally cyclic graph G with minimum degree $\delta(G) \geq 7$.

Part of the proof works in even greater generality. The *local girth* of a graph G is the minimum of the girths of the open neighbourhoods in G. We will show in §3 that G is eventually k-periodic whenever its local girth is at least 7 (Theorem 8). In particular this implies the k-boundedness of the underlying graph of any Whitney triangulation of a compact surface (with or without border) provided that the minimum degree of the interior vertices is at least 7 (Theorem 16). For instance, if G underlies a Whitney triangulation of the disk, then G is k-bounded if the minimum degree of the interior vertices is at least 7, but we conjecture a much stronger result:

Conjecture 1 *Let G be the underlying graph of a Whitney triangulation of the disk. Then G is k-null.*

Notice that here no restrictions on the minimum degree are made. A related conjecture is:

Conjecture 2 *Let $G \neq K_4$ be the underlying graph of a Whitney triangulation of the sphere. Then G is k-divergent.*

2 Clique-Helly graphs

A family $\mathcal{C}$ of sets is said to satisfy the *Helly property* if the intersection of any subfamily $\mathcal{F}$ of mutually intersecting members of $\mathcal{C}$ is non-empty. The following lemma provides an interesting example:

Lemma 3 *Let G be a graph and let $\mathcal{C}$ be any family of edges of G such that no edge of $\mathcal{C}$ is contained in a triangle of G. Then $\mathcal{C}$ satisfies the Helly property.*

PROOF. Fix $\mathcal{F} \subseteq \mathcal{C}$ such that any two edges in $\mathcal{F}$ meet. Assume that $\mathcal{F}$ contains at least two edges $e_1 = \{x, a\}$ and $e_2 = \{x, b\}$. Any other edge $e \in \mathcal{F}$ contains the vertex x, since otherwise $e = \{a, b\}$ would be contained in the triangle $\{x, a, b\}$. □

A graph H is said to be *clique-Helly* if the family $\mathcal{C}$ of all the cliques of H satisfies the Helly property. Given any graph H the relation of *domination* among the vertices of H is defined as follows: the vertex v is said to *dominate* the vertex u if $N[u] \subseteq N[v]$, where $N[v] = N(v) \cup \{v\}$ is the *closed neighbourhood* of v. Thus, v dominates u if and only if either $v = u$ or v and u are neighbours and, apart from v, every neighbour of u is also a neighbour of v. The relation of domination is a preorder in $V(H)$ (i.e., this relation is reflexive and transitive). As usual with preorders, the *mutual domination* is an equivalence relation in $V(H)$, and the domination relation induces a partial order on the quotient set of all the classes of mutual domination of vertices of H. In this language, the results (Satz 1 and Satz 2) of Escalante's paper [2] which we will use are:

Theorem 4 *If H is clique-Helly then its clique graph kH is also clique-Helly.*

Theorem 5 *If H is clique-Helly then its second clique graph k^2H is isomorphic to the subgraph $\overline{E}(H)$ of H induced by any system of representatives of the maximal classes of mutual domination.*

The second of the above results generalize the oldest result about iterated clique graphs, which is due to S. T. Hedetniemi and P. J. Slater [5]:

Proposition 6 *If H is connected, triangleless, and with at least 3 vertices, then $k^2H \cong H - \{v \mid \deg(v) = 1\}$.*

PROOF. By Lemma 3, H is clique-Helly. The vertices which are dominated by another are those of degree 1. The relation of mutual domination is trivial. □

The following result is an easy consequence of Theorems 4 and 5. It is all that we will need of Escalante's paper [2]:

Proposition 7 *Let H be a clique-Helly graph. Then:*

(1) H is eventually k-periodic of period 1 or 2, and
(2) If each vertex of H only dominates itself, then $k^2H \cong H$. □

3 Graphs with local girth at least 7

Recall that the *girth* $g(G)$ of a graph G is the minimum length of a cycle in G (if G does not have cycles, then $g(G) = \infty$). If $v \in V(G)$, the *local girth of G at v* is the girth of the open neighbourhood of v in G; in symbols, $\mathrm{lg}_v(G) = g(N(v))$. The *local girth* of the graph G is the minimum of all the local girths of G, i.e., $\mathrm{lg}(G) = \min\{\mathrm{lg}_v(G) \mid v \in V(G)\}$. For instance any tree has infinite local girth, but also a cycle (or a cycle with a diagonal) has infinite local girth. The main purpose of this section is to prove the following result:

Theorem 8 *If the local girth of the graph G satisfies $\mathrm{lg}(G) \geq 7$, then kG is clique-Helly. In particular, G is eventually k-periodic of period 1 or 2.*

That G is eventually k-periodic of period 1 or 2 is immediate from 7(1), so we need to prove only that kG is clique-Helly. In order to do this, we will first study the cliques of kG in §3.1, and then in §3.2 we will reduce the proof to a simple statement about families of triangles and vertices in a graph G with $\mathrm{lg}(G) \geq 7$.

3.1 The cliques of kG

Assuming only that $\mathrm{lg}(G) \geq 4$ (i.e., that G contains no tetrahedron K_4) the largest possible order for a clique in G is 3, and so the cliques of G are its triangles and those of its edges that are not contained in some triangle. The clique graph kG has these cliques as vertices and the adjacencies are non-empty intersections. Note that an edge and a triangle (or an edge and a different edge) can have only one vertex in common, whereas two different triangles can meet either at a vertex or at an edge. It will be convenient to distinguish two kinds of cliques of kG: the *stars* and the *neckties.*

Given a vertex v of an arbitrary graph G, the *star* of v is the set of all the cliques of G which contain v, i.e. $v^* = \{Q \in V(kG) \mid v \in Q\}$. Then v^* induces a complete subgraph of kG which may or may not be a clique of kG. We call v a *center* of v^* (the centers of v^* are all twins). Notice that a clique $\mathcal{Q}$ of kG is a star if and only if the intersection of all the elements of $\mathcal{Q}$ (which are cliques of G) is non-empty: there is a vertex of G which is contained in all the elements of $\mathcal{Q}$. The set of all the stars which are cliques of kG is a set of vertices in k^2G, and two different stars u^* and v^* have a non-empty intersection (i.e. are adjacent in k^2G) if and only if u and v are neighbours in G. The Helly property holds for stars:

Lemma 9 *Let G be any graph and let $\mathcal{F}$ be a family of mutually intersecting stars of vertices of G. Then the intersection of all the members of $\mathcal{F}$ is non-*

empty.

PROOF. If $\mathcal{Q} \in \mathcal{F}$, then there is some vertex $v \in V(G)$ such that $\mathcal{Q} = v^*$. Since any two members of $\mathcal{F}$ have a non-empty intersection, the subgraph C of G induced by the centers of the stars in $\mathcal{F}$ is complete. Any clique Q of G which contains C is then a member of all the stars in $\mathcal{F}$. □

Any clique of kG which is not a star will be said to be a *necktie*. A clique $\mathcal{Q}$ of kG is a necktie iff $\cap\mathcal{Q} = \varnothing$, iff for every vertex which is contained in some element of $\mathcal{Q}$ there exists another element of $\mathcal{Q}$ which does not contain that vertex. We will give now the construction of certain neckties. These will indeed be neckties if $\lg(G) \geq 4$, and they will be *all* the neckties if $\lg(G) \geq 5$.

Consider an *inner* triangle $T = \{a, b, c\}$ of G, i.e., for each of the three edges of T there exists at least one triangle T' of G such that $T' \neq T$ and T' meets T in that edge. The *necktie of* G *centered at* T is the collection $\mathcal{Q}_T$ consisting of T and all the triangles that meet T at some edge. The triangle T is called the *center* of $\mathcal{Q}_T$ and the remaining triangles of $\mathcal{Q}_T$ are its *ears*.

Let us show that $\mathcal{Q}_T$ is indeed a necktie if $\lg(G) \geq 4$: Recall that every triangle of G is a clique of G, i.e. a vertex of kG. Since any triangle in $\mathcal{Q}_T$ contains at least an edge of T and any two edges of T meet in at least one vertex, $\mathcal{Q}_T$ induces a complete subgraph of kG. Let us show that $\mathcal{Q}_T$ is a clique of kG, assuming to the contrary that there is some clique Q of G which is not in $\mathcal{Q}_T$ but meets every triangle in $\mathcal{Q}_T$. Note that Q can not meet T in an edge, for then Q would be a triangle and so a member of $\mathcal{Q}_T$. Thus, Q meets T in just one vertex, say c. Let $T' = \{a, b, c'\}$ be a triangle of G which meets T precisely in the edge $\{a, b\}$. Hence, $T' \in \mathcal{Q}_T$. Since Q has to meet T', and Q contains neither a nor b, then Q must contain c'. But then $\{b, c, c'\}$ is a triangle in $N(a)$, contradicting that $\lg(G) \geq 4$. This contradiction shows that $\mathcal{Q}_T$ is a clique of kG for each inner triangle T of G. Since the intersection of the ears of $\mathcal{Q}_T$ is empty, $\mathcal{Q}_T$ is a necktie.

Assuming only that $\lg(G) \geq 4$, it is not always true that any necktie of kG is of the form $\mathcal{Q}_T$ for some inner triangle T of G: indeed, the octahedron $\mathcal{O}_3$ has two neckties which are not of this form. However, we have the following result:

Proposition 10 *If* $\lg(G) \geq 5$*, then any necktie of* kG *is of the form* $\mathcal{Q}_T$ *for some inner triangle* T *of* G*.*

PROOF. Let $\mathcal{Q}$ be any clique of kG which is not a star. We will show that $\mathcal{Q} = \mathcal{Q}_T$ for some inner triangle T of G. The elements of $\mathcal{Q}$, being cliques of

G, are either triangles of G or edges of G which are not contained in triangles. It can not be the case that all the elements of $\mathcal{Q}$ are edges, since by Lemma 3 $\mathcal{Q}$ would be a star in that case. Hence there is a triangle $T = \{a, b, c\} \in \mathcal{Q}$.

Now we claim that each element of $\mathcal{Q}$ is a triangle. Indeed, if $\mathcal{Q}$ contained some edge E, then (say) $E = \{a, d\}$ for some vertex d not in T. Since $\mathcal{Q}$ is a necktie, there must be a $Q \in \mathcal{Q}$ such that $a \notin Q$. Then Q contains $d \in E$ and (say) $b \in T$, but in this case there is a triangle $\{a, b, d\}$ which contains E, contradicting that E is a clique. Hence all the elements of $\mathcal{Q}$ are triangles. Since $\mathcal{Q}$ is not a star, there must be more that one triangle in $\mathcal{Q}$.

Furthermore, we claim that there must exist two triangles in $\mathcal{Q}$ which meet at just one vertex: Otherwise, any two triangles of $\mathcal{Q}$ meet at an edge, and from this a contradiction follows: Besides the above mentioned triangle $T = \{a, b, c\} \in \mathcal{Q}$, we know that there exists another triangle $T_1 \in \mathcal{Q}$. Since T_1 meets T in an edge, we can assume that $T_1 = \{a, b, c'\}$ with $c' \notin T$. Since $\mathcal{Q}$ is not a star, there is a triangle $T_2 \in \mathcal{Q}$ such that $b \notin T_2$, but then T_2 would have to meet T and T_1 in the edges $\{a, c\}$ and $\{a, c'\}$ and therefore $T_2 = \{a, c, c'\}$. This is a contradiction because G can not contain a tetrahedron.

We can then take $T = \{a, b, c\}$ so that there is a triangle $T_1 \in \mathcal{Q}$ which meets T in just one vertex, say $T_1 = \{a, b', c'\}$. Since $\mathcal{Q}$ is a necktie, there must be a triangle $T_2 \in \mathcal{Q}$ which does not contain a. The triangle T_2 must contain at least one vertex from each of T and T_1, but it can not contain an edge of any of these triangles: $\{b, c\} \subset T_2$ or $\{b', c'\} \subset T_2$ would imply that G contains a tetrahedron. Without loss of generality we can assume that $T_2 = \{a', b, c'\}$, where the vertex a' is not in $T \cup T_1$. Now the triangle $T_3 = \{a, b, c'\}$ is an interior triangle of G. Supposing that $T_3 \notin \mathcal{Q}$, we will find a contradiction. Under this supposition, there must exist a triangle $T_4 \in \mathcal{Q}$ such that $T_4 \cap T_3 = \varnothing$, but then we would have $T_4 \cap T = \{c\}$, $T_4 \cap T_1 = \{b'\}$, and $T_4 \cap T_2 = \{a'\}$, so $T_4 = \{a', b', c\}$ and G would contain an octahedron, whose local girth is 4. Therefore, $T_3 \in \mathcal{Q}$, and it is now easy to see that $\mathcal{Q} = \mathcal{Q}_{T_3}$: no triangle of $\mathcal{Q}$ can meet T_3 in just one vertex because G does not contain tetrahedra, and $\mathcal{Q}$ must therefore contain all the triangles of G that meet T_3 in some edge. $\square$

3.2 Reduction and Proof of Theorem 8

Let us assume that the graph G satisfies $\mathrm{lg}(G) \geq 7$. Let $\mathcal{F}$ be a family of cliques of kG such that any two members of $\mathcal{F}$ have a non-empty intersection. In order to prove Theorem 8, we need to prove that the intersection of all the members of $\mathcal{F}$ is non-empty. For instance, if every member of $\mathcal{F}$ is a star, we know by Lemma 9 that this is true. Thus we may as well assume that $\mathcal{F}$ contains some neckties. By Proposition 10, we know that each necktie is of the form $\mathcal{Q}_T$ for some inner triangle T of G. Recall that T is the center of $\mathcal{Q}_T$ and all other triangles in $\mathcal{Q}_T$ (at least three) are called the ears of $\mathcal{Q}_T$. Since the neckties are composed entirely of triangles, we need to find a triangle of G which is contained in all the stars of $\mathcal{F}$ and is either the center or an ear of each necktie of $\mathcal{F}$. Our reduction of the problem will be based on the following observations:

We have remarked already that two stars u^* and v^* intersect if and only if their centers u and v are neighbours in G.

A star u^* and a necktie $\mathcal{Q}_T$ in kG have a non-empty intersection if and only if they share some triangle of G. There are two possibilities: either this shared triangle is the center T of $\mathcal{Q}_T$ (if and only if u is a vertex of T), or $u \notin T$ and then the shared triangle is an ear of $\mathcal{Q}_T$ (if and only if $u \notin T$ but u is a neighbour of two vertices of T).

Two different neckties $\mathcal{Q}_T$ and $\mathcal{Q}_{T'}$ in kG have a non-empty intersection if and only if they share some triangle of G. There are again two possibilities: either this shared triangle is the center of one of the neckties and an ear of the other, or the shared triangle is an ear of each of the two neckties. Notice that the first case takes place precisely when the center of each necktie is an ear of the other, and that this happens if and only if the two centers T and T' meet at an edge. The second case takes place precisely when the two centers T and T' meet at a vertex (say $T = \{a, b, v\}$ and $T' = \{a', b', v\}$) and there is an edge e in G joining some vertex in the edge $T - v$ with some vertex in the edge $T' - v$ (say $e = \{a, a'\}$). We say that e is a *crossbar* joining T and T'. Notice that crossbars, when they exist, are unique: indeed, G contains no tetrahedron and no vertex of G can have a cycle of length 4 in its neighbourhood.

In view of these remarks, we can represent the stars and neckties in $\mathcal{F}$ by their centers and thus Theorem 8 will be a consequence of the following result. The hypothesis are the conditions that the centers of the stars and neckties in $\mathcal{F}$ must satisfy if any two of these cliques intersect, and the conclusion translates the fact that the triangle X is in any of these cliques. We have remarked above that we can assume that $\mathcal{F}$ contains some necktie.

Theorem 11 *Assume that* $\lg(G) \geq 7$ *and let* $\mathcal{F}$ *be a family of vertices and triangles of* G *which includes at least one triangle and satisfies the following three conditions:*

(i) Any two vertices in $\mathcal{F}$ *are neighbours.*

(ii) For each triangle and each vertex in $\mathcal{F}$*, either the vertex lies on the triangle or it is a neighbour of two vertices of the triangle.*

(iii) Given two different triangles in $\mathcal{F}$*, either they meet in an edge or they meet in just one vertex and they are joined by a crossbar.*

Then there exists a triangle X *in* G *such that:*

(A) Each triangle of $\mathcal{F}$ meets X in an edge or is equal to X, and
(B) each vertex of $\mathcal{F}$ is in X.

PROOF. Let us study first the case in which any two triangles in $\mathcal{F}$ meet in an edge. Let $T = \{a, b, c\}$ be any triangle in $\mathcal{F}$. If all the vertices of $\mathcal{F}$ lie in T, then $X = T$ will clearly do. Then we can assume that there is a vertex $u \in \mathcal{F}$ such that $u \notin T$. By (ii) we can assume that $\{u, a\}, \{u, b\} \in E(G)$. Notice that u is the only vertex of $\mathcal{F}$ which is not in T: Indeed if $v \in \mathcal{F}$ is not in T and $u \neq v$, we have $\{u, v\} \in E(G)$ by (i), and by (ii) there would be two possibilities: either $\{v, a\}, \{v, b\} \in E(G)$, but then $\{a, b, u, v\}$ would be a tetrahedron, or (say) $\{v, b\}, \{v, c\} \in E(G)$, but then c, a, u, v, c would be a cycle of length four in $N(b)$. Notice also that $c \notin \mathcal{F}$ because otherwise $\{a, b, c, u\}$ would be a tetrahedron by (i). Therefore, $X = \{u, a, b\}$ satisfies (B). We prove (A) by contradiction: Let T' be a triangle of $\mathcal{F}$ such that $T' \neq X$ and T' does not meet X in an edge. Then T' must meet T in an edge other than $\{a, b\}$, say $\{a, c\}$, so $T' = \{a, c, b'\}$. By (ii) we must have either an edge $\{u, c\}$ or an edge $\{u, b'\}$, but $\{u, c\} \in E(G)$ implies that $\{a, b, c, u\}$ is a tetrahedron, and $\{u, b'\} \in E(G)$ implies that u, b, c, b', u is a cycle of length four in $N(a)$. Therefore, $X = \{u, a, b\}$ also satisfies (A).

Now we can assume that there are two triangles in $\mathcal{F}$ which do not meet in an edge: by (iii) they meet in one vertex and are joined by a crossbar. Let $T_1 = \{x, b, a\}$ and $T_2 = \{x, d, c\}$ be these two triangles in $\mathcal{F}$. Without loss of generality we assume that $\{b, c\}$ is the crossbar joining T_1 and T_2. We claim that the triangle $X = \{x, c, b\}$ satisfies (A) and (B).

The proof of (A) will be indirect: assuming that there exists a triangle $T \in \mathcal{F}$ that does not meet X in an edge and T is not X, we will get a contradiction.

First we claim that $x \notin T$: Assuming that $x \in T$, say $T = \{x, u, v\}$, we will get a contradiction. To begin with, T must meet at least one of T_1 and T_2 in an edge: Indeed, if T meets both T_1 and T_2 in just the vertex x, we have two crossbars joining T with T_1 and T_2, i.e., joining $\{u, v\}$ with $\{a, b\}$ and also with $\{c, d\}$; but this implies the existence in $N(x)$ of a cycle of length at most 6. We can therefore assume that T meets T_1 in some edge and, since this edge cannot be $\{x, b\} \subset X$, it must be $\{x, a\}$ and so $T = \{x, a, v\}$. Now T must also meet T_2 in an edge: Indeed, if T were to meet T_2 in just the vertex x, there would exist a crossbar joining $\{a, v\}$ with $\{c, d\}$, but since v, a, b, c, d is a path in $N(x)$, it would follow that there exists in $N(x)$ of a cycle of length at most 5. Therefore, T meets T_2 in an edge which contains x, and this edge cannot be $\{x, c\} \subset X$, so T meets T_2 in the edge $\{x, d\}$ and thus $T = \{x, a, d\}$. Now a, b, c, d, a is a cycle of length 4 in $N(x)$, and this contradiction shows that $x \notin T$.

Now that we know that $x \notin T$, we get that T must contain at least one vertex from $\{a, b\}$ and at least one from $\{c, d\}$. But then these two vertices are neighbours and therefore the edge joining them is a crossbar joining T_1 and T_2. By the uniqueness of crossbars we obtain that $\{b, c\}$ is an edge of T, and thus either $T = X$ or T meets X in this edge: this is the desired contradiction, and (A) is proved.

Now we prove (B): Let v be a vertex of $\mathcal{F}$ and suppose that $v \notin X$. We will get a contradiction. If $v \in T_1 \cup T_2$, then v must be either a or d, say $v = a$. By (ii), we must have either $\{a, c\} \in E(G)$ or $\{a, d\} \in E(G)$, but the first of these gives a tetrahedron $\{a, b, c, x\}$, and the second a cycle a, d, c, b, a in $N(x)$. Assuming that $v \notin T_1 \cup T_2$, we will also derive a contradiction. If v is a neighbour of x, by (ii) there must be one edge joining v with $\{a, b\} \subset T_1$ and another edge joining v with $\{c, d\} \subset T_2$ but, since a, b, c, d is a path in $N(x)$, it follows that there exists in $N(x)$ a cycle of length at most 5. Therefore v is not a neighbour of x, but then (ii) now implies that v is a neighbour of both a and b, and also v is a neighbour of c and d. Therefore v, a, x, c, v is a cycle in $N(b)$, and this contradiction completes the proof. □

4 Whitney triangulations

Let us consider a triangulation (i.e., simplicial decomposition) $\mathcal{T}$ of some compact (connected) surface. If we call G the underlying graph (i.e., the 1-skeleton) of $\mathcal{T}$, then every 2-simplex of $\mathcal{T}$ is a triangle (three-vertex complete subgraph) of G. A triangulation in which, conversely, every triangle of G is a face of $\mathcal{T}$ is called a *Whitney triangulation.* In 1931, H. Whitney proved that the underlying graph of any such triangulation of the sphere is a Hamiltonian graph [15]. With other names, Whitney triangulations have been considered also by W. T. Tutte in [14] (Simple triangulations) and by N. Hartsfield and G. Ringel in [4] (Clean triangulations).

Whitney triangulations are quite amenable for graph-theoretical considerations because they are determined by their underlying graph: the two-dimensional faces are just the triangles of the graph. In other words, we can think of a Whitney triangulation as an object wearing two hats: on one hand it is just a graph, and on the other hand it is a 2-dimensional simplicial complex which in turn can be considered either as a purely combinatorial object or as a topological surface with a fixed simplicial decomposition.

In §4.1 we will characterize the underlying graphs of Whitney triangulations of closed surfaces as being precisely the locally cyclic graphs. As an instance of the amenability of Whitney triangulations, notice that the Euler characteristic χ of the closed surface triangulated by a locally cyclic graph G is very easy to calculate: if G has n vertices and its average degree is $\overline{d} \in \mathbb{Q}$, then $\chi = \frac{n}{6}(6 - \overline{d}) \in \mathbb{Z}$. Of course, it is not true that every surface triangulation is Whitney: for instance, the Heawood map in the torus has K_7 as underlying graph and this is not locally cyclic. However, the class of Whitney triangulations is a wide one: the first barycentric subdivision of any triangulation of a compact surface is always Whitney.

Proving the following result is the main purpose of this section:

Theorem 12 *If G is a locally cyclic graph with minimum degree $\delta(G) \geq 7$, then $k^3G \cong kG$.*

This will be proved in §4.3. Notice that for a locally cyclic graph G the conditions "G is regular of degree d" and "G is a locally C_d graph" are equivalent. Therefore, the above theorem contains the following result, which, as remarked in the Introduction, completes the determination of the k-behaviour of all the locally C_t graphs for $t \geq 3$:

Theorem 13 *If G is a locally C_d graph and $d \geq 7$, then $k^3G \cong kG$.* □

4.1 Characterization of Whitney Triangulations

We refer to [7] for the elements of surface triangulations.

Proposition 14 *The graph G is the* 1*-skeleton of a Whitney triangulation of some closed surface if and only if G is a locally cyclic graph.*

PROOF. Let G be the 1-skeleton of a triangulation $\mathcal{T}$ of some closed surface. If $v \in V(G)$, the faces around v yield a cycle C with $V(C) = N(v)$. If $\mathcal{T}$ is Whitney this cycle must be induced because if C had a chord some edge would belong to more than two faces.

Conversely, assume that G is locally cyclic and put a 2-cell on the boundary of each triangle. Since each edge is on exactly 2 triangles, each interior point in an edge has a neighbourhood homeomorphic to $\mathbb{R}^2$. The locally cyclic condition also guarantees that each vertex has a neighbourhood homeomorphic to $\mathbb{R}^2$. □

Proposition 15 *The graph G is the* 1*-skeleton of a Whitney triangulation of some compact surface if and only if the open neighbourhood of any vertex of G is either a cycle or a path.*

PROOF. It is analogous to the previous proof. □

If G is the 1-skeleton of some Whitney triangulation of a compact surface with border, there are two types of vertices of G: The vertices lying in the interior of the surface (i.e. the *interior* vertices) have cyclic open neighbourhood, and so the local girth at these vertices is their degree. For the vertices lying on the border of the surface the open neighbourhood is a path, and so the local girth at the *exterior* vertices is infinite. Thus in this case the local girth is the minimum of the degrees of the interior vertices (if there are no interior vertices, the local girth is infinite). Therefore, the following result is a consequence of Theorem 8:

Theorem 16 *If G is the 1-skeleton of some Whitney triangulation of a compact surface (with or without border) and the minimum degree of the interior vertices of G is at least 7, then kG is clique-Helly and therefore G is eventually k-periodic of period 1 or 2.* □

4.2 Existence of Regular Whitney Triangulations

As mentioned before, for each $d \in \{3, 4, 5\}$ there is a unique locally C_d graph, namely the tetrahedron, octahedron and icosahedron. These three are triangulations of the sphere ($\chi = 2$). Also, for $d = 6$, we mentioned that there is an infinite number of locally C_6 graphs (see [9]), each of them triangulating either the torus or the Klein bottle ($\chi = 0$).

The first example of a locally C_t graph with $t \geq 7$ that we studied on a computer (using Gap [3]) was a locally C_{10} graph which is very easy to construct using the product $G \times H$ with (g, h) adjacent to (g', h') iff $\{g, g'\} \in E(G)$ and $\{h, h'\} \in E(H)$. We know after Proposition 5 of [6] that $G \times H$ is locally $G_0 \times H_0$ whenever G is locally G_0 and H is locally H_0. Therefore, since $C_5 \times K_2 = C_{10}$, the product of the icosahedron and a triangle is a locally C_{10} graph of order 36.

For each $d \geq 7$ there is an infinite number of locally C_d graphs, as will be shown in the next proposition. All this graphs triangulate surfaces of negative Euler characteristic $\chi = \frac{n}{6}(6 - d)$ and, at variance with the case $d \leq 6$, the Euler characteristic is a function of the number of vertices. We will restrict to orientable surfaces and we will use triangular covering maps and liftings of graphs, for which we refer to [9].

Proposition 17 *Let d be any integer greater than or equal to 7. Then there exists an infinite number of non-isomorphic locally C_d graphs. Furthermore, there is an infinite number of non-homeomorphic orientable closed surfaces admitting a Whitney triangulation which is regular of degree d.*

PROOF. M. Brown and R. Connelly proved in [1] that there exists at least one finite locally C_d graph G. If the closed surface $\mathcal{S}$ associated with G is not orientable, we can lift G to a graph $\widetilde{G}$ whose associated surface is the orientation cover $\widetilde{\mathcal{S}}$ of $\mathcal{S}$. In that case, we have a 2-to-1 triangular covering map of graphs $\widetilde{G} \to G$ and, since triangular covering maps of graphs are local isomorphisms, $\widetilde{G}$ is also a finite locally C_d graph. Thus, we can assume that $\mathcal{S}$ is orientable. Being orientable and with negative Euler characteristic, the surface $\mathcal{S}$ is then a multiple torus, say a sphere with $g > 1$ handles.

Now, for each integer $n \geq 1$, there exists an orientable closed surface $\mathcal{S}^{(n)}$ (namely a sphere with $n(g-1)+1$ handles) such that there exists an n-to-1 covering map $\mathcal{S}^{(n)} \to \mathcal{S}$. Indeed, there is an n-to-1 covering map $\varphi = z^n \times id$ from the 2-dimensional torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$ to itself. Remove a small open disk D from $\mathbb{T}$ and use it to form the connected sum $\mathbb{T} \# \mathcal{S}'$ of $\mathbb{T}$ and an orientable surface $\mathcal{S}'$ of genus $g-1$. This connected sum is homeomorphic to $\mathcal{S}$. Consider now $\varphi^{-1}(D) \subset \mathbb{T}$, which is the disjoint union of n open disks because D is small. Removing these n disks from $\mathbb{T}$ and attaching n copies of $\mathcal{S}'$ (each with a small disk removed) we obtain the surface $\mathcal{S}^{(n)}$.

For each $n > 1$, we can use the covering map $\mathcal{S}^{(n)} \to \mathcal{S}$ to lift the graph G to a graph $G^{(n)}$ which underlies a triangulation of $\mathcal{S}^{(n)}$. Since we then have a triangular covering map of graphs $G^{(n)} \to G$ and these are local isomorphisms, $G^{(n)}$ is also a finite locally C_d graph. □

4.3 Proof of Theorem 12

If G is locally cyclic, the local girth at a given vertex is the degree of the vertex, so the local girth of G is its minimum degree $\delta(G)$. Therefore, if $\delta(G) \geq 7$, we already know by Theorem 8 that kG is clique-Helly. Using Theorem 7(2), Theorem 12 will then be a consequence of the following result, which does not need the hypothesis on the minimum degree:

Proposition 18 *If G is locally cyclic, then no vertex of kG dominates another different vertex of kG.*

PROOF. The only locally cyclic graph with minimum degree 3 is the tetrahedron K_4. Since kK_4 has just one vertex, the result is true when $\delta(G) = 3$. We can therefore assume that $\delta(G) \geq 4$, and hence G can not contain a tetrahedron.

The cliques of G are clearly its triangles, so kG has a vertex for each triangle of G and two different vertices T and T' of kG are adjacent if and only if these triangles meet at either a vertex or an edge.

Let $T \neq T'$ be two triangles of G. We will show that T does not dominate T', i.e., that either T and T' are not neighbours, or they are but there is a neighbour T'' of T' which is not a neighbour of T. Let us assume then that T and T' are neighbours in kG.

Assume first that T and T' meet at an edge, say $T = \{a, b, c\}$ and $T' = \{a, b, c'\}$. Since $\delta(G) > 3$, in the cycle of neighbours of c' there is an edge $\{d, e\}$ which is disjoint from $\{a, b\}$. Consider the triangle $T'' = \{c', d, e\}$, which meets T' in c'. We claim that $T \cap T'' = \varnothing$. Indeed, if this were not the case we would have $c \in \{d, e\}$, and G would contain a tetrahedron.

Assume now that T and T' meet at a vertex, say $T = \{a, b, c\}$ and $T' = \{a, b', c'\}$. Since each edge of G is contained in exactly two triangles, there must exist a triangle $T'' = \{a', c', b'\}$ which meets T' in the edge $\{b', c'\}$. Once again we must have that $T \cap T'' = \varnothing$, because otherwise we would have $a' \in \{b, c\}$ and G would contain a tetrahedron. $\square$

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