On bicliques and the second clique graph of suspensions

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Abstract

The clique graph $K(G)$ of a graph $G$ is the intersection graph of the set of all (maximal) cliques of $G$. The second clique graph $K^2(G)$ of $G$ is defined as $K^2(G) = K(K(G))$. The main motivation for this work is to attempt to characterize the graphs $G$ that maximize $|K^2(G)|$, as has been done for $|K(G)|$ by Moon and Moser in 1965.

The suspension $S(G)$ of a graph $G$ is the graph that results from adding two non-adjacent vertices to the graph $G$, that are adjacent to every vertex of $G$. Using a new biclique operator $B$ that transforms a graph $G$ into its biclique graph $B(G)$, we found the characterization $K^2(S(G)) \cong B(K(G))$. We also found a characterization of the graphs $G$, that maximize $|B(G)|$.

Here, a biclique $(X,Y)$ of $G$ is an ordered pair of subsets of vertices of $G$ (not necessarily disjoint), such that every vertex $x \in X$ is adjacent or equal to every vertex $y \in Y$, and such that $(X,Y)$ is maximal under component-wise inclusion. The biclique graph $B(G)$ of the graph $G$, is the graph whose vertices are the bicliques of $G$ and two vertices $(X,Y)$ and $(X',Y')$ are adjacent, if and only if $X \cap X' \neq \emptyset$ or $Y \cap Y' \neq \emptyset$.

Keywords: graph theory, graph dynamics, clique graphs, bicliques.

1. Introduction

All our graphs are finite, simple and non-empty. We identify induced subgraphs with their vertex set, in particular, we shall write $x \in G$, instead of $x \in V(G)$. A clique of a graph $G$ is a maximal complete subgraph. The clique graph $K(G)$ of a graph $G$, is the intersection graph of the set of all cliques of $G$ (i.e. the graph whose vertices are the cliques of $G$ and two vertices are adjacent if and only if they share at least one vertex).

Evidently, the clique operator $K$ can be iterated: $K^{n+1}(G) = K(K^n(G))$, for $n \in \mathbb{N}$. Clique graphs and iterated clique graphs have been studied extensively [12, 17, 21] and have found applications to the study of the Fixed Point Property of partially ordered sets [10] and to Loop Quantum Gravity [18, 19, 20].

If $x$ and $y$ are vertices of the graph $G$ that are adjacent or equal, we shall write $x \simeq y$. We shall denote by $N_G(x)$ and $N_G[x]$ the open neighborhood and closed neighborhood of the vertex $x$, respectively. The subscript will be omitted whenever it is clear from the context which graph is $G$.

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We shall denote by $\overline{G}$ to the complement graph of the graph $G$ and by $K_n$, $I_n$, $C_n$ and $P_n$, to the complete graph, the edgeless graph, the cycle and the path of $n$ vertices, respectively. We also use standard notation for graph isomorphism $G \cong H$ and for the direct product of graphs $G \times H$. We define the circle product by $G \circ H = \overline{G} \times \overline{H}$ (in [9] this product is simply denoted as the complementary of the direct product: $\overline{\times}$). Note that in $G \circ H$, the vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent-or-equal if and only if $g_1 \simeq g_2$ in $G$ or $h_1 \simeq h_2$ in $H$; for this reason some authors call this product the or-product, hence the symbol used here.

We denote by $G + H$ the Zykov sum (also know as the join of graphs) of the graphs $G$ and $H$ (i.e. the graph which is obtained from the disjoint union of $G$ and $H$ by adding all the possible edges between $G$ and $H$). We define the suspension $S(G)$ of a graph $G$ by $S(G) = I_2 + G$ and the $d$-dimensional octahedral graph by $O_d = I_2 + I_2 + \cdots + I_2$ ($d$ times).

In this work, we define a biclique of a graph $G$ by considering the set $\mathcal{B}_G = \{(X,Y) \in 2^G \times 2^G \mid x \simeq y, \text{ for every } x \in X \text{ and } y \in Y\}$ and defining a partial order on $\mathcal{B}_G$, by $(X_1,Y_1) \preceq (X_2,Y_2)$ if and only if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. A biclique $(X,Y)$ of $G$ is a maximal element of $\mathcal{B}_G$ under $\preceq$. The biclique graph $B(G)$ of $G$, is the graph whose vertices are the bicliques of $G$ with $(X_1,Y_1) \simeq (X_2,Y_2)$, if and only if $X_1 \cap X_2 \neq \emptyset$ or $Y_1 \cap Y_2 \neq \emptyset$.

Several variations of a biclique operator have been studied in the literature by Prisner [17], Zelinka [22], Figueroa and Llano [2], and by Groshaus et al. [8, 7, 5, 6, 4]. The variation studied here is new as far as we know.

A well known result from Moon and Moser [13] is that for any given order $n = |G| \geq 2$, the graphs $G$ that maximize the number of cliques (i.e. $|K(G)|$) are defined by $G = H + I_3 + I_3 + \cdots + I_3$, where $H \in \{I_2, I_3, I_2 + I_2, I_4\}$ is taken such that $|H| = |G| - 3\lfloor \frac{n-2}{3} \rfloor$. Intuitively, one would expect that these same graphs also maximize $|K^m(G)|$ for all $m$, but experiments with small graphs show that that is not the case. The experiments (we used GAP and YAGS [3, 1]) show that, for $n \in \{2, 3, \cdots, 9\}$, the number of cliques of $K(G)$ (i.e. $|K^2(G)|$) is maximized when $G = H + I_2 + I_2 + \cdots + I_2$, where $H \in \{I_2, I_3\}$, except for a few expected and understandable exceptions for small $n$ ($I_4$ for $n = 4$ and $I_5$, $C_5$ and $C_4 \cup I_1$ for $n = 5$) also attain the maximum of $|K^2(G)|$ for the given order $n$.

In view of the previous evidence and in an attempt to advance towards a characterization of the graphs maximizing $|K^2(G)|$, we study here the graphs which are suspensions $G = S(G_0) = G_0 + I_2$ and since we characterized $K^2(S(G_0)) \cong B(K(G_0))$ (Theorem 1), we also study the graphs $H$ that maximize $|B(H)|$. Our Theorem 14 gives a full characterization of such graphs.

A preliminary version of this work was published in [16], which only contained proofs for the even order case (next section) and Lemma 4. For the reader’s convenience, we include all the proofs here.

2. The even order case

Let us characterize first, $K^2(S(G))$ in terms of the biclique operator $B$.

**Theorem 1.** $K^2(S(G)) \cong B(K(G))$.

**Proof.** From the definition of $S(G)$, there are vertices $x, y \in S(G) \setminus G$, $x \neq y$, such that $N(x) = N(y) = G$. Note that if $p \in K(S(G))$, then $x \in p$ or $y \in p$, by maximality of $p$, and
We proceed to show that

Proof. Define \( \tau : B(K(G)) \to K^2(S(G)) \) by

\[
\tau((X,Y)) = \left( \bigcup_{q \in X} \{q \cup \{x\}\} \right) \cup \left( \bigcup_{q \in Y} \{q \cup \{y\}\} \right).
\]

We claim that \( \tau \) is an isomorphism of graphs. Let \( P = (\bigcup_{i=1}^r \{q_i \cup \{x\}\} \bigcup (\bigcup_{i=1}^s \{q_i \cup \{y\}\}) \) be a clique of \( K(S(G)) \) with \( q_i \in K(G) \) for all \( i \in \{1,2,\ldots,r\} \) and \( q_i' \in K(G) \) for all \( i \in \{1,2,\ldots,s\} \), then the ordered pair \( W = (\bigcup_{i=1}^r \{q_i\}, \bigcup_{i=1}^s \{q_i'\}) \) is a biclique of \( K(G) \), it follows that \( \tau(W) = P \), thus \( \tau \) is surjective. 

Let \( W_1 = (X_1,Y_1) \) and \( W_2 = (X_2,Y_2) \) be bicliques of \( K(G) \), if \( \tau(W_1) = \tau(W_2) \) then for \( A_1 = \bigcup_{q \in X_1} \{q \cup \{x\}\} \) and \( A_2 = \bigcup_{q \in X_2} \{q \cup \{x\}\} \) we have that \( A_1 = A_2 \), hence \( X_1 = X_2 \). A similar argument shows that \( Y_1 = Y_2 \), consequently \( W_1 = W_2 \) and \( \tau \) is injective. Finally, the bicliques \( W_1 \) and \( W_2 \) are adjacent in \( B(K(G)) \) if and only if there is a clique \( q \in K(G) \) such that \( q \in (X_1 \cap X_2) \cup (Y_1 \cap Y_2) \), since the last statement is true if and only if \( q \cup \{x\} \in \tau(W_1) \cap \tau(W_2) \) or \( q \cup \{y\} \in \tau(W_1) \cap \tau(W_2) \) (i.e. when \( \tau(W_1) \) and \( \tau(W_2) \) are adjacent in \( K^2(S(G)) \)), we conclude that \( \tau \) is an isomorphism. \( \square \)

It is well known that the clique graph of a Zykov sum is a circle product, \( K(G + H) \cong K(G) \circ K(H) \) \cite{14, 15, 11}. The biclique operator also shares this property:

**Theorem 2.** For any graphs \( G \) and \( H \), we have: \( B(G + H) \cong B(G) \circ B(H) \).

**Proof.** Define \( \phi : B(G + H) \to B(G) \circ B(H) \) by \( \phi((G_1 \cup H_1, G_2 \cup H_2)) = ((G_1, G_2), (H_1, H_2)) \), where \( G_1 \) and \( G_2 \) are subsets of vertices of \( G \) and \( H_1 \) and \( H_2 \) are subsets of vertices of \( H \). We proceed to show that \( \phi \) is an isomorphism of graphs.

If \( T = ((G_1, G_2), (H_1, H_2)) \in B(G) \circ B(H) \) then \( W = (G_1 \cup H_1, G_2 \cup H_2) \) is a biclique of \( G + H \), hence \( \phi(W) = T \) and \( \phi \) is surjective. It is straightforward to verify that \( \phi \) is injective. Now, the bicliques \( W_1 = (G_1 \cup H_1, G_1' \cup H_1') \) and \( W_2 = (G_2 \cup H_2, G_2' \cup H_2') \) of \( G + H \) are adjacent or equal in \( B(G + H) \) if and only if \( (G_1 \cup H_1) \cap (G_2 \cup H_2) = \emptyset \) or \( (G_1' \cup H_1') \cap (G_2' \cup H_2') = \emptyset \), but this last statement is true if and only if \( G_1 \cup G_1' \simeq G_2 \cup G_2' \) in \( B(G) \) or \( H_1 \cup H_1' \simeq H_2 \cup H_2' \) in \( B(H) \), therefore \( \phi(W_1) \simeq \phi(W_2) \) in \( B(G) \circ B(H) \) (i.e. \( \phi \) preserves adjacency). \( \square \)

Given a graph \( G \) and a subset \( S \subseteq G \), define \( N[S] = \bigcap_{x \in S} N_G[x] \) (with \( N[\emptyset] = V(G) \)). Note that for every \( S \subseteq G \), we have that \( (N[N[S]], N[S]) \) is a biclique in \( B(G) \). Also, for every biclique \( (X,Y) \in B(G) \), we have that \( (X,Y) = (N[N[X]], N[X]) \). Define \( \beta : 2^G \to B(G) \) by \( \beta(S) = (N[N[S]], N[S]) \). Observe that \( \beta \) is surjective, and in particular, \( |B(G)| \leq 2^{|G|} \). The equality holds exactly when \( G \) is an octahedral graph:

**Theorem 3.** The following statements are equivalent:

1. \( \beta \) is injective.
2. \( N[X] \neq N[X'] \) for all \( X, X' \subseteq G \) with \( X \neq X' \).
3. \( N[G \setminus \{x\}] \neq N[G] \) for all \( x \in G \).
4. For all \( x \in G \), there is some \( y \in G \) such that \( x \neq y \) and \( y \simeq z \) for all \( z \in G \setminus \{x\} \).
5. \( n = |G| \) is even and \( G \cong O_d \) for \( d = \frac{n}{2} \).
Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are straight forward. For (4) ⇒ (5), observe that x is the only vertex of G satisfying y ∉ x and hence we must have x = z for all z ∈ G \ {y}. It follows that G = I_2 + I_2 + · · · + I_2 (d times). Finally, for (5) ⇒ (1), consider two different X_1, X_2 ⊆ O_d. Assume, without loss of generality, that X_1 \ X_2 ≠ ∅, then there is a vertex z ∈ X_1 \ X_2 and a vertex w ∈ O_d with w ∉ z. Note that N[w] = G \ {z}, hence w ∈ N[X_2] and w ∉ N[X_1]. Therefore N[X_1] ≠ N[X_2].

3. The odd order case

By the previous theorem, in the even order case, |G| = n = 2d, the maximum value of |B(G)| is 2^n and it is achieved exactly when G = O_d. In the odd order case, n = 2d + 3, we shall show that G maximizes |B(G)| if and only if G ∼ I_3 + O_d (Theorem 14). Let us start by computing |B(I_3 + O_d)| and hence by obtaining a lower bound for max{|B(G)| : |G| = n}.

Lemma 4. Let n = 2d + 3 = |I_3 + O_d|, then |B(I_3 + O_d)| = \( \frac{5}{8} \cdot 2^n \)

Proof. Observe that the bicliques of I_3 are: (\( \emptyset \), I_3), (I_3, \( \emptyset \)), and (\{x\}, \{x\}), for each x ∈ I_3; hence |B(I_3)| = 5. By theorems 2 and 3, B(I_3 + O_d) ∼ B(I_3) ∘ B(O_d), hence |B(I_3 + O_d)| = |B(I_3) ∘ B(O_d)| = |B(I_3)| · |B(O_d)| = 5 · 2^{2d} = \( \frac{5}{8} \cdot 2^n \).

In what follows, we shall denote the collections of subsets of G that contain and do not contain the vertex x ∈ G, by \( S^G_x \) and \( S^G_x \) respectively. Similarly, the collection of subsets that contain the vertex x and do not contain the vertex y will be denoted by \( S^G_{xy} \) and so on. The superscript may be omitted when, from context, it is clear which graph is G.

Given a collection \( C \) of subsets of vertices of G, we define \( \mathbb{B}(C) = \{ \beta(S) \mid S ∈ C \} \) (recall that \( \beta(S) \) is a biclique of G, for every \( S ⊆ G \)). The following lemma gives a useful upper bound for |\( \mathbb{B}(C) \)|.

Lemma 5. Let \( S \) be a subset of the graph G, and let \( C \) be a collection of subsets of G such that for every \( X ∈ C \), we have that \( N[X] ⊆ S \), then |\( \mathbb{B}(C) \)| ≤ 2^{|S|}.

Proof. Recall that for every every \( Z ⊆ G \), (\( N[N[Z]], N[Z] \)) is a biclique of G (and that every biclique has that form). Therefore, a biclique of G is uniquely determined by a subset \( N[Z] ⊆ G \). Since for every \( X ∈ C \), we have that \( N[X] ⊆ S \), then |\( \mathbb{B}(C) \)| = |\( \{N[X] \mid X ∈ C\} \)| ≤ 2^{|S|}.

From now on, let \( G \) be a graph that maximizes |\( B(G) \)| when |\( G | = n = 2d + 3$. Note that by Lemma 4, we have that |\( B(G) \)| ≥ \( \frac{5}{8} \cdot 2^n \). We shall use this observation extensively in order to characterize the graph \( G \).

Lemma 6. \( G \) can not have universal vertices (i.e. \( N[x] ≠ G \) for all \( x ∈ G \)).

Proof. Suppose that \( G \) has an universal vertex \( x \), define \( κ : S_x ↪ S_τ \) by \( κ(S) = S \setminus \{x\} \), note that \( κ \) is a bijection, moreover, for any \( X ∈ S_x \) we have that \( N[X] = N[X \setminus \{x\}] ∩ N[x] = N[X \setminus \{x\}] ∩ G = N[X \setminus \{x\}] = N[κ(X)] \), this implies that \( B(S_x) = B(S_τ) \), consequently

|\( B(G) \)| = |\( B(2^G) \)| = |\( B(S_x ∪ S_τ) \)| = |\( B(S_x) ∪ B(S_τ) \)| = |\( B(S_x) \)|

\leq |\( S_x \)| = \( 2^n - 1 \) \leq \( \frac{1}{2} \cdot 2^n < \frac{5}{8} \cdot 2^n \),
which is a contradiction to the maximality of $\mathcal{G}$ by Lemma 4. Therefore, $\mathcal{G}$ can not have universal vertices. \hfill \Box

Recall that $\overline{\mathcal{G}}$ is the complement of $\mathcal{G}$ (same vertices, complementary set of edges).

**Lemma 7.** If $\overline{\mathcal{G}}$ has a vertex of degree $r$, then $|B(\mathcal{G})| \leq 2^n \left(\frac{1}{2} + \frac{1}{2^r}\right)$. Hence $\Delta(\overline{\mathcal{G}}) \leq 3$ and if $\Delta(\overline{\mathcal{G}}) = 3$ then $|B(\mathcal{G})| \leq \frac{5}{8} \cdot 2^n$.

**Proof.** Let $x \in \overline{\mathcal{G}}$ be a vertex of degree $r$. Since for every set $X \subseteq S_x$ we have that $N[X] \subseteq N[x]$, it follows from Lemma 5 that $|B(S_x)| \leq 2^{|N[x]|} = 2^{n-r}$. Therefore

$$|B(\mathcal{G})| = |\mathcal{B}(S_\tau) \cup \mathcal{B}(S_x)| \leq |\mathcal{B}(S_\tau)| + |\mathcal{B}(S_x)| \leq |S_\tau| + |\mathcal{B}(S_x)| \leq 2^{n-1} + 2^{n-r} = 2^n \left(\frac{1}{2} + \frac{1}{2^r}\right).$$

For $r \geq 4$, the previous inequality gives $|B(\mathcal{G})| < \frac{5}{8} \cdot 2^n$, which is not possible by Lemma 4 ($\mathcal{G}$ maximizes $|B(\mathcal{G})|$), therefore $\Delta(\overline{\mathcal{G}}) \leq 3$ and if $\Delta(\overline{\mathcal{G}}) = 3$, we must have $|B(\mathcal{G})| \leq \frac{5}{8} \cdot 2^n$. \hfill \Box

**Lemma 8.** $\overline{\mathcal{G}}$ is the disjoint union of cycles and paths.

**Proof.** In view of Lemma 6, it will be enough to show that $\Delta(\overline{\mathcal{G}}) \leq 2$. Suppose that $\overline{\mathcal{G}}$ has a vertex $x$ of degree 3 and let $N_{\overline{\mathcal{G}}}(x) = \{y, r, s\}$. We shall show that this assumption leads to $|B(\mathcal{G})| < \frac{5}{8} \cdot 2^n$, a contradiction to Lemma 4.

Suppose, then, that no vertex of $N_{\overline{\mathcal{G}}}(x)$ has neighbors in $\overline{\mathcal{G}} \setminus \{x, y, r, s\}$, and let $H$ be the induced subgraph of $\mathcal{G}$ with vertices $N_{\overline{\mathcal{G}}}(x)$, then $\mathcal{G} = H + (G \setminus H)$. Since $|H| = 4$ and $H \neq O_2$, we have that $|B(H)| < |B(O_2)|$ by Theorem 2 that

$$|B(\mathcal{G})| = |B(H + (G \setminus H))| = |B(H) \circ B(G \setminus H)| = |B(H)||B(G \setminus H)| < |B(O_2)||B(G \setminus H)| = |B(O_2) \circ B(G \setminus H)| = |B(O_2 + (G \setminus H))|.$$

A contradiction to the maximality of $\mathcal{G}$, thus, without loss of generality, there is a vertex $z \in \mathcal{G} \setminus \{x, y, r, s\}$ such that $z$ is not adjacent to $y$. Note that the graph $G_0 = \mathcal{G} \setminus \{x, y\}$ has odd order and by Theorem 3, item 3, there is a vertex $t \in G_0$ such that $U_0 = N_{G_0}[G_0] = N_{G_0}[G_0 \setminus \{t\}]$. Thus, we have the following two cases:

a) Case $t \neq z$. From the proof of Lemma 7, we know that

$$|B(\mathcal{G})| \leq |\mathcal{B}(S_\tau)| + |\mathcal{B}(S_x)| \leq |S_\tau| + |\mathcal{B}(S_x)| \leq 2^{n-1} + 2^{n-3} = \frac{5}{8} \cdot 2^n,$$

with $|\mathcal{B}(S_\tau)| \leq |S_\tau|$. Thus, if we can show that there are two sets $S, S' \in S_\tau$ such that $N_{\overline{\mathcal{G}}}[S] = N_{\overline{\mathcal{G}}}[S']$, then $\beta(S) = \beta(S')$ and $|\mathcal{B}(S_\tau)| < |S_\tau|$, from this it will follow that $|B(\mathcal{G})| < \frac{5}{8} \cdot 2^n$. For this, let $S = G_0$ and $S' = G_0 \setminus \{t\}$. Since neither $x$ nor $y$ are in $N[S] = N_{\overline{\mathcal{G}}}[S]$ or in $N[S'] = N_{\overline{\mathcal{G}}}[S']$, we have that $N[S] = N_{G_0}[S] = U_0$ and $N[S'] = N_{G_0}[S'] = U_0$. Hence $N[S] = N[S']$, which is the desired contradiction.
b) Case $t = z$. Again, from the proof of Lemma 7 we know that
\[
|B(G)| \leq |\mathcal{B}(S_x)| + |\mathcal{B}(S_x)| \leq |\mathcal{B}(S_x)| + 2^{|N[x]|} \leq 2^{n-1} + 2^{n-3} = \frac{5}{8} \cdot 2^n,
\]
with $|\mathcal{B}(S_x)| \leq 2^{|N[x]|}$. Therefore, if we can show that there are two sets $S$, $S' \subseteq S_x$ such that $N[S], N[S'] \subseteq 2^{|N[x]|}$ and $N[S] = N[S']$, then $|B(S_x)| < 2^{|N[x]|}$ and it will follow that $|B(G)| < \frac{5}{8} \cdot 2^n$. For this, let $S = G_0 \cup \{x\}$ and $S' = (G_0 \setminus \{t\}) \cup \{x\}$. Again, we have that $N[G_0] = N_{G_0}[G_0]$, thus
\[
N[S] = N[G_0] \cap N[x] = N_{G_0}[G_0] \cap N[x] = U_0 \cap N[x] = U_0 \setminus \{r, s\}.
\]
Since $r \in S'$, we have that $x \notin N[S']$. Moreover, we have that $N[G_0 \setminus \{t\}] \cap G_0 = N_{G_0}[G_0 \setminus \{t\}] = U_0$, therefore
\[
N[S'] = N[S'] \setminus \{x\} = N[G_0 \setminus \{t\}] \cap N(x) = N[G_0 \setminus \{t\}] \cap (G_0 \setminus \{r, s\})
\]
\[
= (N[G_0 \setminus \{t\}] \cap G_0) \setminus \{r, s\} = U_0 \setminus \{r, s\}.
\]
Hence $N[S] = N[S']$ and $N[S], N[S'] \subseteq 2^{|N[x]|}$, thus $|B(G)| < \frac{5}{8} \cdot 2^n$, which is the desired contradiction.

It follows that $\Delta(\overline{G}) \leq 2$. Therefore, $\overline{G}$ is as claimed. \hfill \square

From Lemma 8, we have that $G = P_n + P_{n_2} + \cdots + P_{n_r} + C_{m_1} + C_{m_2} + \cdots + C_{m_s}$. We leave the calculation of the exact number of bicliques of the complements of paths and cycles as an open problem, nevertheless, an upper bound will suffice to characterize $G$. For this, we have the following lemmas:

**Lemma 9.** Let $G$ be a graph with $n = |G|$ and let $\mathcal{C} = \{Z \subseteq G \mid (N[Z], Z) \notin B(G)\}$, then $|B(G)| \leq 2^n - |\mathcal{C}|$.

**Proof.** Recall that if $(X, Y)$ is a biclique of $G$, then $(X, Y) = (N[N[X]], N[X])$, therefore $(X, Y) = (N[W], W)$ for $W = N[X]$. Since $W \subseteq G$, each set $Z \in \mathcal{C}$ reduces by 1 the total of possible bicliques (which is at most $2^n$), therefore $|B(G)| \leq 2^n - |\mathcal{C}|$. \hfill \square

The next lemma is useful to construct collections of sets that satisfy Lemma 9.

**Lemma 10.** Let $G$ be a graph, then for any $Z \subseteq G$ for which there is no set $W \subseteq G$ such that $Z = \cup_{w \in W} N_G(w)$, we have that $(N_{\overline{G}}[Z], Z) \notin B(\overline{G})$.

**Proof.** Note that $N_G(x) = N_{\overline{G}}[x]$ for any $x \in G$, then, given a set $S \subseteq G$ we have that
\[
N_{\overline{G}}[S] = \bigcap_{x \in S} N_G[x] = \bigcup_{x \in S} N_{\overline{G}}[x] = \bigcup_{x \in S} N_G(x).
\]
Hence, for any $(X, Y) \in B(\overline{G})$, there is a $W \subseteq G$ such that $(X, Y) = (N_{\overline{G}}[N_{\overline{G}}[W]], N_{\overline{G}}[W]) = (N_{\overline{G}}[Z], Z)$, with $Z = \bigcup_{w \in W} N_G(w)$. Hence, for any set $Z$ that doesn’t satisfy this condition, we have that $(N_{\overline{G}}[Z], Z) \notin B(\overline{G})$. \hfill \square
From Lemma 10 we have that \( (N_G[Z], Z) \) is not a biclique of \( \overline{G} \), if \( Z \) cannot be expressed as the union of open neighborhoods of \( G \). For brevity, we shall call the sets \( Z \) that have this property as **non-coverable sets in \( G \)**. In order to construct non-coverable sets in \( G \), we have the following lemma:

**Lemma 11.** Let \( x \) be a vertex of the graph \( G \) such that for every \( y \in N(x) \), we have that \( |N(y) \setminus \{x\}| \geq 1 \). Let \( Z = \{z_1, \ldots, z_r\} \) be a set of vertices of \( G \) such that \( Z \supseteq \left( \bigcup_{y \in N(x)} N(y) \right) \setminus \{x\} \). Then every set in the collection \( S_{x, z_1, \ldots, z_r} \) is non-coverable in \( G \).

**Proof.** Let \( S \) be a set of the collection \( S_{x, z_1, \ldots, z_r} \), and let \( y \) be a vertex of \( G \) such that \( N(y) \supseteq \{x\} \). Since \( |N(y) \setminus \{x\}| \geq 1 \) and \( Z \) contains the union of all the open neighborhoods of neighborhoods of \( x \) (excluding \( x \)), there is a vertex \( z_i \in Z \) such that \( z_i \) is also in \( N(y) \), thus, any union of open neighborhoods that includes \( x \), also includes some vertex in the set \( Z \). Therefore, \( S \) is a non-coverable set in \( G \).

Using Lemma 11, we get the following upper bounds for the number of bicliques of complements of paths and cycles:

**Lemma 12.** The number of bicliques of the complements of paths satisfies:

1. \( |B(\overline{P}_2)| = 4 = 2^2 \)
2. \( |B(\overline{P}_n)| < \frac{5}{8} \cdot 2^n \) for \( n \geq 3 \).

**Proof.** Let \( \{0, \ldots, n-1\} \) be the set of vertices of \( P_n \). For \( n \leq 4 \), is easy to check manually that \( |B(\overline{P}_2)| = 4, |B(\overline{P}_3)| = 4 < 5 = \frac{5}{8} \cdot 2^3 \) and \( |B(\overline{P}_4)| = 9 < 10 = \frac{5}{8} \cdot 2^4 \).

For \( n \geq 5 \), the collections \( C_1 = S_{0,2}^{P_n}, C_2 = S_{2,0}^{P_n} \) and \( C_3 = S_{n-1,0}^{P_n} \cdot S_{2,n-3}^{P_n} \) (if \( n = 5 \), take \( C_3 = S_{4,0}^{P_n} \)) are collections of non-coverable sets by Lemma 11, respectively (in particular, \( C_3 \) is non-coverable because it is a subset of the collection \( S_{n-1,n-3}^{P_n} \)). Moreover, the collections \( C_1, C_2 \) and \( C_3 \) are pairwise disjoint because their sets differ in including/excluding the vertices \( 0 \) and \( 2 \), also note that \( |C_3| > 1 \). It follows from Lemmas 9 and 10 that

\[
|B(\overline{P}_n)| \leq 2^n - |C_1 \cup C_2 \cup C_3| = 2^n - |S_{0,2}^{P_n}| - |S_{2,0}^{P_n}| - |S_{n-1,0}^{P_n} \cdot S_{2,n-3}^{P_n}|
\]

\[
< 2^n - 2^{n-2} - 2^{n-3} - 1 < \frac{5}{8} \cdot 2^n.
\]

**Lemma 13.** The number of bicliques of the complements of cycles satisfies:

1. \( |B(\overline{C}_3)| = 5 = \frac{5}{8} \cdot 2^3 \).
2. \( |B(\overline{C}_n)| < \frac{5}{8} \cdot 2^n \) for \( n \geq 4 \).

**Proof.** Let \( \{0, \ldots, n-1\} \) be the set of vertices of \( C_n \). For \( n \leq 5 \), is straightforward to check manually that \( |B(\overline{C}_5)| = 5 = \frac{5}{8} \cdot 2^3, |B(\overline{C}_4)| = 4 < 10 = \frac{5}{8} \cdot 2^4 \) and \( |B(\overline{C}_5)| = 17 < 20 = \frac{5}{8} \cdot 2^5 \).

For \( n \geq 6 \), the following are collections of non-coverable sets in \( C_n \) by Lemma 11:

- \( C_1 = S_{2,0}^{C_n}, 1 \).
- \( C_2 = S_{0,(n-2),2}^{C_n} \).
• $C_3 = S_{1,5}^n, 2, 3, \overline{n-1}$ (non-coverable because of $\overline{n-1}$, 1 and 3).

• $C_4 = S_{0, 1, 2, 3, \overline{n-1}}^n$ (non-coverable because of $\overline{n-1}$, 1 and 3).

• $C_5 = S_{3, 4, 5, 2, \overline{t}}^n$. If $n = 6$, take $t = 0$, otherwise take $t = 6$ (non-coverable because of 2, 4 and 7).

• $C_6 = S_{0, 2, 3, 1, 5}^n$ (non-coverable because of 1, 3 and 5).

• $C_7 = S_{2, 3, 4, 5, 1, 5}^n$ (non-coverable because of 1, 3 and 5).

Note that collections $C_1$, $C_2$, $C_3$ and $C_4$ are pairwise disjoint because their sets differ in including/excluding vertices 0 and 2. The collection $C_5$ is pairwise disjoint with $C_1$, $C_2$ and $C_4$ because of the vertices 0 and 2, and with $C_3$ because of the vertex 3. The collection $C_6$ is pairwise disjoint with $C_1$, $C_2$, $C_3$ and $C_5$ because of the vertices 0 and 2, and with $C_4$ because of the vertex 3. The collection $C_7$ is pairwise disjoint with $C_2$, $C_3$, $C_4$, $C_5$ and $C_6$ because of the vertices 0 and 2, and with $C_1$ because of the vertex 4. Therefore, the collections from $C_1$ to $C_7$ are pairwise disjoint.

Note that $|C_7| \geq 1$, it follows from Lemmas 9 and 10 that

$$|B(C_n)| \leq 2^n - |\cup_{i=1}^7 C_i| = 2^n - \left( \sum_{i=1}^7 |C_i| \right) \leq 2^n - (|S_{2, 5, 7}^n| + |S_{0, 1, 5, 7}^n| + |S_{0, 1, 2, 3, \overline{n-1}}^n| + |S_{3, 4, 5, 2, \overline{t}}^n| + |S_{3, 4, 5, 1, 5}^n| + |S_{2, 3, 4, 5, 1, 5}^n|)$$

$$\leq 2^n - (2 \cdot 2^{n-3} + 4 \cdot 2^{n-5} + 1) < \frac{5}{3} \cdot 2^n.$$

\[\square\]

**Theorem 14.** Let $G$ be a graph of order $n > 1$, maximizing $|B(G)|$. Then, if $n = 2d$, we have that $G \cong O_d$; otherwise, $n = 2d + 3$ and $G \cong I_3 + O_d$.

**Proof.** The even order case, $n = 2d$, is covered by Theorem 3. For the odd order case, $n = 2d + 3$, we have that $G = C_{m_1} + C_{m_2} + \cdots + C_{m_s} + P_{n_1} + P_{n_2} + \cdots + P_{n_r}$ by Lemma 8 and from Theorem 2, we have that

$$|B(G)| = |B(C_{m_1} + C_{m_2} + \cdots + C_{m_s} + P_{n_1} + P_{n_2} + \cdots + P_{n_r})|$$

$$= |B(C_{m_1}) \circ B(C_{m_2}) \circ \cdots \circ B(C_{m_s}) \circ B(P_{n_1}) \circ B(P_{n_2}) \circ \cdots \circ B(P_{n_r})|$$

$$= |B(C_{m_1})| \cdot |B(C_{m_2})| \cdots |B(C_{m_s})| \cdot |B(P_{n_1})| \cdots |B(P_{n_r})|.$$

Since $I_2 = P_2$, $I_3 = C_3$ and $O_d = I_2 + I_2 + \cdots + I_2$ ($d$ times), it follows from lemmas 12 and 13 that $G$ can only attain the maximum when $G \cong I_3 + I_2 + I_2 + \cdots + I_2 = I_3 + O_d$ as claimed. \[\square\]

Neumann-Lara proved in [14] that the clique graph of the $d$-dimensional octahedral graph $O_d$, is again an octahedral graph (indeed: $K(O_d) \cong O_{2d-1}$). Since $K^2(S(G)) \cong B(K(G))$ by Theorem 1, it follows that $K^2(O_d) = K^2(S(O_{d-1})) \cong B(K(O_{d-1})) \cong B(O_{2d-2})$. This result, together with Theorem 14, suggest that if $G$ has even order, the graphs maximizing $|K^2(G)|$
are the same as those maximizing $|B(G)|$ (except when $n = 4$, in this case, $I_4$ also maximizes $|K^2(G)|$). Certainly, we have the following bound:

$$|K^2(G)| \geq |K^2(O_d)| = |B(O_{2d-2})| = 2^{2d-1} = \sqrt{2}^{\sqrt{2^n}}.$$  

For the odd order case, the experimental evidence mentioned before, suggest that the graphs maximizing $|K^2(G)|$ are also the same as those maximizing $|B(G)|$ (except for the few exceptions mentioned earlier in Section 1).

**Acknowledgment** We thank the anonymous referees for their comments which helped us improve the presentation of this paper.

**References**


