EDGE CONTRACTION AND EDGE REMOVAL ON ITERATED CLIQUE GRAPHS

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Abstract. The clique graph $K(G)$ of a graph G is the intersection graph of all its (maximal) cliques. We explore the effect of operations like edge contraction, edge removal and others on the dynamical behaviour of a graph under the iteration of the clique operator K. As a consequence of this study, we can now prove the clique divergence of graphs for which no previously known technique would yield the result. In particular, we prove that every clique divergent graph is a spanning subgraph of a clique divergent graph with diameter two.

1. Introduction

Our graphs are simple, finite and non-empty. We identify induced subgraphs with their vertex sets, so we usually write $v \in G$ instead of $v \in V(G)$; however, we may also use $V(G)$ for emphasis. A *clique* of a graph is a maximal complete subgraph. The *clique graph* $K(G)$ of a graph G is the intersection graph of its cliques. *Iterated clique graphs* are defined recursively by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. The *K-behaviour* of a graph can be *K-divergent* ($\lim_{n\to\infty} |K^n(G)| = \infty$) or *K-convergent* ($K^n(G) \cong K^m(G)$ for some $n < m$). In the latter case, the K-behaviour can be *K-null* (i.e. $K^n(G)$ has no edges for some n) or not. We shall also say *clique divergent*, *clique convergent* and *clique null* instead of K-divergent, K-convergent and K-null respectively. Two graphs G and H have the *same K-behaviour* if both are K-divergent, or both are K-convergent but not K-null, or both are K-null. Extensive literature on clique graphs can be found in [31, 36]. In recent years, there has been an increased interest in clique graphs [1–6, 8, 9, 16–26, 35] and even applications of iterated clique graphs to Loop Quantum Gravity have been found [32–34].

Edge contraction and edge removal had received little attention (so far only in [13]) in the context of clique graphs, perhaps because these operations may have a dramatic impact on the clique-related properties of a graph. For instance, we know by Neumann-Lara that the octahedron is clique divergent [10, 27], but any edge contraction/removal/addition yields a clique null graph. Here we report the results of our study on edge contraction/removal and their opposite operations in connection with clique behaviour. Specifically, we studied conditions guaranteeing that these operations leave the K-behaviour of a graph invariant or at least make it change in a controlled way (see 6.1 and 6.2).

Usually the study of clique graphs is restricted to connected graphs, since each connected component can be analyzed independently. However, here we found it convenient not to assume our graphs to be connected, as we shall consider operations that may disconnect a graph. It should be

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Key words and phrases. iterated clique graphs, local cutpoints, edge contraction, edge removal.

mentioned that the usual definition for a graph G to be clique null is that $K^n(G)$ is the one-vertex graph K_1 for some n. Note that both definitions coincide in the connected case, so all that is known about connected clique null graphs is still true under the new definition.

In §2 we quickly review the needed known notions and results. In §3 and §4 we study marked graphs, thus providing the central techniques behind the main theorems in §5 and §6. An extended abstract reporting part of this work without proofs was published in [11].

2. Preliminaries

Morphisms, retractions. Here, a *morphism* of graphs $f : G \to H$ is a vertex mapping such that for every $u, v \in V(G)$ we have $u \simeq v \Rightarrow f(u) \simeq f(v)$, where "$\simeq$" is the relation of *adjacency-or-equality*: $u \simeq v \Leftrightarrow u \sim v$ or $u = v$. In particular, note that a morphism can identify adjacent vertices. A morphism $\rho : G \to H$ is a *retraction* if there is another morphism (called a *section*) $\sigma : H \to G$ satisfying that $\rho \circ \sigma$ is the identity map in H. Under these circumstances, we say that H is a *retract* of G. Note that if H is a retract of G, then G contains an induced subgraph $\sigma(H)$ which is isomorphic to H.

If $f : G \to H$ is a morphism and $q \in K(G)$ is a clique of G, we always have that $f(q)$ is a complete subgraph of H, but possibly not a a clique (i.e. not maximal). There is, however, a natural (but usually not unique) way to define a morphism $f_K : K(G) \to K(H)$, namely selecting for each $q \in K(G)$ any fixed clique $f_K(q) \in K(H)$ containing $f(q)$. Iterating this construction, we get morphisms $f_{K^n} : K^n(G) \to K^n(H)$ for every $n \geq 2$. This new morphism f_K (no matter which selections were made) has many useful properties, the first one discovered being the following *Retraction Theorem* of Neumann-Lara:

Theorem 2.1. [27,31,36] *If $\rho : G \to H$ is a retraction, $\rho_K : K(G) \to K(H)$ is also a retraction. In particular, if H is a retract of G and H is K-divergent, then so is G.*

Domination, dismantlings, hash arrows. A vertex $v \in G$ is *dominated by* $u \in G$ if the closed neighbourhood of v is contained in that of u: $N[v] \subseteq N[u]$. Every vertex is dominated by itself, but we say that v is *dominated* (not specifying u) only when v is dominated by some vertex other than itself. In other words, v is dominated iff the open neighbourhood $N(v)$ is a cone (with apex u). We say that G is *dismantlable to* H if there is a succession of graphs $G_0, G_1, \ldots, G_r$ satisfying $G_0 = G$, $G_r \cong H$ and $G_{i+1} = G_i - v_i$ where v_i is a dominated vertex of G_i. Also, G is *dismantlable* if G is dismantlable to a graph with no edges (to the one-vertex graph K_1 in the connected case). Since the relation of domination among vertices of a graph is a preorder, mutual domination is an equivalence relation, and domination induces a partial order in the quotient set. The *pared graph* of G is the subgraph $P(G)$ induced in G by any set of representatives of the maximal equivalence classes. For example, the pared graph of the n-path ($n \geq 3$) is the $(n-2)$-path. It is easy to show that the pared graph is well defined up to an isomorphism. Escalante [10] introduced and Prisner [30] used different, but obviously equivalent, definitions of pared graphs.

We write $G \xrightarrow{\#} H$ if G contains an induced subgraph H_0 which is isomorphic to H and such that every vertex $v \in G$ is dominated by some (not necessarily different) vertex $u \in H_0$ (see [12] for details). For instance, we always have that $G \xrightarrow{\#} G$, $G \xrightarrow{\#} P(G)$ and, whenever v is dominated in G, we also have $G \xrightarrow{\#} G - v$. Furthermore, $G \xrightarrow{\#} H$ implies $H \xrightarrow{\#} P(G)$. It is easy to show that G is dismantlable to H iff $G \xrightarrow{\#} G_1 \xrightarrow{\#} G_2 \xrightarrow{\#} \cdots \xrightarrow{\#} G_r = H$ for some graphs G_i. We abbreviate this last condition as $G \xrightarrow{\#_r} H$. Note that $G \xrightarrow{\#} H$ implies that H is a retract of G (σ is the isomorphism from H to H_0 and ρ is defined, for $v \in G$, by $\rho(v) = \sigma^{-1}(u)$ where $u \in H_0$ is any vertex which dominates v). The main reason for studying this *hash arrow* relation is that, with great frequency, it is much easier to prove theorems using hash arrows instead of dismantlings, pared graphs, or removal of dominated vertices. The next two results and 5.3 are examples of this claim.

Theorem 2.2. [12, Thm.3] *If $G \xrightarrow{\#} H$, then $K(G) \xrightarrow{\#} K(H)$.*

Theorem 2.3. [12, Thm.5] *If G is dismantlable to H, then G and H have the same K-behaviour. In particular, if v is a dominated vertex of G, then G and $G - v$ have the same K-behaviour.*

Stars, normal vertices, local cutpoints. Let G be a graph. If one wanted to represent the vertex $v \in G$ by some vertex $q_v \in K(G)$, one might choose any clique q_v of G with $v \in q_v$, but in general there is no useful way to do this. For instance, if we do the same for another vertex $w \in G$, it can well happen (take $G = P_4$) that $v \sim w$ in G, but $q_v \not\simeq q_w$ in $K(G)$.

One fares better trying to represent $v \in G$ by a vertex $Q_v \in K^2(G)$ (i.e. a clique Q_v of $K(G)$, or a *clique of cliques* Q_v of G), and for this we will use stars. The *star* of $v \in G$ is the set $v^* = \{q \in K(G) \mid v \in q\}$, which is a complete subgraph of $K(G)$. This is not always a clique of $K(G)$, for instance $v^* \notin K^2(P_3)$ if v is terminal. More generally, $v^* \subseteq u^*$ iff u dominates v, so $v^* \notin K^2(G)$ if v does not dominate its dominator u, but this is not the only way in which v^* could fail to be a clique of cliques. Now, if we choose any clique Q_v of $K(G)$ that contains v^* and do the same for $w \in G$, then $Q_v \simeq Q_w$ in $K^2(G)$ whenever $v \sim w$ in G. We could still have $Q_v \sim Q_w$ in $K^2(G)$ with $v \not\simeq w$ in G, as can be observed for $G = P_6^2$. This won't happen for normal vertices.

A vertex v in a graph G is said to be *normal* if v^* is a clique of cliques. In this case we represent $v \in G$ by its star $Q_v = v^* \in K^2(G)$ and, if w is another normal vertex of G, we have that $v^* \simeq w^*$ in $K^2(G)$ iff $v \simeq w$ in G. That v^* can be equal to w^* for some adjacent normal vertices $v, w \in G$ is unavoidable, as $v^* = w^*$ iff v and w are *twins*, i.e. $N[v] = N[w]$ (iff v and w dominate each other). For example, all the vertices of the complete graph K_n are normal, but they are all twins.

Our graphs are not necessarily connected, so a *cutpoint* is a vertex u whose removal increases the number of connected components, i.e. the removal of u disconnects its connected component. More generally, a *local cutpoint* $u \in G$ is a vertex with disconnected open neighbourhood $N(u)$, so u is a cutpoint of $N[u]$ and not necessarily of G.

Lemma 2.4. *If $u \in q_1, q_2 \in Q \in K^2(G)$ and $(q_1 \cup q_2) - u$ is disconnected, then $Q = u^*$.*

Proof. If $Q \neq u^*$, there is some $q \in Q$ with $u \notin q$, but $q_1 \cap q \neq \varnothing \neq q_2 \cap q$ would contradict that $(q_1 \cup q_2) - u$ is disconnected. □

Lemma 2.5. *If $u \in G$ is a local cutpoint, then u is normal.*

Proof. Let v_1, v_2 belong to different connected components of $N(u) = N[u] - u$, and let q_1, q_2 be cliques of G containing the edges uv_1, uv_2. Note that $(q_1 \cup q_2) - u \subseteq N(u)$ is disconnected, and that $q_1, q_2 \in u^*$. Let $Q \in K^2(G)$ be such that $u^* \subseteq Q$. By 2.4, $Q = u^*$ is a clique of $K(G)$. □

We can produce local cutpoints as follows. Take two non-isolated vertices $v \neq w$ in a graph H, and suppose that $d_H(v, w) \geq 4$. Now let G be the graph obtained from H by identifying v and w into a new vertex u, which is adjacent to all vertices in $N_H(v) \cup N_H(w)$. Then $u \in G$ is a local cutpoint, and u is a cutpoint of G iff either $d_H(v, w) = \infty$ or one of v, w was a cutpoint of H.

We can look at this the other way around: If $u \in G$ is a local cutpoint and $C_1, C_2, \ldots, C_r$ are the connected components of $N_G(u)$, by *cutting through* u in G we mean choosing a partition of

$\{1, 2, \ldots, r\}$ into two non-empty disjoint subsets I, J, replacing u by two new vertices v, w and making v adjacent to all vertices in $\cup\{C_i \mid i \in I\}$ and w to all those in $\cup\{C_j \mid j \in J\}$. This is similar to splitting the vertex u, only that $v \nsim w$. In the resulting graph H, we have that v and w are non-isolated vertices with $d_H(v, w) \geq 4$. One could say that we just described a *simple cutting*, and that we can also cut *totally* trough u replacing it by r new vertices $v_1, v_2, \ldots, v_r$ and making each v_i adjacent to the vertices in "its own" component C_i of $N_G(u)$. However, this total cut can clearly be achieved by a sequence of $r - 1$ simple cuts, so studying simple cuts will be enough. If the local cutpoint $u \in G$ is as well a cutpoint of G, we say that a simple cut through u *disconnects* G if the resulting graph H has more components than the original G. This certainly happens if $N_G(u)$ has just two components, but for more than two it depends on the particular simple cut. We can now observe that, for a simple cut through the local cutpoint $u \in G$, not only the new, non-isolated vertices $v, w \in H$ satisfy $d_H(v, w) \geq 4$, but also that $d_H(v, w) = \infty$ iff u is a cutpoint of G and the cut disconnects G.

We are interested in local cutpoints because their stars are often local cutpoints again. Let $v \in G$ be a local cutpoint. We know by 2.5 that $v^* \in K^2(G)$. It turns out that the number of connected components of $N_{K^2(G)}(v^*)$ equals the number of connected components of $N_G(v)$ containing some vertex that is not dominated by v, hence v^* is a local cutpoint if and only if v has at least one non dominated vertex in at least two connected components of $N(v)$. Any shortest path in $G - v$ from one connected component of $N(v)$ to another starts with a vertex which is not dominated by v. In particular, v^* is a local cutpoint of $K^2(G)$ if there is a cut through v that does not disconnect G, for instance if v is not a cutpoint of G.

Distances in the second clique graph. Let $Q \in V(K^2(G))$, say $Q = \{q_1, q_2, \ldots, q_s\}$. We define the *basement* of Q as the set $\mathcal{B}(Q) = q_1 \cup q_2 \cup \cdots \cup q_s$. Thus $\mathcal{B}(Q)$ is always a set of vertices of G. For instance, if $v \in G$ is normal, we know that $v^* \in K^2(G)$, and we clearly have that $\mathcal{B}(v^*) = N_G[v]$. Basements for iterated cliques (i.e. $Q \in V(K^n(G))$ for $n \geq 1$) were introduced with a different name and terminology by Bornstein and Szwarcfiter in [7]. The one presented here is a simplified version of that in [29]. Given two sets of vertices $R, S \subseteq V(G)$, we define their *distance set* as $\mathcal{D}(R, S) = \{d_G(r, s) \mid r \in R, s \in S\}$.

The following is a particular case ($n = 2$) of the *Distance Formula* of [29]:

Theorem 2.6. [29, Thm.4] *Let $Q_1, Q_2 \in V(K^2(G))$. Then:*

$$\max \mathcal{D}(\mathcal{B}(Q_1), \mathcal{B}(Q_2)) - 2 \leq d_{K^2(G)}(Q_1, Q_2) \leq \min \mathcal{D}(\mathcal{B}(Q_1), \mathcal{B}(Q_2)) + 2.$$

3. Marked Graphs I: Definitions and Main Lemma

Suppose we have two disjoint graphs A and B and suppose we know their clique behaviours. If C is obtained identifying one vertex of A with one vertex of B, can we say what is the clique behaviour of C? It is not possible: the clique behaviour of C can depend on which particular vertices are identified (see Example B.3). This makes it impossible, for instance, to study the clique behaviour of a graph in terms of the clique behaviours of its blocks. We need more information, at least we need to know which vertices are the ones to be identified, so we should think not in terms of graphs but in terms of graphs with some specially marked vertices: those that will be identified.

A *marked graph* (A, X) is a graph A together with some set $X \subseteq V(A)$ of *marked vertices* of A. If X is understood, we can simply speak of "the marked graph A", and not bother with (A, X). We identify any graph G with the marked graph $(G, \varnothing)$ with no marked vertices.

However, if (A, X) is a marked graph, there seems to be no useful way to mark vertices in $K(A)$. Thus, we can not define the clique operator for marked graphs, and this is problematic since we are studying clique behaviour. What we can do is to define a new operator ξ for marked graphs that is very closely related to K^2 in the sense that $\xi(G, \varnothing) = (K^2(G), \varnothing)$, that the number of marked vertices is always the same in $\xi^n(G, X)$ for all n, and that we can actually study the clique behaviour of graphs in terms of the ξ-behaviours of, say, their marked blocks, among many other interesting properties. This section and the next are devoted to the main properties of the operator ξ for marked graphs. It is advisable to keep in mind that our main subject is clique behaviour, and that ξ-behaviour will only be a powerful but auxiliary tool.

Let (A, X) be a marked graph. In order to define $\xi(A, X)$, we define first the *wire-haired* graph $W(A, X)$: for each marked vertex $x \in X$, create a new vertex x' and attach the edge xx' to A; the new graph thus obtained is $W(A, X)$. See Fig. 1, where encircled vertices are marked. We say that $h_x = xx'$ is the *hair* at x. Notice that $W(A, X)$ does not have any marked vertices.

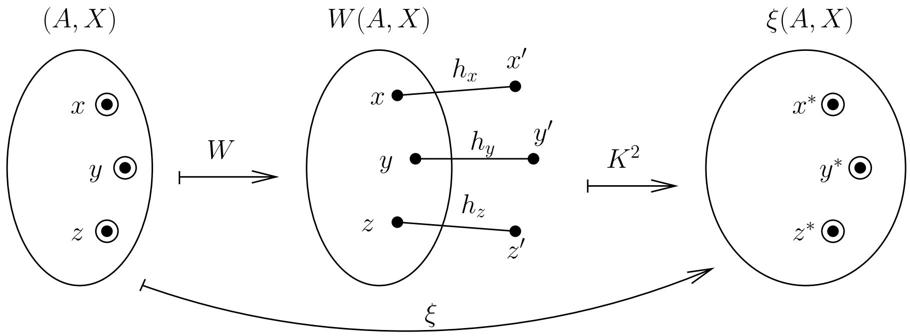


FIGURE 1. (A, X), $W(A, X)$ and $\xi(A, X)$ for $X = \{x, y, z\}$

If $x \in A$ is not normal, x^* is not a vertex of $K^2(A)$. However, if $x \in X$, x is a cutpoint of $W(A, X)$ due to the hair h_x, so x is normal in $W(A, X)$ by 2.5, and thus x^* is a vertex in $K^2(W(A, X))$. Then, putting $X^* = \{x^* \mid x \in X\}$, we can finally define $\xi(A, X)$:

$$\xi(A, X) = (K^2(W(A, X)), X^*).$$

As in the case of the clique operator we define, in a completely analogous way, iterated ξ-graphs, ξ-divergence, ξ-convergence, ξ-nullity, ξ-behaviour, and so on. We observe that $\xi(G, X)$ has the same number of marked vertices as (G, X): Indeed, if $x_1, x_2 \in X$ are distinct, $h_i \in x_i^* \in \xi(G, X)$, and no clique of cliques can contain more than one hair (all hairs are disjoint) so $x_1^* \neq x_2^*$. Also, if $x \in X$, then x^{**} is a marked vertex of $\xi^2(G, X)$ and x^{***} is a marked vertex of $\xi^3(G, X)$, etc. When we are not particularly interested in indicating the number of stars in the exponent, we simply write $x^{\star}$ instead of $x^{**\cdots*}$. For instance, we say that $x^{\star}$ is a marked vertex of $\xi^n(G, X)$ for each $x \in X$ and $n \geq 0$. Since $\xi(G, \varnothing) = (K^2(G), \varnothing)$, it follows that for graphs without marked vertices the notions of ξ-behaviour and K-behaviour coincide.

We found it convenient to develop intuitive notation for several simple operations on marked graphs that otherwise would require long sentences or expressions to be specified. Our notation is compact and affords short statements and quasi-algebraic proofs, but it also embodies visual information about the graphical operations and marks involved, which makes it easy to read.

Consider for instance the expression "$A_x B \approx {}_{\bar{x}}A\, B_{\bar{x}}$" in Theorem 4.3. It means: "Given the two disjoint marked graphs (A, X) and (B, Y), each of which contains a vertex labeled x, the disjoint union $C = (A \dot\cup B, (X \cup \{x\}) \dot\cup (Y \cup \{x\}))$ has the same ξ-behaviour as the marked graph obtained from C by identifying both copies of vertex x and unmarking the resulting new vertex x".

The idea is simple (see Fig. 2): All our graphs here are marked graphs. Subindices on any side, as in A_{xy}, ${}_{xy}B_z$, or C_y indicate that the corresponding graph has (different) vertices bearing those names. A bar over a subindex, as in ${}_{\bar{x}}A$ and $B_{y\bar{z}}$, indicates that you must mark the vertex (if it was not already marked), and the absence of such a bar indicates that you should unmark that vertex (if it was previously marked). A small space between marked graphs, as in ${}_xA\,B_x$, indicates disjoint union, and shared subindices, as in $A_{x\bar{y}}B$, indicate gluing two disjoint graphs by identifying the vertices bearing those names in both graphs. Also we use $A_{[x=y]}$ (or $A_{[x=y]}B$) to indicate that two different vertices $x, y \in A$ (or $x \in A$, $y \in B$) have been identified, so $A_{\overline{[x=y]}}$ means that x and y have been identified and the resulting vertex has been marked. The vertices not indicated in subindices may be marked or not, so $A_{\bar{x}}$ has at least one marked vertex, but may have more.

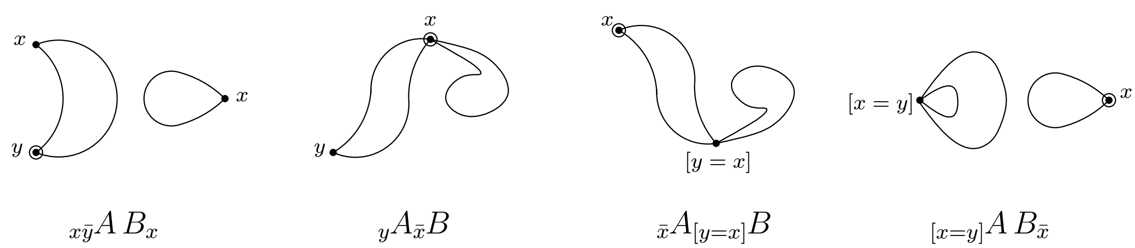


FIGURE 2. Pictorial and notational representations of four marked graphs.

As suggested above, we use $A \approx B$ to indicate that two marked graphs have the same ξ-behaviour. Similarly, we write $A \prec B$ to indicate that B has a strictly wilder behaviour than A, as in:

$$\xi\text{-null} \prec \xi\text{-convergent-but-not-}\xi\text{-null} \prec \xi\text{-divergent}.$$

As usual, $A \preccurlyeq B$ means "$A \prec B$ or $A \approx B$". Likewise, we use this linear order of behaviours to define the maximum among the behaviours of two marked graphs, as in the first of the following two simple observations:

Lemma 3.1. *The ξ-behaviour of $A\,B$ is the maximum of those of A and B.* □

Lemma 3.2. *For $Q \in \xi(A_{\bar{x}})$, if $h_x \in Q$ then $Q = x^*$.* □

The following lemma further clarifies the relation between the operators ξ and K^2 and confirms that marking vertices make them behave under ξ as normal vertices do under K^2. Indeed, marking a normal vertex $x \in A$ makes no difference after applying the ξ operator, except that x^* gets also marked. Of course, isomorphism of marked graphs means that there is an isomorphism of graphs that maps marked vertices onto marked vertices bijectively.

Lemma 3.3. *If $x \in A$ is normal, then $\xi(A_x)_{\overline{x^*}} \cong \xi(A_{\bar{x}})$.*

Proof. The only difference between $W(A_x)$ and $W(A_{\bar{x}})$ is the pending vertex x' and the corresponding hair $h_x = xx'$. Hence $K(W(A_x))$ and $K(W(A_{\bar{x}}))$ differ only by an additional vertex in $K(W(A_{\bar{x}}))$ (and its incident edges), but by 3.2 the only clique of cliques affected by the addition is x^*, which obviously has exactly the same adjacencies as before. The only possible difference here could be that x^* is always a vertex of $K^2(W(A_{\bar{x}}))$, while x^* is a vertex of $K^2(W(A_x))$ only when x is normal in A. But that was precisely our hypothesis. □

Lemma 3.4. $d_{\xi(A_{\bar{x}\bar{y}})}(x^*, y^*) = d_A(x, y)$.

Proof. Clearly, the statement holds when $d_A(x, y) = \infty$ or $d_A(x, y) = 0$. If $d_A(x, y) = 1$, take a clique $q \supseteq xy$ and observe that x^* and y^* are different ($h_x \in x^*, h_x \notin y^*$) but adjacent ($q \in x^* \cap y^*$). Otherwise, notice first that:

$$\begin{array}{rcl} \max \mathcal{D}(N_{W(A_{\bar{x}\bar{y}})}[x], N_{W(A_{\bar{x}\bar{y}})}[y]) & = & d_A(x,y) + 2 \quad \text{and} \\ \min \mathcal{D}(N_{W(A_{\bar{x}\bar{y}})}[x], N_{W(A_{\bar{x}\bar{y}})}[y]) & = & d_A(x,y) - 2, \end{array}$$

and then apply 2.6. □

Corollary 3.5. *If* $0 < d_A(x, y) < \infty$, $A_{\bar{x}\bar{y}}$ *is never* ξ*-null.* □

The following lemma is central to this work, most of our main results are strongly based on it.

Main Lemma 3.6. *If* x *and* y *are not isolated vertices of* A*, and* $d_A(x, y) \geq 4$*, then:*

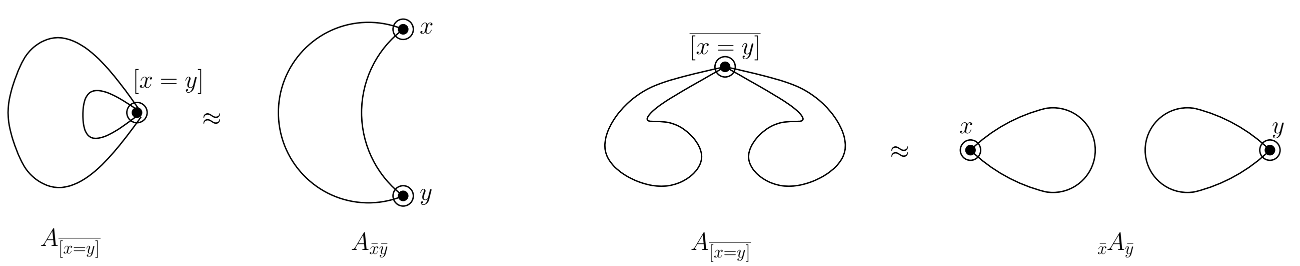
$$\xi(A_{[x=y]}) \cong \xi(A_{\bar{x}\bar{y}})_{[x^*=y^*]}.$$

We defer the proof to Appendix A; it only uses already introduced material, so it could be read now if so wished. Note that the conclusion fails when one of x, y is isolated (take $P_3 \cup K_1$), and when $d(x, y) < 4$ (take P_4). Like Janus Bifrons, the lemma has two faces that look in opposite directions. In one direction, it says that if the marked, non-isolated vertices x, y lie at distance at least four, the same graph will be obtained either if we first identify x with y, then unmark the resulting new vertex z (= "$[x = y]$"), and finally apply ξ, or if we first apply ξ, then identify x^* and y^*, and finally unmark the resulting vertex $z^* = [x^* = y^*]$. In short, the operations "identify-and-unmark" and "apply ξ" commute. But by our remarks in §2 this can be read the other way around as saying: cut through the unmarked local cutpoint $z \in A$, mark the new vertices x and y, apply ξ, identify x^* with y^* and unmark the resulting vertex: all this gives the same as $\xi(A_z)$. Thus our main lemma is, modulo judicious markings or unmarkings, a result on the relation of ξ with identifications of vertices on the one hand, and with cutting through local cutpoints on the other hand.

4. Marked Graphs II: Local Cutpoints and ξ-Behaviour

If the marked graph D has a local cutpoint z, cutting (simply) through z will yield another marked graph with the same ξ-behaviour as D. We only need to know when to mark or not to mark the two new vertices x and y. The main cases to be considered are whether z is marked or not, and whether the cut disconnects D or not (see §2). If the local cutpoint $z \in D$ is marked, both x and y must be marked, regardless of whether the cut disconnects D or not:

Theorem 4.1. *If* $d_A(x, y) \geq 4$*, then* $A_{\overline{[x=y]}} \approx A_{\bar{x}\bar{y}}$.



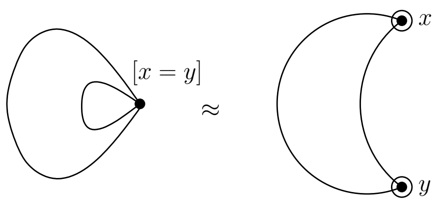
Proof. We prove that $\xi^n(A_{\overline{[x=y]}}) \cong \xi^n(A_{\bar{x}\bar{y}})_{\overline{[x^{\star}=y^{\star}]}}$. Let $n = 1$. If x is isolated, $\xi(A_{\bar{x}\bar{y}})_{\overline{[x^*=y^*]}} \cong \xi((A - x)_{\bar{y}}) \cong \xi(A_{\overline{[x=y]}})$. If x, y are not isolated, $\xi(A_{[x=y]}) \cong \xi(A_{\bar{x}\bar{y}})_{[x^*=y^*]}$ by 3.6. Therefore

$\xi(A_{[x=y]})_{\overline{[x=y]^*}} \cong \xi(A_{\bar{x}\bar{y}})_{\overline{[x^*=y^*]}}$ (marking vertices paired by the previous isomorphism, see the proof of 3.6). By 3.3, $\xi(A_{\overline{[x=y]}}) \cong \xi(A_{[x=y]})_{\overline{[x=y]^*}}$ and we get $\xi(A_{\overline{[x=y]}}) \cong \xi(A_{\bar{x}\bar{y}})_{\overline{[x^\star=y^\star]}}$ as required.

Now assume that $n > 1$ and that $\xi^{n-1}(A_{\overline{[x=y]}}) \cong \xi^{n-1}(A_{\bar{x}\bar{y}})_{\overline{[x^\star=y^\star]}}$. Our distance hypothesis is preserved by 3.4. Applying ξ to both sides we get $\xi^n(A_{\overline{[x=y]}}) \cong \xi(\xi^{n-1}(A_{\bar{x}\bar{y}})_{\overline{[x^\star=y^\star]}})$, but by the base case we know that $\xi(\xi^{n-1}(A_{\bar{x}\bar{y}})_{\overline{[x^\star=y^\star]}}) \cong \xi(\xi^{n-1}(A_{\bar{x}\bar{y}}))_{\overline{[x^\star=y^\star]}}$. Therefore, we have obtained that $\xi^n(A_{\overline{[x=y]}}) \cong \xi^n(A_{\bar{x}\bar{y}})_{\overline{[x^\star=y^\star]}}$ as required. □

We now turn to study the remaining case of an unmarked local cutpoint $z \in D$. If the cut does not disconnect D, again both x and y must be marked after cutting to get the same ξ-behaviour:

Theorem 4.2. *If $4 \leq d_A(x,y) < \infty$, then $A_{[x=y]} \approx A_{\bar{x}\bar{y}}$.*

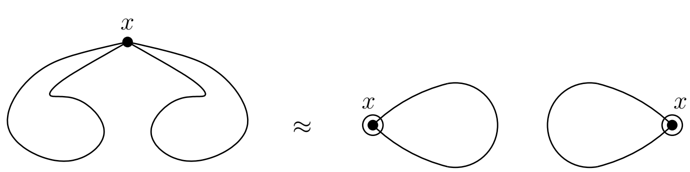


$A_{[x=y]}$ $A_{\bar{x}\bar{y}}$

Proof. Note first that x and y are not isolated. By 3.6, $\xi(A_{[x=y]}) \cong \xi(A_{\bar{x}\bar{y}})_{[x^*=y^*]}$. By 3.4 we have $4 \leq d_{\xi(A_{\bar{x}\bar{y}})}(x^*,y^*) < \infty$, so x^* and y^* are not isolated in $\xi(A_{\bar{x}\bar{y}})$. Then, applying 3.6 again, we have that $\xi^2(A_{[x=y]}) \cong \xi(\xi(A_{\bar{x}\bar{y}})_{[x^*=y^*]}) \cong \xi^2(A_{\bar{x}\bar{y}})_{[x^{**}=y^{**}]}$ with $4 \leq d_{\xi^2(A_{\bar{x}\bar{y}})}(x^{**},y^{**}) < \infty$ as before. Iterating the argument we get $\xi^n(A_{[x=y]}) \cong \xi^n(A_{\bar{x}\bar{y}})_{[x^\star=y^\star]}$ for all n, and we are done. □

Only the case of an unmarked cutpoint $z \in D$ and a cut that disconnects D remains to be studied, but this falls apart into two subcases: when we cut through z, its connected component $\mathrm{cc}(z,D)$ breaks in two, and it can be that one of the new components is ξ-null, or not. In the latter case, as perhaps expected, again both x and y must be marked after cutting to get the same ξ-behaviour:

Theorem 4.3. *If $\mathrm{cc}(x, {}_{\bar{x}}A)$ and $\mathrm{cc}(x, B_{\bar{x}})$ are not ξ-null, then $A_xB \approx {}_{\bar{x}}A\, B_{\bar{x}}$.*



A_xB ${}_{\bar{x}}A\, B_{\bar{x}}$

Proof. By 3.1 we can assume that ${}_{\bar{x}}A$ and $B_{\bar{x}}$ are connected, i.e. $\mathrm{cc}(x, {}_{\bar{x}}A) = {}_{\bar{x}}A$, $\mathrm{cc}(x, B_{\bar{x}}) = B_{\bar{x}}$. Since the operators W and K (hence ξ) preserve connectedness, it follows from our hypotheses that $x^\star$ is not an isolated vertex in $\xi^n({}_{\bar{x}}A)$ nor in $\xi^n(B_{\bar{x}})$. As in the proof of 4.2, it follows from 3.6 that $\xi^n(A_xB) \cong \xi^n({}_{\bar{x}}A)_{x^\star}\xi^n(B_{\bar{x}})$ for all n, which implies the statement. □

In the last subcase, when one of the two new components created by the cut is ξ-null, x and y must not be marked after cutting. We will see this in Theorem 4.10, but we need first to generalize some known results (2.1, 2.2 and 2.3) to marked graphs, thus obtaining 4.4, 4.6 and 4.7.

A *marked retraction* $\rho : (A, X) \to (B, Y)$ is a graph retraction $\rho : A \to B$ admitting a section $\sigma : B \to A$ (recall: ρ and σ are graph morphisms and $\rho \circ \sigma = 1_B$) such that $\sigma(Y) \subseteq X$. In this case, we say that (B, Y) is a *marked retract* of (A, X) and that σ is a *marked section*. Notice that $\rho(X) \subseteq Y$ does not necessarily hold for a marked retraction $\rho : (A, X) \to (B, Y)$.

Theorem 4.4. *If (B, Y) is a marked retract of (A, X), then $\xi(B, Y)$ is a marked retract of $\xi(A, X)$. In particular $(B, Y) \preccurlyeq (A, X)$.*

Proof. It is clear that $W(B, Y)$ is also a retract of $W(A, X)$, let σ and ρ be the corresponding section and retraction satisfying $\sigma(h_y) = h_{\sigma y}$ for every $y \in Y$. It follows by 2.1 that $K^2(W(B, Y))$ is a retract of $K^2(W(A, X))$ and the corresponding retraction and section are ρ_{K^2} and σ_{K^2}. To show that $\xi(B, Y)$ is a marked retract of $\xi(A, X)$, it only remains to be proved that $\sigma_{K^2}(Y^*) \subseteq X^*$: Take $y^* \in Y^*$, since $y^* \ni h_y$ we have $\sigma_{K^2}(y^*) \ni \sigma_K(h_y) = h_{\sigma y}$, which implies that $\sigma_{K^2}(y^*) = (\sigma y)^* \in X^*$ by 3.2. Therefore, $\xi^n(B, Y)$ is a marked retract of $\xi^n(A, X)$ for all n. Since then $\xi^n(B, Y)$ is isomorphic to an induced subgraph of $\xi^n(A, X)$ for all n, it follows that $(B, Y) \preccurlyeq (A, X)$. □

Clearly, A_x is a marked retract of $A_{\bar{x}}$:

Theorem 4.5. *For any marked graph A, $A_x \preccurlyeq A_{\bar{x}}$.* □

Example B.1 shows that the previous inequality can indeed be strict.

By $(A, X) \xrightarrow{\#} (B, Y)$ (or even $A \xrightarrow{\#} B$ if X, Y are understood) we mean that there exist two morphisms $\rho : A \to B$ and $\sigma : B \to A$ such that $\rho \circ \sigma = 1_B$, $\sigma(Y) = X$ and x is dominated by $\sigma\rho(x)$ for all $x \in A$. Notice that this definition of the hash arrow relation for marked graphs implies $|X| = |Y|$, and that it also implies that (B, Y) is a marked retract of (A, X).

Theorem 4.6. *If $(A, X) \xrightarrow{\#} (B, Y)$, then $\xi(A, X) \xrightarrow{\#} \xi(B, Y)$.*

Proof. Clearly, also $W(A, X) \xrightarrow{\#} W(B, Y)$. Consider the morphisms $\rho : W(A, X) \to W(B, Y)$ and $\sigma : W(B, Y) \to W(A, X)$ of the definition of the hash arrow. As in the proof of 4.4, we obtain that $\rho_{K^2} : \xi(A, X) \to \xi(B, Y)$ is a marked retraction. Moreover, since $\sigma(Y) = X$, we also have that $\sigma_{K^2}(Y^*) = X^*$. By 2.2, Q is dominated by $\sigma_{K^2}\rho_{K^2}(Q)$ for all $Q \in \xi(A, X)$. □

Theorem 4.7. *If $(A, X) \xrightarrow{\#} (B, Y)$, then $(A, X) \approx (B, Y)$.*

Proof. Since $\sigma\rho(x)$ dominates x for all $x \in A$, it follows that both belong to the same connected component of A. Therefore ρ induces a bijection between the connected components of A and those of B, and by 3.1 we may assume without loss that A and B are connected. Since for $|Y| = 0$ the statement is just a particular case of 2.3, we may assume $|Y| \geq 1$.

Assume first that $B = (B, Y)$ is not ξ-null. By 4.4 $A = (A, X)$ is not ξ-null. We shall prove the statement by induction on $|Y| \geq 1$. Take $y \in Y$, $x = \sigma(y) \in X$ and let C be a cycle of length at least 4 without marked vertices. Since $C_{\bar{z}}$ is ξ-convergent but not ξ-null (indeed $\xi(C_{\bar{z}}) \cong C_{\bar{z}}$), it follows by 3.1 and 4.3 that ${}_{\bar{x}}A \approx {}_{\bar{x}}A\, C_{\bar{x}} \approx A_x C$ and ${}_{\bar{y}}B \approx {}_{\bar{y}}B\, C_{\bar{y}} \approx B_y C$. Since $A_x C \xrightarrow{\#} B_y C$, we conclude this case by the inductive hypothesis, as $B_y C$ has fewer marked vertices than B.

Finally, assume (B, Y) to be ξ-null. For some n, since B is connected, $\xi^n(B, Y)$ is a graph on a single vertex $y^{\star}$. By 4.6 we have $\xi^n(A, X) \xrightarrow{\#} \xi^n(B, Y)$. Now every vertex $v \in \xi^n(A, X)$ is dominated by $\sigma\rho(v) = \sigma(y^{\star}) = x^{\star}$, so $\xi^n(A, X)$ is a cone with apex $x^{\star}$, and only $x^{\star}$ is marked in $\xi^n(A, X)$. Then $W(\xi^n(A, X))$ is still a cone with apex $x^{\star}$ and, since every clique of $W(\xi^n(A, X))$ contains $x^{\star}$, $\xi^{n+1}(A, X) = (K^2(W(\xi^n(A, X))), X^{\star})$ has only one vertex. □

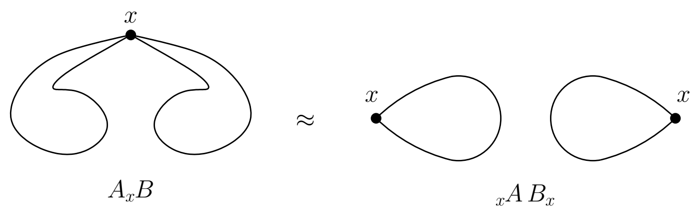
Lemma 4.8. $\xi^n(A_{\bar{x}})_{x^{\star}} \xrightarrow{\#_n} \xi^n(A_x)$.

Proof. Assume $n = 1$. Let E be an edge. Note first that $\xi(A_x E) = \xi(A_{\bar{x}})_{x^*}$ and $A_x E \xrightarrow{\#} A_x$, then apply 4.6. Now assume $n > 1$. By the inductive hypothesis, $\xi^{n-1}(A_{\bar{x}})_{x^{\star}} \xrightarrow{\#_{n-1}} \xi^{n-1}(A_x)$. Using the base case and 4.6, we have that $\xi^n(A_{\bar{x}})_{x^{\star}} = \xi(\xi^{n-1}(A_{\bar{x}}))_{x^{\star}} \xrightarrow{\#} \xi(\xi^{n-1}(A_{\bar{x}})_{x^{\star}}) \xrightarrow{\#_{n-1}} \xi^n(A_x)$. □

Lemma 4.9. $A_x \approx \xi^n(A_{\bar{x}})_{x^{\star}}$

Proof. Clearly $A_x \approx \xi^n(A_x)$. Now use 4.8 and 4.7. ☐

Theorem 4.10. *If either* $\mathrm{cc}(x, {}_{\bar{x}}A)$ *or* $\mathrm{cc}(x, B_{\bar{x}})$ *is* ξ*-null, then* $A_xB \approx {}_xA\,B_x$*.*



Proof. Without loss: A, B are connected (by 3.1) and $\xi^n({}_{\bar{x}}A) \cong K_1$ is trivial, but neither $\xi^{n-1}({}_{\bar{x}}A)$ or $\xi^{n-1}(B_{\bar{x}})$ are trivial. By 3.6 and 4.9, $\xi^n(A_xB) \cong \xi^n({}_{\bar{x}}A)_{x^{\star}}\xi^n(B_{\bar{x}}) = \xi^n(B_{\bar{x}})_{x^{\star}} \approx B_x$. ☐

Theorems 4.1, 4.2, 4.3 and 4.10 motivate us to define a *persistent vertex* as a local cutpoint $z \in D$ such that either z is a marked vertex of D, or z is not a cutpoint of D, or $D_z = A_zB$ and none of the connected components containing z in ${}_{\bar{z}}A$ and $B_{\bar{z}}$ is ξ-null. Also, we define ✂D as the marked graph obtained from D by cutting trough all the local cutpoints (totally, or simply but including those arising at previous cuttings) and marking all the vertices which come from a persistent vertex. Then we have already the following result:

Theorem 4.11. $D \approx$ ✂D *for any marked graph* D*.* ☐

5. The Problem of the Kissing Nullities

So far we we are unable to tell the K-behaviour of G_vH when we know G and H to be K-null. Specifically, we have been trying to solve the following problem since at least 2000:

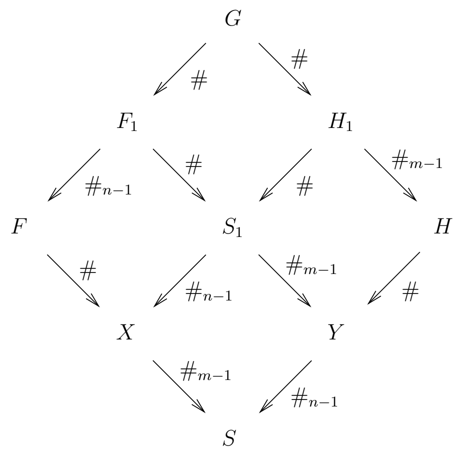
Problem 5.1. *Are there* K*-null graphs* G, H *such that* G_vH *is not* K*-null?*

The corresponding question of whether there are K-convergent graphs G, H such that G_vH is K-divergent is affirmatively answered in Example B.2. The following characterization is not completely satisfactory because (assuming A_xB not to have any marked vertices) we do not know whether "$A_{\bar{x}}$ is ξ-null" is equivalent to "A_x is ξ-null" or not, hence it could be that we are asking for the disjunction of two equivalent conditions.

Theorem 5.2. A_xB *is* ξ*-null if and only if* $A_{\bar{x}}$ *and* B_x *are* ξ*-null OR* A_x *and* $B_{\bar{x}}$ *are* ξ*-null.*

Proof. By 3.1 we can assume that A_xB is connected. If none of ${}_{\bar{x}}A$ or $B_{\bar{x}}$ is ξ-null, then by 4.3 $A_xB \approx {}_{\bar{x}}A\,B_{\bar{x}}$, which is not ξ-null by 3.1. Otherwise, assume without loss that ${}_{\bar{x}}A$ is ξ-null. By 4.5, ${}_xA$ is also ξ-null. Then, we have $A_xB \approx {}_xA\,B_x \approx B_x$ by 4.10 and 3.1. ☐

Theorem 5.3. *If* $G \xrightarrow{\#_n} F$ *and* $G \xrightarrow{\#_m} H$ *then there is a graph* S *such that* $F \xrightarrow{\#_m} S$ *and* $H \xrightarrow{\#_n} S$*. In particular, if* G *is dismantlable and* $G \xrightarrow{\#_n} H$ *then* H *is also dismantlable.*



Proof. The result clearly holds if $n = 0$ or $m = 0$. By hypothesis we have graphs F_1 and H_1 with $G \xrightarrow{\#} F_1 \xrightarrow{\#_{n-1}} F$ and $G \xrightarrow{\#} H_1 \xrightarrow{\#_{m-1}} H$. We use induction on $n + m$. The existence of S_1 is just the base case, and for it one can take $S_1 = P(G)$. The existence first of X, Y and then of S in the diagram is guaranteed by the inductive hypothesis. □

A dismantlable graph G is *uniquely startable at* v if v is the only dominated vertex of G. An example of a dismantlable, uniquely startable graph is drawn in Fig. 3.

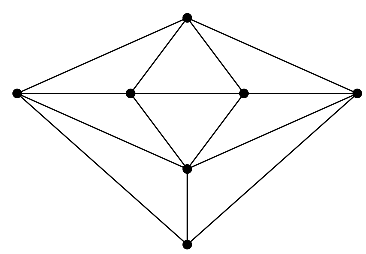


FIGURE 3. Minimum dismantlable, uniquely startable graph.

The following result summarizes our best findings so far in our attempt to solve Problem 5.1.

Theorem 5.4. *The following conditions are equivalent:*

(1) *There are K-null graphs G, H such that G_vH is not K-null.*
(2) *There is a K-null graph G such that G_vG is not K-null.*
(3) *There is a K-null graph G_v such that $G_{\bar{v}}$ is not ξ-null.*
(4) *There is a K-null graph H_w which is dismantlable, uniquely startable at w, and such that $H_{\bar{w}}$ is not ξ-null.*

Proof. We shall prove (1) ⇒ (2) ⇒ (3) ⇒ (4). The converse implications are all trivial.

(1) ⇒ (2): By 5.2, none of ${}_{\bar{v}}G$ or $H_{\bar{v}}$ is ξ-null, hence G_vG is not ξ-null also by 5.2.
(2) ⇒ (3): Use 5.2.
(3) ⇒ (4): Let n be such that $\xi^n(G_v) = K^{2n}(G) \cong K_1$. Let H be obtained from $\xi^n(G_{\bar{v}})_{v^\star}$ by successively removing all the dominated vertices save for vertex $v^\star$. Put $w = v^\star \in V(H)$. Then for some m we have $\xi^n(G_{\bar{v}})_{v^\star} \xrightarrow{\#_m} H_w$. It follows that H_w is K-null because of 4.9 and 4.7. Since we also have $\xi^n(G_{\bar{v}}) \xrightarrow{\#_m} H_{\bar{w}}$ and $\xi^n(G_{\bar{v}})$ is not ξ-null, it follows by 4.7 that $H_{\bar{w}}$ is not ξ-null. By 4.8, $\xi^n(G_{\bar{v}})_{v^\star} \xrightarrow{\#_n} \xi^n(G_v) \cong K_1$, and since we also have $\xi^n(G_{\bar{v}})_{v^\star} \xrightarrow{\#_m} H_w$, it follows by 5.3 that H_y is dismantlable. By construction, H_w can not contain any dominated vertex other than w. □

6. Some Applications

In this section, except within proofs, all our graphs are graphs without marked vertices. As it is usual in the literature, given a graph G and an edge uv, we denote contraction and removal of that edge by G/uv and $G\backslash uv$ respectively. A *local bridge* $uv \in E(G)$ is a bridge in the subgraph of G induced by $N[u] \cup N[v]$; this is equivalent to asking that $d_{G\backslash uv}(u, v) \geq 4$.

Theorem 6.1. *Let G be a graph and let $uv \in E(G)$ be a local bridge. Then $G \approx G/uv$.*

Proof. Let $E_{uv} \cong K_2$ be an edge. Assume first that $4 \leq d_{G\backslash uv}(u, v) < \infty$. Let $H = G\backslash uv$. By 4.2 we have $H_{[u=v]} \approx H_{\bar u\bar v}$. By 3.5, $H_{\bar u\bar v}$ is not ξ-null, but $E_{\bar u\bar v}$ is ξ-convergent, hence by 3.1 we obtain $H_{\bar u\bar v} \approx {}_{\bar u\bar v}H\ E_{\bar u\bar v}$. Now 3.5 and 4.3 give ${}_{\bar u\bar v}H\ E_{\bar u\bar v} \approx {}_{\bar v}H_u E_{\bar v}$, and 4.2 implies ${}_{\bar v}H_u E_{\bar v} \approx H_{uv}E$. Hence $H_{[u=v]} \approx H_{uv}E$, which means the same as $G \approx G/uv$.

Now assume $d_{G\backslash uv}(u, v) = \infty$, say $G = H_u E_v F$. Assume also: $\mathrm{cc}(u, H_{\bar u})$ is ξ-null. Since $E_{\bar v}$ and ${}_{uv}E$ are ξ-null we have by 4.10 and 3.1 that $G \approx {}_uH\ (E_vF)_u \approx {}_uH\ ({}_vE\ F_v)_u \approx {}_uH\ ({}_{uv}E\ F_v) \approx {}_uH\ F_v$. Applying 4.10 again we get $G \approx {}_uH\ F_v \approx H_{[u=v]}F = G/uv$. On the other hand, if none of $\mathrm{cc}(u, H_{\bar u})$ or $\mathrm{cc}(v, {}_{\bar v}F)$ is ξ-null, we can apply 4.3 twice to get $G = H_uE_vF \approx {}_{\bar u}H\ (E_vF)_{\bar u} \approx {}_{\bar u}H\ ({}_{\bar u\bar v}E\ F_{\bar v})$. Since ${}_{\bar u\bar v}E$ is ξ-convergent, by 3.1 and 4.3 we get $G \approx {}_{\bar u}H\ ({}_{\bar u\bar v}E\ F_{\bar v}) \approx {}_{\bar u}H\ F_{\bar v} \approx H_{[u=v]}F = G/uv$. □

By Example B.4, removing a local bridge can alter the K-behaviour. We can only show that it does not worsen. The hypothesis, however, is weaker: $d_{G\backslash uv}(u, v) \geq 3$ iff uv lies in no triangle.

Theorem 6.2. *Let G be a graph and let $uv \in E(G)$ be in no triangle. Then $G\backslash uv \preccurlyeq G$.*

Proof. Suppose first $3 \leq d_{G\backslash uv}(u, v) < \infty$. Let $H = G\backslash uv$. Now apply 3.1, 4.5, 4.3 (with 3.5) and 4.2 to obtain ${}_{uv}H \approx {}_{uv}H\ E_{uv} \preccurlyeq {}_{\bar u\bar v}H\ E_{\bar u\bar v} \approx {}_{\bar u}H_v E_{\bar u} \approx H_{uv}E = G$. Thus $G\backslash uv \preccurlyeq G$ in this case.

Suppose now $d_{G\backslash uv}(u, v) = \infty$, and let $G = H_uE_vF$. If none of $\mathrm{cc}(u, H_{\bar u})$ and $\mathrm{cc}(v, {}_{\bar v}F)$ is ξ-null, apply 4.3 twice, 3.1 and then 4.5, otherwise apply 4.10 twice, 3.1 and then 4.10 again. □

The following is a corollary of Theorem 6.2:

Theorem 6.3. *Each clique divergent graph is a spanning subgraph of some clique divergent graph of diameter two.* □

The behaviour of many new graphs can be determined thanks to the techniques developed in this paper; for instance, most of the diameter two graphs obtained in 6.3. For a particular example, take the icosahedron plus an edge between two antipodal vertices. The new edge shortens the distance between them, invalidating the techniques used to determine that the icosahedron is clique divergent in [28] or [16]. However, using the clique divergence of the icosahedron and 6.2, we get that the icosahedron plus that edge (or all six of them) is indeed clique divergent.

Theorem 6.4. *G is K-null if, and only if, G does not have any persistent vertex and every connected component of ✂G is K-null.*

Proof. Without loss assume G to be connected. We already know by 4.11 that $G \approx$ ✂G, hence we only need to show that, if G is K-null, it does not have persistent vertices. Suppose G has a persistent vertex w. If w is a local cutpoint such that $G_w = H_{[u=v]}$ with $4 \leq d_H(u, v) < \infty$, we apply 4.2 and 3.5 to get a contradiction. Otherwise we must have $G_w = H_wF$, where $\mathrm{cc}(w, {}_{\bar w}H)$ and $\mathrm{cc}(w, F_{\bar w})$ are not ξ-null, but then by 4.3 we get that $G_w \approx {}_{\bar w}H\ F_{\bar w}$, a contradiction by 3.1. □

We define $\Diamond G$ as the graph obtained from G after cutting (totally) through all its local cutpoints and attaching a 4-cycle to each of the vertices which come from a persistent vertex of G.

Theorem 6.5. $G \approx \Diamond G$

Proof. If G does not have any persistent vertex, clearly ✂$G \cong \Diamond G$, and we know by 4.11 that $G \approx$ ✂G. On the other hand, if G has persistent vertices, then ✂G has some (say m) marked vertices. Let $C_{\bar{z}}$ be the 4-cycle with one marked vertex. By 6.4 G is not K-null, hence by 4.11 ✂G is also not ξ-null and therefore, by 3.1, ✂G has the same behaviour as the disjoint union of ✂G and m disjoint copies of $C_{\bar{z}}$, but this last graph is exactly ✂$\Diamond G$, which has the same behaviour as $\Diamond G$ by 4.11. Hence $G \approx$ ✂$G \approx$ ✂$\Diamond G \approx \Diamond G$. □

Appendix A. Proof of Main Lemma 3.6

Let x and y be non-isolated vertices of the marked graph A, and assume that $d_A(x,y) \geq 4$. We need to prove that $\xi(A_{[x=y]}) \cong \xi(A_{\bar{x}\bar{y}})_{[x^*=y^*]}$. In order to do this, we shall define a morphism $f : W(A_{\bar{x}\bar{y}}) \to W(A_{[x=y]})$ such that the induced morphism at the level of cliques of cliques $f_{K^2} : K^2(W(A_{\bar{x}\bar{y}})) \to K^2(W(A_{[x=y]}))$ has the following properties:

(1) f_{K^2} is vertex-surjective.
(2) f_{K^2} is edge-surjective.
(3) $f_{K^2}(Q_1) = f_{K^2}(Q_2)$ iff either $Q_1 = Q_2$ or $\{Q_1, Q_2\} = \{x^*, y^*\}$.
(4) f_{K^2} sends marked vertices of $\xi(A_{\bar{x}\bar{y}})_{[x^*=y^*]}$ onto marked vertices of $\xi(A_{[x=y]})$.

Properties (1) and (2) say that $K^2(W(A_{[x=y]}))$ is a quotient graph of $K^2(W(A_{\bar{x}\bar{y}}))$ and Property (3) tells us that the only identification involved in this quotient is $[x^* = y^*]$, so therefore we have that $K^2(W(A_{[x=y]})) \cong K^2(W(A_{\bar{x}\bar{y}}))_{[x^*=y^*]}$. Property (4) says that this isomorphism is an isomorphism of marked graphs, which is what we had to prove. Let us start then by defining f:

Please refer to Fig. 4. Since x and y are not isolated in A, we can fix vertices $x_1, y_1 \in A$ such that xx_1 and yy_1 are edges of A. Let $f : W(A_{\bar{x}\bar{y}}) \to W(A_{[x=y]})$ be the morphism defined by $f(x) = f(y) = [x = y]$, $f(x') = y_1$, $f(y') = x_1$ and $f(w) = w$ for $w \notin \{x, y, x', y'\}$. In what follows we shall rename $W(A_{\bar{x}\bar{y}}) = W$, $W(A_{[x=y]}) = W'$ and $[x = y] = z$.

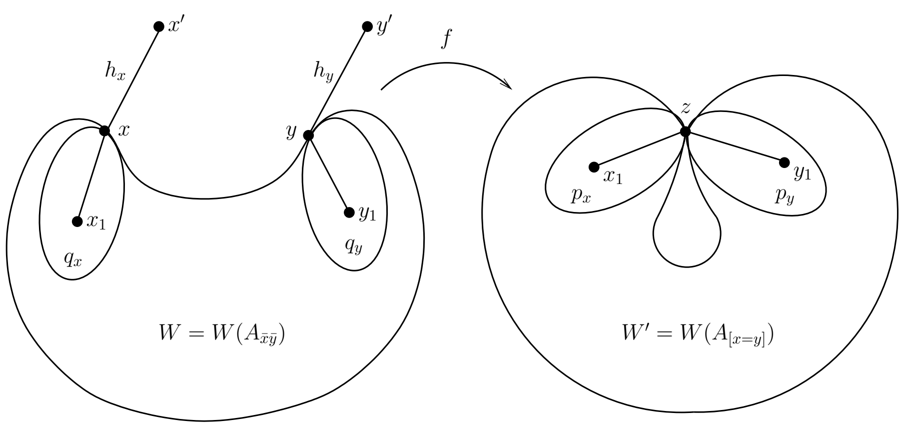


Figure 4. The mapping $f : W(A_{\bar{x}\bar{y}}) \to W(A_{[x=y]})$.

Before proving (1)-(4), we shall analyze some properties of f, f_K and f_{K^2}:

(5) **The restriction $f_| : W - \{x, y, x', y'\} \to W' - z$ is an isomorphism:**
Obvious.

(6) $\boldsymbol{f(q) = f_K(q)}$ **for** $\boldsymbol{q \in K(W)}$**,** $\boldsymbol{q \neq h_x, h_y}$**:**
Assume first that $z \notin f(q)$. It follows that $x, y, x', y' \notin q$. If $f(q)$ is not already a clique of W' it follows by (5) that $f(q) + z$ is a clique of W', but that means that for every vertex $w \in q$, we have that either $w \sim x$ or $w \sim y$. By the distance hypothesis, we can not have that $w \sim x$ and $w' \sim y$ for some $w, w' \in q$. Then it follows that either $q + x$ or $q + y$ is a *complete* (i.e. a complete subgraph) properly containing q, which is a contradiction.

Now, if $z \in f(q)$ we have that either $x \in q$ or $y \in q$ (but not both). Assume without loss that $x \in q$. If $f(q) + w$ is a bigger complete, it follows by (5) that w is adjacent (in W) to every vertex in $q - x$. Since q is a clique, we know that $w \nsim x$, but then $w \sim y$, contradicting our distance hypothesis.

(7) $\boldsymbol{f_K}$ **is vertex-surjective (even excluding** $\boldsymbol{h_x, h_y}$ **from the domain):**
Let p be a clique of W'. If $z \notin p$, p is also a clique of W and we have $f_K(p) = f(p) = p$ by (6). Now, if $z \in p$, then $p - z$ is a complete of W and either x or y (but not both) is adjacent to every vertex in $p-z$. It follows that either $f_K(p-z+x) = p$ or $f_K(p-z+y) = p$.

(8) **The only cliques of** $\boldsymbol{W'}$ **with more than one preimage are the following two:** $\boldsymbol{p_y := f_K(h_x)}$ **and** $\boldsymbol{p_x := f_K(h_y)}$**:**
Let $p_y := f_K(h_x)$ and $p_x := f_K(h_y)$. Let q_x and q_y be such that $f_K(q_x) = p_x$ and $f_K(q_y) = p_y$ as constructed in (7). We have that $x \in q_x \neq h_y$ and $y \in q_y \neq h_x$ but $f_K(q_x) = f_K(h_y)$ and $f_K(q_y) = f_K(h_x)$. Now, let $q_1, q_2 \in K(W)$, $q_1, q_2 \neq h_x, h_y$ with $f_K(q_1) = f_K(q_2)$. By (6), $f(q_1) = f_K(q_1) = f_K(q_2) = f(q_2)$, but then it follows that $q_1 - \{x, y\} = q_2 - \{x, y\}$ and, by the distance hypothesis, it follows that $q_1 = q_2$.

(9) **For** $\boldsymbol{q_1, q_2 \in K(W)}$**,** $\boldsymbol{q_1, q_2 \neq h_x, h_y}$ **we have that** $\boldsymbol{f_K(q_1) \simeq f_K(q_2)}$ **if, and only if, either** $\boldsymbol{q_1 \simeq q_2}$ **or (**$\boldsymbol{x \in q_1}$ **and** $\boldsymbol{y \in q_2}$**) or (**$\boldsymbol{y \in q_1}$ **and** $\boldsymbol{x \in q_2}$**):**
Obvious.

(10) $\boldsymbol{f_{K^2}(Q) = f_K(Q)}$ **for all** $\boldsymbol{Q \in K^2(W)}$**,** $\boldsymbol{Q \neq x^*, y^*}$**:**
Let $Q = \{q_1, q_2, \ldots, q_r\}$, $Q \neq x^*, y^*$. By 3.2 we have that $q_i \neq h_x, h_y$ for all i. Now let $p_i = f_K(q_i)$, $P = \{p_1, p_2, \ldots, p_r\}$. Obviously P is a complete. Assume there is a $p_0 \in K(W')$, $p_0 \neq p_1, p_2, \ldots, p_r$ such that $p_0 \sim p_1, p_2, \ldots, p_r$ and let $q_0 \in K(W)$ be such that $f_K(q_0) = p_0$ as constructed in (7). Then by (9) $q_0 \sim q_1, q_2, \ldots, q_r$ (for otherwise, say, $x \in q_0$, $y \in q_i$ for some i, but that implies $y \in q_j$ for all j, and therefore $Q = y^*$). Hence, $Q + q_0$ would be a complete properly containing the clique Q, a contradiction.

(11) $\boldsymbol{f_{K^2}(x^*) = f_{K^2}(y^*) = z^*}$**:**
If $Q = x^*$, then $q_x, h_x \in Q$. Hence, $f_K(q_x), f_K(h_x) \in f_{K^2}(Q)$ implies that $p_x, p_y \in f_{K^2}Q$, but $(p_x \cup p_y) - z$ is disconnected and then, by 2.4, $f_{K^2}(x^*) = z^*$. Obviously, the same happens when $Q = y^*$.

Now we can proceed to the proof of (1)-(4).

Proof of (1): Let $P \in K^2(W')$, we shall show that there is some $Q \in K^2(W)$ such that $f_{K^2}(Q) = P$. If $P = \{p_1, p_2, \ldots, p_r\}$, let $q_i \in f_K^{-1}(p_i)$ as constructed in (7). It follows by (9) that either $Q = \{q_1, q_2, \ldots, q_r\}$ is a complete (hence a clique, and by (10) $f_{K^2}(Q) = P$ and we are done), or there are some i, j with $x \in q_i$ and $y \in q_j$, but then $(p_i \cup p_j) - z = (f_K(q_i) \cup f_K(q_j)) - z$ is disconnected which implies by 2.4 that $P = z^*$ and we are done by (11).

Proof of (2): Let $P_1, P_2 \in K^2(W')$, $P_1 \sim P_2$, and let $p \in P_1 \cap P_2$. Assume first that none of P_1, P_2 is z^*; then their preimages Q_1, Q_2 as constructed in the proof of (1) will contain a clique $q \in f_K^{-1}(p)$, hence $Q_1 \sim Q_2$. Now assume $P_1 = z^* \neq P_2$. Let Q_2 be as in the previous case. Let

$p \in z^* \cap P_2$, then $z \in p \in P_2$. Now taking $q \in f_K^{-1}(p) \cap Q_2$ we have that either $x \in q$ or $y \in q$, hence either $Q_1 = x^*$ or $Q_1 = y^*$ has a nonempty intersection with Q_2.

Proof of (3): Immediate from (10) and (11).

Proof of (4): Obvious. □

Appendix B. Examples & Counterexamples

Example B.1. *There are (connected, unmarked) graphs G such that $G_v \prec G_{\bar v}$.*

Proof. (Sketch) The graph R_4 in Fig. 5 (identify vertices bearing equal labels) is an example of a clockwork graph. Clockwork graphs were introduced in [14] and further studied in [15]. In [15], it is shown that the clique graph of any clockwork graph can be obtained by applying two simple operations called *tick* and *tock*: tick removes some vertices and tock adds some vertices. So, for a clockwork graph G, we have: $K(G) \cong \text{tock} \circ \text{tick}(G)$. Let $G_v{=}R_4$. Then G_v is ξ-convergent (indeed $\xi(G_v) \cong G_v$) but a straightforward verification using clockwork graph techniques shows that $\xi(\text{tock}^n(G), \{v^{\star}\}) \cong (\text{tock}^{n+1}(G), \{v^{\star}\})$, which implies that $G_{\bar v}$ is ξ-divergent. □

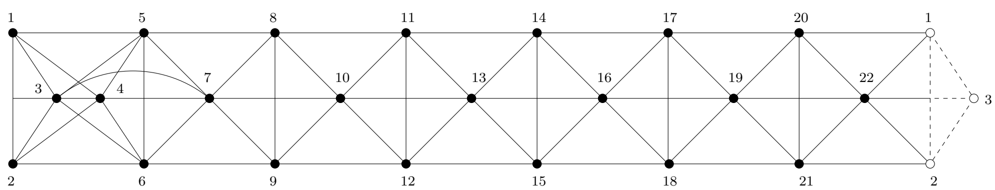


Figure 5. A K-convergent (but not K-null) graph R_4. However $R_{\bar 4}$ is ξ-divergent, while the unmarked graphs $R_{[4=14]}$, $R+e$ and R_4C are K-divergent, where e is the edge joining the vertices 4 and 14, and C is a 4-cycle.

Example B.2. *There are K-convergent graphs G, H such that G_vH is K-divergent.*

Proof. Take $G_v = R_4$ (see Fig. 5) and let $H_v = C_4$ be a 4-cycle. Then both G and H are K-convergent, but by 4.3 $G_vH \approx {}_{\bar v}G\,H_{\bar v}$ and we know that $G_{\bar v} = R_{\bar 4}$ is divergent by Example B.1, hence G_vH is K-divergent by 3.1. □

Example B.3. *There are graphs G, H such that such that the clique behaviour of G_vH depends on the chosen vertex v.*

Proof. We know by Example B.2 that R_4C is clique divergent, but a direct calculation shows that R_1C is clique convergent. □

Example B.4. *There are graphs G such that $G\backslash uv \prec G$ and uv is a local bridge of G.*

Proof. Add the edge uv ($u = 4$ and $v = 14$) to the graph R_4 in Fig. 5 to get $G = R + uv$. Then we know that $G\backslash uv = R$ is K-convergent, but G is K-divergent by 4.2, 4.3, 3.1, 4.5 and B.1. □

Example B.5. *There are connected graphs G_{uv} such that $G_{uv} \prec G_{[u=v]}$.*

Proof. As before take $G_{uv} = R_{4,14}$. Then G itself is clique convergent, but $G_{[u=v]}$ is clique divergent by 4.2, 4.5 and B.1. □

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