Diclique digraphs

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Abstract

Given a digraph D, a *diclique* is a maximal pair of vertex sets (X, Y) satisfying $x \in X, y \in Y \implies x \to y$. The *diclique digraph* of D, $\overrightarrow{K}(D)$, has the dicliques of D as vertices two of them being adjacent, $(X, Y) \to (X', Y')$, if and only if $Y \cap X' \neq \varnothing$. *Iterated diclique digraphs* are defined by $\overrightarrow{K}^0(D) = D$ and $\overrightarrow{K}^n(D) = \overrightarrow{K}(\overrightarrow{K}^{n-1}(D))$. Here we study the *diclique operator* $\overrightarrow{K}$ from the perspective of *graph dynamics* [21] and hence we are interested in $\overrightarrow{K}$*-divergence* ($|\overrightarrow{K}^n(D)| \to \infty$), $\overrightarrow{K}$*-convergence* ($\overrightarrow{K}^n(D) \cong \overrightarrow{K}^m(D)$ for some $n < m$), $\overrightarrow{K}$*-invariance* ($\overrightarrow{K}(D) \cong D$) and similar properties. In particular, we provide here the first examples of simple digraphs which are $\overrightarrow{K}$-divergent.

1 Introduction

Unless otherwise stated, our digraphs are *simple*, i.e. without loops or symmetric edges. We mostly use standard terminology here [1, 21], but for the reader convenience, we make explicit several definitions and terminology conventions in the following section.

Graph dynamics [21] is concerned with a given graph (or digraph) operator Φ and the sequence of graphs resulting from applying it iteratively to graphs: $G, \Phi(G), \Phi^2(G), \ldots$. Many natural questions appear in this context like, whose graphs are Φ*-invariant*, which graphs belong to the image of Φ, which ones are Φ*-divergent*, which classes of graphs are Φ-invariant and so on. Several graph operators have been extensively studied in this way, including the *clique graph operator* [2, 13, 15, 17–19, 26, 27] and the *biclique graph operator* [8–12]. Other operators concerning bicliques as the *strong-factor*, the *weak-factor* and the *clean-factor* (a refinement of the weak-factor) *operators* have been studied in [3, 4].

The *diclique digraph* and the *diclique operator* were introduced by E. Prisner inspired by the clique graph operator [21]. In the beginning, this operator was defined as the biclique digraph for general digraphs with possible loops and symmetric (but not multiple) arcs (see Definition 19.23 of Prisner's book [21]). The diclique operator have been investigated where

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loops where implicitly [5, 28] or explicitly [6] considered. In the case of simple digraphs (no loops, no symmetric arcs), as is our case, the definition of the diclique operator excludes intersections of the parts of a diclique (see Section 2 for the details).

Graph dynamics have been applied to Loop Quantum Gravity [22–25] whereas dicliques have been applied, for example, in general systems theory to represent binary relations [14], in economic based decision-making [7] and in search techniques to find internally densely connected groups of nodes in directed networks [20].

In this paper, we study the dynamics of the diclique operator $\overrightarrow{K}$ (detailed definitions will be given in the next section). Concerning the diclique operator, B. Zelinka [28] exhibited a self-diclique digraph that is not a directed cycle (the circulant $\overrightarrow{C}_6(1,2)$). This result partially answered a question posed by Prisner (see [21], open problem 39, p.207) on the existence of a $\overrightarrow{K}$-periodic strongly connected digraphs. In [5], an infinite family of self-diclique circulant digraphs was exhibited generalizing the example given by Zelinka. A characterization of the self-diclique circulant digraphs and an infinite family of non-circulant self-diclique digraphs were provided in [6].

Among other results, in Sections 3 and 4, we characterize the dicliques of a digraph (Theorem 4), define a pair of canonical morphisms $\alpha, \beta : D \to \overrightarrow{K}(D)$ (Theorem 6) and show that the diclique operator is surjective for the class of simple digraphs (Corollary 9). We also study the $\overrightarrow{K}$-preserved properties like transitivity, strong connectivity, girth, acyclicity and also the digraph properties that vary in a controlled way under $\overrightarrow{K}$ such as diameter and height (Theorems 11 and 13). We present numerous examples to show that many of our theorems cannot be improved.

In Section 5, we present an infinite family of examples of $\overrightarrow{K}$-divergent digraphs. These examples are the first ones since Prisner [21] (1995). Also they are the first ones which are

- simple,
- not strongly connected and
- of linear growth.

Observe that Prisner's examples grow exponentially at each step, and hence they grow super-exponentially. In the last section, we show that the diclique operator $\overrightarrow{K}$ distributes over the categorical (or tensor) product (Theorem 19) and that there exist $\overrightarrow{K}$-divergent digraphs with polynomial growth rate (Theorem 21).

2 Preliminaries and terminology

As usual, given a digraph D, $V(D)$ and $A(D)$ denote the vertex set and the arc set of D, respectively. When the digraph of reference is clear, we use V and A instead of $V(D)$ and $A(D)$ respectively. The expression "$u \to v$" denotes both, the arc "(u, v)" and the proposition "(u, v) is an arc (of the considered digraph)".

Let X and Y be nonempty subsets of $V(D)$. The pair (X, Y) is a *disimplex* of D if for every $x \in X$ and every $y \in Y$, $(x, y) \in A(D)$. Note that if (X, Y) is a disimplex, then we have that $X \cap Y = \varnothing$ since otherwise, we would have some $z \in X \cap Y$ and hence a loop

$z \to z$ at z, contrary to our assumption that D is simple. For disimplices (X, Y) and (Z, W) of D, we say that (X, Y) is *included* in (Z, W), denoted by $(X, Y) \preceq (Z, W)$, if and only if $X \subseteq Z$ and $Y \subseteq W$. Clearly, this is a partial order relation among disimplices of $V(D)$. Note that every arc $u \to v = (u, v)$ determines a disimplex $(X, Y) = (\{u\}, \{v\})$.

A disimplex (X, Y) is a *diclique* of D if it is maximal with respect to the order relation $\preceq$. Note that (the disimplex determined by) any arc is included in some diclique. We emphasize that by our definitions, we must have $X \neq \varnothing \neq Y$ and $X \cap Y = \varnothing$. The set of dicliques of D is denoted $\mathcal{D}(D)$.

The *diclique digraph* of D, denoted by $\overrightarrow{K}(D)$, is the digraph whose vertex set is the set of dicliques of D, $\mathcal{D}(D)$, and $(X, Y) \to (Z, W)$ is an arc of $\overrightarrow{K}(D)$ if and only if $Y \cap Z \neq \varnothing$. We say that $\overrightarrow{K}$ is the *diclique operator*. Observe that if $A(D) = \varnothing$, there are no dicliques in D. In this case, $\overrightarrow{K}(D)$ is the *empty digraph*, the digraph with no vertices and no arcs. As usual, we will denote the empty digraph by $\varnothing$. Hence, in this case, we write $\overrightarrow{K}(D) = \varnothing$. Obviously, we also have $\overrightarrow{K}(\varnothing) = \varnothing$. By convention we will assume that the empty digraph is simple and acyclic.

From the general theory of graph operators [21], we recall the following definitions: A digraph is $\overrightarrow{K}$*-divergent* if $\lim_{n\to\infty} |\overrightarrow{K}^n(D)| = \infty$. Otherwise, it is $\overrightarrow{K}$*-convergent*, which necessarily means $\overrightarrow{K}^t(D) \cong \overrightarrow{K}^{t+p}(D)$ for some $t \geq 0$, $p \geq 1$. When t and p are the smallest possible integers satisfying that, we say that t is the *transition index* of D (under $\overrightarrow{K}$) and p is the *period* of D (under $\overrightarrow{K}$). In this case, we say that D *converges to* the set $\{\overrightarrow{K}^t(D), \overrightarrow{K}^{t+1}(D), \ldots, \overrightarrow{K}^{t+p-1}(D)\}$. If $p = 1$ we simply say that D *converges to* $\overrightarrow{K}^t(D)$. When D converges to $\varnothing$, D is said to be $\overrightarrow{K}$*-mortal*. When $t = 0$ we say that D is *periodic* of period p and when $t = 0$ and $p = 1$, we say that D is *self-diclique* or $\overrightarrow{K}$*-invariant*.

For instance, let $D = TT_3$ be a digraph such that $V(D) = \{1, 2, 3\}$ and $A(D) = \{(1, 2), (1, 3), (2, 3)\}$, (see Figure 1). The disimplex $(\{1\}, \{2\})$ of D is not a diclique, since $(\{1\}, \{2\}) \preceq (\{1\}, \{2, 3\})$ and

$$\mathcal{D}(D) = \{(\{1\}, \{2, 3\}), (\{1, 2\}, \{3\})\}.$$

Therefore, $\overrightarrow{K}(D)$ is isomorphic to an arc.

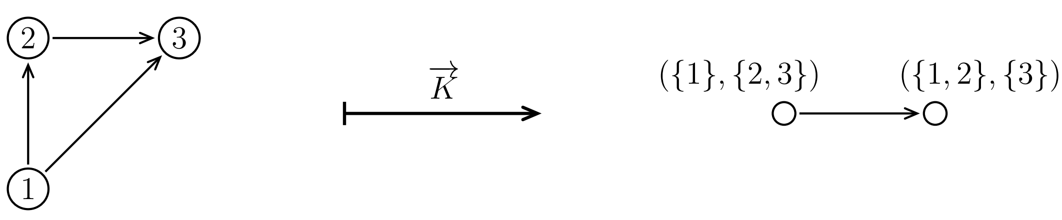

Figure 1: TT_3 and its diclique digraph.

For every $x \in V$ we define $N^+(x) = \{y \in V : (x, y) \in A\}$ and $N^-(x) = \{y \in V : (y, x) \in A\}$ to be the *out-* and the *in-neighborhood* of x, respectively. The *height* of a digraph D is the length of a directed path of maximum length in D and it is denoted by $h(D)$. Recall that if D has an (x,y)-walk W, then D contains an (x,y)-path P such that $A(P) \subseteq A(W)$. The *girth* of a digraph D, denoted by $\overrightarrow{g}(D)$, is the length of a shortest directed cycle contained in the digraph. If D is acyclic, then $\overrightarrow{g}(D) = \infty$. Recall that the empty graph $\varnothing$ is acyclic by convention and therefore $\overrightarrow{g}(\varnothing) = \infty$. Since we only consider simple digraphs, $\overrightarrow{g}(D) \geq 3$.

We denote by $d_D(x,y)$ the *distance* from a vertex x to a vertex y of D. If y is reachable from x, then $d_D(x,y)$ is finite, otherwise $d_D(x,y) = \infty$. Recall that the distance from a set X to a set Y of vertices in D is defined by

$$d_D(X,Y) = \max\{d_D(x,y) : x \in X, y \in Y\}$$

and the *diameter* of D is $\operatorname{diam}(D) = d_D(V,V)$. Clearly, D has finite diameter if and only if D is strongly connected. Moreover, $\operatorname{diam}(\varnothing) = \infty$.

A digraph D is *transitive* if $(x,y),(y,z) \in A(D)$ implies $(x,z) \in A(D)$. D is *acyclic* if there are no directed cycles in D. A *tournament* on n vertices is an orientation of the complete graph K_n. Notice that a tournament is transitive if and only if it is acyclic.

As we previously mentioned, the clique operator of a graph G inspired the definition of the diclique operator. Besides, the biclique operator resembles the diclique operator since dicliques can be considered as directed bicliques (the orientation goes from one part to the other in bipartition). We explore the similarities and the differences between the three operators throughout the paper.

Let $G = (V,E)$ be a graph. For every $x \in V$, the sets $N(x)$ and $N[x] = N(x) \cup \{x\}$ are the *neighborhood* and the *closed neighborhood* of x, respectively. A *clique* of a graph G is a maximal complete subgraph, which we often identify with its corresponding vertex set. The *clique graph* of G is the intersection graph $K(G)$ of the cliques of G, that is, $V(K(G)) = \mathcal{C}(G)$ and $X, Y \in \mathcal{C}(G)$ are adjacent if and only if $X \cap Y \neq \varnothing$, where $\mathcal{C}(G)$ is the set of cliques of G. A *biclique* of a graph G is a maximal induced complete bipartite subgraph. The *biclique graph* of G is the intersection graph $KB(G)$ of the bicliques of G.

3 Dicliques and simple digraphs

For the following theorem, assume that the digraph D may contain loops or symmetric arcs. When we have loops in a digraph, the definition of dicliques allows that a diclique (X,Y) may have intersecting parts $X \cap Y \neq \varnothing$. We prove that the property of being a simple digraph is invariant under $\overrightarrow{K}$:

Theorem 1. *D is a simple digraph if, and only if, $\overrightarrow{K}(D)$ is a simple digraph.*

Proof. Suppose that $\overrightarrow{K}(D)$ has a loop at a diclique (X,Y) of D, that is, $\overrightarrow{K}(D)$ has an arc of the form $(X,Y) \to (X,Y)$. Then $Y \cap X \neq \varnothing$. Take $z \in Y \cap X$, since (X,Y) is a diclique, we have $z \to z$ and hence a loop in D, contrary to our hypothesis.

On the other hand, if we had a symmetric arc $(X,Y) \rightleftarrows (W,Z)$ in $\overrightarrow{K}(D)$, then there exist $y \in Y \cap W$ and $x \in Z \cap X$. Since (X,Y) and (W,Z) are dicliques, we have $x \to y \to x$. Then we either have that $x \neq y$, which implies a symmetric arc $x \rightleftarrows y$ in D, or we have that $x = y$, which implies a loop at x in D. Both possibilities are contrary to our hypothesis.

If D is not simple, then there is either a loop $z \to z$ or a symmetric edge $x \rightleftarrows y$. In the first case, the arc $z \to z$ is included in some diclique (Z, Z') and since $z \in Z \cap Z'$, we have a loop $(Z, Z') \to (Z, Z')$ in $\overrightarrow{K}(D)$. In the second case, the arcs $x \to y$ and $y \to x$ must be included in some dicliques (X, Y) and (Y', X'), since $x \in X' \cap X$ and $y \in Y \cap Y'$ we have that there is either a loop at (X, Y) (whenever $(X, Y) = (Y', X')$) or a symmetric edge $(X, Y) \rightleftarrows (Y', X')$ in $\overrightarrow{K}(D)$. $\square$

The symbol $\overleftarrow{D}$ denotes the *converse* of D ($\overleftarrow{D}$ is obtained from D by reversing all the arcs). Note that $\overleftarrow{\overleftarrow{D}} = D$. We prove that the operations of reversing arcs and the diclique operator commute:

Theorem 2. *Let D be a digraph. Then $\overrightarrow{K}(\overleftarrow{D}) = \overleftarrow{\overrightarrow{K}(D)}$.*

Proof. From the definition of the converse, $(A, B) \in \mathcal{D}(D)$ if and only if $(B, A) \in \mathcal{D}(\overleftarrow{D})$. Therefore, there is a bijection between $V(\overrightarrow{K}(D))$ and $V(\overrightarrow{K}(\overleftarrow{D})) = V(\overleftarrow{\overrightarrow{K}(\overleftarrow{D})})$. Moreover, notice that $(A, B) \to (A', B')$ is an arc of $\overrightarrow{K}(D)$ if and only if $B \cap A' \neq \varnothing$, if and only if $(B', A') \to (B, A)$ is an arc of $\overrightarrow{K}(\overleftarrow{D})$, if and only if $(A, B) \to (A', B')$ is an arc of $\overleftarrow{\overrightarrow{K}(\overleftarrow{D})}$. Therefore $\overrightarrow{K}(D) \cong \overleftarrow{\overrightarrow{K}(\overleftarrow{D})}$, equivalently, $\overrightarrow{K}(\overleftarrow{D}) \cong \overleftarrow{\overrightarrow{K}(D)}$. $\square$

Cliques of graphs satisfy the following straightforward property: $X \subseteq V(G)$ is a clique of G if and only if $X = \bigcap_{x \in X} N[x]$. We shall see in Theorem 3 ($1 \implies 2$) that a similar property holds for dicliques. Given a vertex set $Z \subseteq V = V(D)$. We define

$$N^+(Z) = \textstyle\bigcap_{z \in Z} N^+(z),\ N^-(Z) = \bigcap_{z \in Z} N^-(z).$$

$$\alpha(Z) = (N^-(N^+(Z)), N^+(Z)) \text{ and } \beta(Z) = (N^-(Z), N^+(N^-(Z))).$$

By convention, $N^+(\varnothing) = V = N^-(\varnothing)$. Since D has no loops, we also have $N^+(V) = \varnothing = N^-(V)$. Then, $N^+(Z)$ is the set of all common out-neighbors of (the vertices of) Z, and $N^-(Z)$ is the set of all common in-neighbors of Z. It should be clear then that for all Z, we have $Z \subseteq N^-(N^+(Z))$ and $Z \subseteq N^+(N^-(Z))$. Clearly, both $\alpha(Z)$ and $\beta(Z)$ are disimplices as long as both of their components X and Y are non-empty, but note that $\alpha(\varnothing) = \beta(V) = (\varnothing, V)$ and $\alpha(V) = \beta(\varnothing) = (V, \varnothing)$ are never disimplices.

Theorem 3. *Let $D = (V, A)$ be a digraph and $X, Y \subseteq V$ with $X \neq \varnothing \neq Y$. Then, the following conditions are equivalent:*

1. *(X, Y) is a diclique of D.*
2. *$N^+(X) = Y$ and $N^-(Y) = X$.*
3. *$(X, Y) = \alpha(X)$.*
4. *$(X, Y) = \beta(Y)$.*

Proof. **(1) $\implies$ (2):** Since (X,Y) is a disimplex, we have $Y \subseteq N^+(X)$ and $X \subseteq N^-(Y)$. The maximality of (X,Y) under $\preceq$ gives the result.

(2) $\implies$ (1): Clearly, (X,Y) is a disimplex. But none of X and Y can be be enlarged preserving this condition. Hence (X,Y) is maximal under the partial order $\preceq$ and (X,Y) is a diclique.

(2) $\implies$ (3): $\alpha(X) = (N^-(N^+(X)), N^+(X)) = (N^-(Y), Y) = (X,Y)$.

(3) $\implies$ (2): $\alpha(X) = (X,Y) \implies N^+(X) = Y$ and $X = N^-(N^+(X)) = N^-(Y)$.

The remaining cases are similar. □

Theorem 4. *The dicliques of $D = (V, A)$ are:*

$$\begin{aligned}\mathcal{D}(D) &= \{\alpha(Z) : Z \subseteq V, Z \neq \varnothing \neq N^+(Z)\} \\ &= \{\beta(Z) : Z \subseteq V, Z \neq \varnothing \neq N^-(Z)\}.\end{aligned}$$

Proof. All the dicliques (X,Y) are of the form $(X,Y) = \alpha(X)$, with $X \neq \varnothing \neq Y = N^+(X)$ according to Theorem 3($1 \implies 3$). Hence $\mathcal{D}(D) \subseteq \{\alpha(Z) : Z \subseteq V, Z \neq \varnothing \neq N^+(Z)\}$.

Now, take $(X,Y) = \alpha(Z) = (N^-(N^+(Z)), N^+(Z))$ for some $Z \subseteq V$ with $Z \neq \varnothing \neq N^-(Z)$. Then $Y = N^+(Z)$ and $X = N^-(N^+(Z)) = N^-(Y)$, therefore X is the set of common in-neighbors of (the vertices of) Y. Note that $Z \subseteq N^-(N^+(Z)) = X$. It follows that the set of common out-neighbors of X satisfy $N^+(X) = N^+(N^-(Y)) \supseteq Y$, but $N^+(X) \subseteq N^+(Z) = Y$. Therefore $N^+(X) = Y$. Since $X \supseteq Z \neq \varnothing$ and $Y = N^+(Z) \neq \varnothing$, it follows by Theorem 3 ($2 \implies 1$) that (X,Y) is a diclique and hence $\{\alpha(Z) : Z \subseteq V, Z \neq \varnothing \neq N^+(Z)\} \subseteq \mathcal{D}(D)$.

By the previous two paragraphs we have that

$$\mathcal{D}(D) = \{\alpha(Z) : Z \subseteq V, Z \neq \varnothing \neq N^+(Z)\}.$$

The argument for $\mathcal{D}(D) = \{\beta(Z) : Z \subseteq V, Z \neq \varnothing \neq N^-(Z)\}$ is entirely analogous. □

In the singleton cases, $Z = \{x\}$, we write $\alpha(x)$ instead of $\alpha(\{x\})$ and $\beta(x)$ instead of $\beta(\{x\})$. The singleton cases are specially interesting, since the corresponding dicliques $\alpha(x)$ (and $\beta(x)$) tend to reproduce much of the structure of D within $\vec{K}(D)$ as we shall see in Theorem 6.

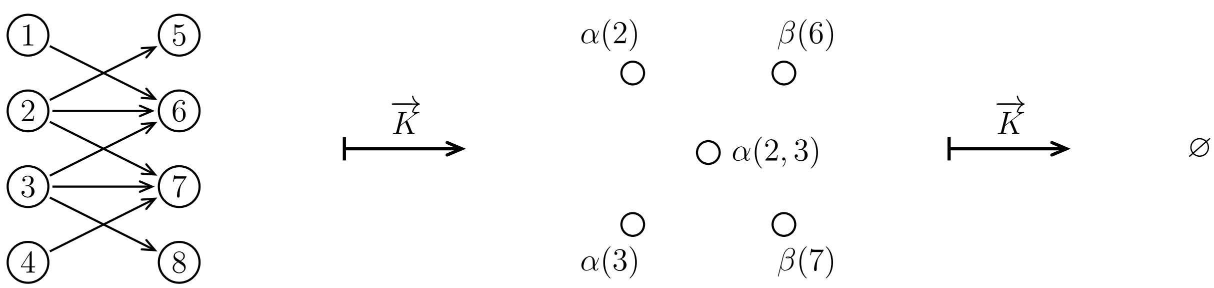

Figure 2: Examples of dicliques and diclique digraphs.

To illustrate all of these consider the digraph in Figure 2: All the dicliques of the first digraph are: $\alpha(2) = (\{2\}, \{5, 6, 7\})$, $\alpha(3) = (\{3\}, \{6, 7, 8\})$, $\beta(6) = (\{1, 2, 3\}, \{6\})$, $\beta(7) = (\{2, 3, 4\}, \{7\})$ and $\alpha(\{2, 3\}) = (\{2, 3\}, \{6, 7\})$. Note that $\alpha(\{2, 3\})$ is the only diclique not coming from a singleton. For any other $Z \subseteq V$, we either have a non-simplex (like $\alpha(6) = \alpha(\{4, 6\}) = (V, \varnothing)$ and $\beta(3) = \beta(\{1, 3, 7\}) = (\varnothing, V)$) or a diclique that has been previously considered (like $\alpha(\{1, 3\}) = \beta(6)$ or $\alpha(4) = \beta(7)$).

We say that a digraph D is a *Neumann-Lara digraph*, in brief, an *NL-digraph* if every diclique of D is $\alpha(x)$ or $\beta(x)$ for some $x \in V(D)$. The following properties are straightforward from the definitions.

Remark 5. *Let D be a digraph and $x, y \in V(D)$, then*

1. *$\alpha(x)$ is a diclique if and only if $N^+(x) \neq \varnothing$,*
2. *$\beta(x)$ is a diclique if and only if $N^-(x) \neq \varnothing$,*
3. *$\alpha(x) = \alpha(y)$ if and only if $N^+(x) = N^+(y)$,*
4. *$\beta(x) = \beta(y)$ if and only if $N^-(x) = N^-(y)$ and*
5. *$\alpha(x) = \beta(y)$ if and only if $x \to y$ and $\alpha(x) = (N^-(y), N^+(x))$.*

Whenever the minimum out-degree and the minimum in-degree of a digraph are not zero, thanks to claims (1) and (2) in the previous remark, $\alpha(x)$ and $\beta(x)$ are always dicliques for all $x \in V(D)$. In this case, α and β are morphisms of digraphs from D to $\overrightarrow{K}(D)$, according to claim (1) in the following theorem:

Theorem 6. *Let D be a digraph, $x, y \in V(D)$ and $\alpha(x)$, $\alpha(y)$, $\beta(x)$ and $\beta(y)$ dicliques of D.*

1. *If $x \to y$, then $\alpha(x) \to \alpha(y)$ and $\beta(x) \to \beta(y)$.*
2. *If D is a tournament, then $x \to y$ if and only if $\alpha(x) \to \alpha(y)$ (resp. $\beta(x) \to \beta(y)$).*
3. *If $x \not\to y$ but $\alpha(x) \to \alpha(y)$ (resp. $\beta(x) \to \beta(y)$), then there exists $z \in N^+(x)$ with $N^+(y) \subseteq N^+(z)$ (resp. $z \in N^-(y)$ with $N^-(x) \subseteq N^-(z)$).*

Proof. If $x \to y$, then clearly, $y \in N^+(x)$ and $x \in N^-(y)$. Hence, $\alpha(x) \to \alpha(y)$ and $\beta(x) \to \beta(y)$. For the second claim, suppose that D is a tournament, if $x = y$ the claim is obvious (as D is a simple digraph and hence, does not contain loops), otherwise assume $x \neq y$ and either $\alpha(x) \to \alpha(y)$ or $\beta(x) \to \beta(y)$. For a contradiction, assume further that $x \not\to y$, then $y \to x$. Thus $\alpha(y) \to \alpha(x)$ and $\beta(y) \to \beta(x)$, which is a contradiction (since the diclique digraph of a simple digraph is again simple). For the third claim, suppose that $\alpha(x) \to \alpha(y)$. Then $N^+(x) \cap N^-(N^+(y)) \neq \varnothing$. Thus, there exists $z \in N^+(x) \cap N^-(N^+(y))$. Since $z \in N^-(N^+(y))$, we have that $N^+(y) \subseteq N^+(z)$. We proceed analogously for $\beta(x) \to \beta(y)$. □

Observe that the converse of claim 1 of the above theorem does not hold in general, for instance, see Figure 3 (note that $\alpha(5) = \alpha(3)$).

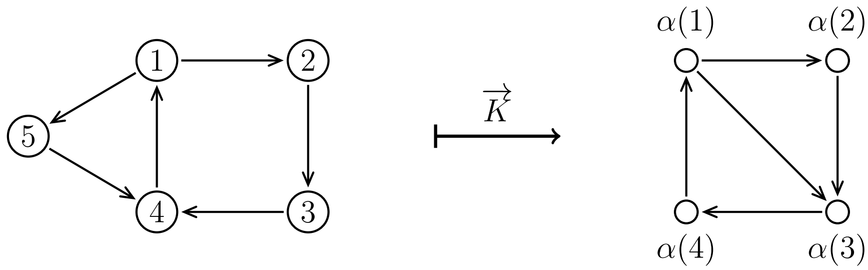

Figure 3: A diclique digraph.

4 Surjectivity, subdigraphs, walks, paths and cycles

It was proved in [13] that there exist graphs which are not the clique graph of any graph, in other words, K is not a surjective operator. In addition, the biclique operator is also not surjective (see [12]). In our case the diclique operator is indeed surjective. Prisner proved this for the class of (not necessarily simple) digraphs. We show here that every simple digraph is the diclique digraph of some simple NL-digraph. Recall that a simple digraph D is an NL-digraph if every diclique of D is $\alpha(x)$ or $\beta(x)$ for some $x \in V(D)$. We need to reproduce a sketch of Prisner's proof to adapt it for our purposes.

First, recall that if D is a digraph, the *line digraph* of D, denoted by $L(D)$, is the digraph whose vertex set is $A(D)$ and its arc set consists of $(x, y) \to (z, w)$ whenever $(x, y), (z, w) \in A(D)$ and $y = z$.

We define D' to be the digraph obtained from D by adding two new vertices s and t and all the arcs $s \to x$ and $y \to t$ for every former source x and every former sink y of D.

Theorem 7 (Prisner [21])**.** *The diclique operator is surjective for the class of (not necessarily simple) digraphs.*

Proof. (Sketch) Let D be a digraph and $\tilde{D}$ the digraph obtained from D by deleting every source and sink of D. It is proved that $\overrightarrow{K}(L(D)) = \tilde{D}$. The surjectivity follows from the definition of the digraph D'. We obtain that $D = \overrightarrow{K}(L(D'))$. □

Theorem 8. *Let D be a digraph. Then $L(D)$ is a NL-digraph.*

Proof. Let w be a vertex of D neither a source nor a sink. Let $A^+(w)$ (resp. $A^-(w)$) be the set of arcs of D starting (resp. ending) on w. Observe that the pairs $(A^-(w), A^+(w))$ (for w as stated), are exactly the dicliques of $L(D)$. If $a \in A^-(w)$, then $(A^-(w), A^+(w)) = \alpha(a)$. □

By the proof of Theorem 7, $D = \overrightarrow{K}(L(D'))$, by Theorem 8, $L(D')$ is a NL-digraph and by Theorem 1, $L(D')$ is simple if and only if D is simple, hence:

Corollary 9. *Let D be a simple digraph. Then there exists a simple NL-digraph H such that $\overrightarrow{K}(H) = D$. Moreover, $K(L(D')) = D$.*

As with clique and biclique graphs, the diclique operator does not preserve induced subdigraphs. Let D be the digraph of Figure 4 and let H be the subdigraph induced by $\{1, 3, 4, 5\}$. Observe that $\overrightarrow{K}(H)$ is not an induced subdigraph of $\overrightarrow{K}(D)$. However, the following result is valid.

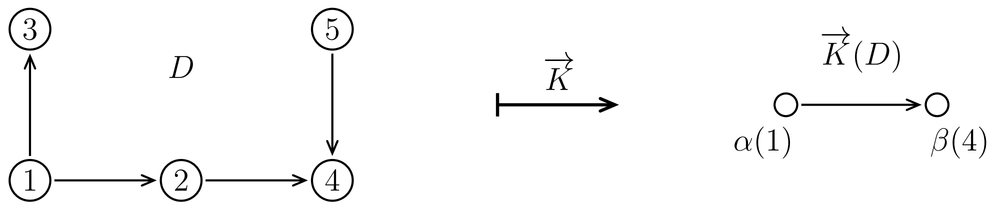

Figure 4: The diclique operator does not preserve induced subgraphs.

Theorem 10. *Let H be an induced subdigraph of a digraph D. Then $\overrightarrow{K}(H)$ is a subdigraph of $\overrightarrow{K}(D)$ (not induced in general).*

Proof. Let (X, Y) be a diclique of H. Clearly, it is a disimplex of D. Thus there exists a diclique (X', Y') of D such that $(X, Y) \preceq (X', Y')$. Since H is an induced subdigraph of D, two different dicliques of H can not be included in the same diclique of D. Furthermore, if $(X, Y) \to (Z, W)$ is an arc of $\overrightarrow{K}(H)$ and (X', Y') and (Z', W') are dicliques of D such that $(X, Y) \preceq (X', Y')$ and $(Z, W) \preceq (Z', W')$, then $(X', Y') \to (Z', W')$ is an arc of $\overrightarrow{K}(D)$. $\square$

It is well-known that the image under the clique operator of a complete graph K_n is a vertex, which is a complete graph ($K(K_n) = K_1$). For digraphs without symmetric arcs, the notion of tournament is analogous to the concept of a complete graph. Note that if D is a tournament, $\overrightarrow{K}(D)$ is not necessary a tournament. For instance, see the tournament on Figure 5.

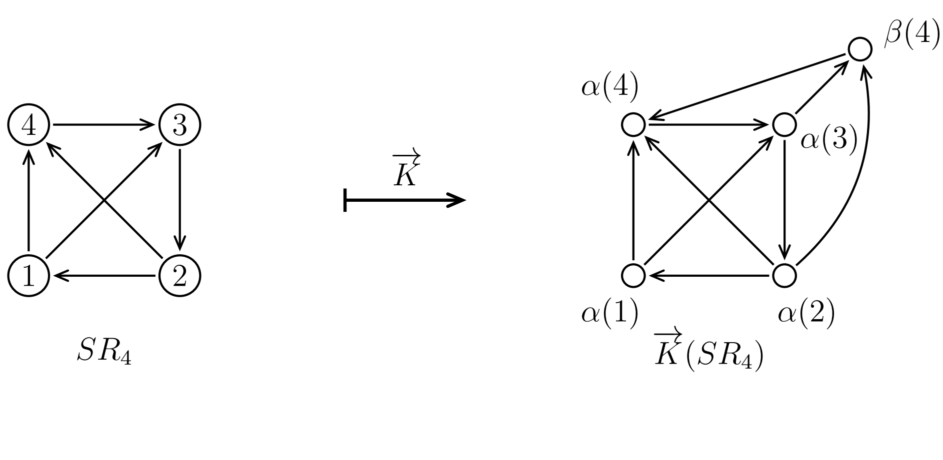

Figure 5: The image of a tournament.

However, if T is a transitive tournament then $\overrightarrow{K}(T)$ is a transitive tournament (we consider the empty digraph to be transitive). First, we study the behavior of the diclique operator for transitive digraphs.

Theorem 11. *If D is a transitive digraph, then $\overrightarrow{K}(D)$ is a transitive digraph.*

Proof. Let (X, Y), (Z, W) and (T, U) be vertices of $\overrightarrow{K}(D)$ such that $(X, Y) \to (Z, W)$ and $(Z, W) \to (T, U)$, that is, $Y \cap Z \neq \varnothing$ and $W \cap T \neq \varnothing$. Let y be a vertex in $Y \cap Z$, then for every $x \in X$ and $w \in W$ we get a directed path $x \to y \to w$ (since (X, Y), (Z, W) are dicliques of D). Since D is a transitive digraph, it follows that $x \to w$ for every $x \in X$ and $w \in W$. Clearly $(X, Y \cup W)$ is a disimplex of D. By the maximality of (X, Y), we get that $Y \cup W \subseteq Y$. Thus, $W \subseteq Y$ and it follows that $Y \cap T \supseteq W \cap T \neq \varnothing$. Hence, there exists an arc $(X, Y) \to (T, U)$ of $\overrightarrow{K}(D)$. We conclude that $\overrightarrow{K}(D)$ is a transitive digraph. □

Note that the converse of the above theorem is not true. For instance, consider the non-transitive digraph D in Figure 4 and its transitive image.

We denote by TT_n the transitive tournament on n vertices. We assume its vertices $V(TT_n) = \{1, 2, \ldots, n\}$ to be ordered from the source to the sink. Note that the empty digraph is the transitive tournament $TT_0 = \varnothing$.

Theorem 12. *For $n \geq 1$ we have $\overrightarrow{K}(TT_n) \cong TT_{n-1}$. Also $\overrightarrow{K}(TT_0) = \varnothing = TT_0$.*

Proof. Let $I_k = \{1, \ldots, k\}$. Clearly, the dicliques of TT_n are $\alpha(k) = (I_k, \overline{I_k})$ for $k \in \{1, \ldots, n-1\}$. By Theorem 6, we obtain that $\overrightarrow{K}(TT_n) = TT_{n-1}$. □

Since every arc is included in some diclique, walks can be *transported* back and forth between D and $\overrightarrow{K}(D)$: Whenever you have a walk $x_0 \to x_1 \to \cdots \to x_r$ of length r in D, you can get a walk in $(A_0, B_0) \to (A_1, B_1) \to \cdots \to (A_{r-1}, B_{r-1})$ of length $r-1$ in $\overrightarrow{K}(D)$ with $x_i \to x_{i+1}$ included in (A_i, B_i) and hence with $x_i \in B_{i-1} \cap A_i$ for $i = 1, 2, \ldots, r-1$ and $x_0 \in A_0$, $x_r \in B_{r-1}$. Conversely, starting with a walk $(A_0, B_0) \to (A_1, B_1) \to \cdots \to (A_r, B_r)$ of length r in $\overrightarrow{K}(D)$ you can get a walk $x_0 \to x_1 \to \cdots \to x_{r-1}$ of length $r-1$ in D, by taking $x_i \in B_i \cap A_{i+1}$. This latter walk can always be extended with two extra vertices $x_{-1} \in A_0$ and $x_r \in B_r$ to a walk $x_{-1} \to x_0 \to \cdots \to x_r$ of length $r+1$.

Closed walks like $x_0 \to x_1 \to \cdots \to x_r = x_0$ and $(A_0, B_0) \to (A_1, B_1) \to \cdots \to (A_r, B_r) = (A_0, B_0)$ can also be transported back and forth in the same way, but this time, the length of the transported closed walk is the same as the original one.

We emphasize that path and cycles can not in general be transported readily in this way since two of the dicliques (or vertices) in the transported path or cycle could end up being equal, and hence the transported "path" becomes a walk. It is easy to construct examples showing this for all cases. For instance, an explicit example can be seen in Figure 6 where the path $1 \to 2 \to \cdots \to 8$ gets transported to the walk $\alpha(1) \to \alpha(2) \to \cdots \to \alpha(7)$ which is not a path, since $\alpha(2) = \alpha(6)$. For this to happen, it is necessary that the original path has chords and hence chordless paths and cycles can indeed be transported cleanly into (not necessarily chordless) paths and cycles. Recall that any walk (closed walk) contains a path (cycle), and that any walk in an acyclic digraph is a path.

With all these observations in mind, it is straight forward to prove the following theorem.

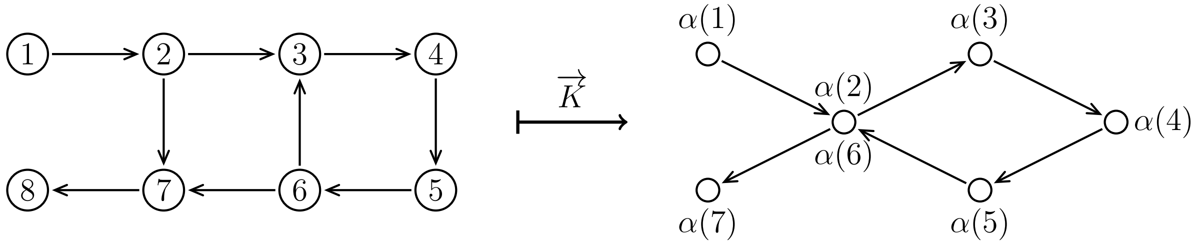

Figure 6: A path transported to a non-path

Theorem 13. *Let D be a digraph.*

1. *If there is an (x, y)-path in D of length r, then there is a $((X, W), (Z, Y))$-path in $\overrightarrow{K}(D)$ of length $s \leq r - 1$ for some dicliques (X, W), (Z, Y) with $x \in X$ and $y \in Y$.*

2. *If there is a $((X, W), (Z, Y))$-path in $\overrightarrow{K}(D)$ of length s, then there is an (x, y)-path of length $r \leq s + 1$ in D for any $x \in X$ and $y \in Y$.*

3. *Chordless paths and cycles get transported into paths and cycles.*

4. *If D is strongly connected, then $\overrightarrow{K}(D)$ is strongly connected and*

$$\operatorname{diam}(D) - 1 \leq \operatorname{diam}(\overrightarrow{K}(D)) \leq \operatorname{diam}(D) + 1.$$

5. *$\overrightarrow{g}(D) = \overrightarrow{g}(\overrightarrow{K}(D))$. In particular, D is acyclic if and only if $\overrightarrow{K}(D)$ is acyclic.*

6. *If D is acyclic with $A(D) \neq \varnothing$, then $h(\overrightarrow{K}(D)) = h(D) - 1$. In particular, this is true when D is transitive.* □

Observe that a graph G is a connected graph if and only if $K(G)$ is also a connected graph. The analogous concept for digraphs is the strong connectivity. By Theorem 13.4 the strong connectivity of D implies the strong connectivity of $\overrightarrow{K}(D)$, but the converse is not true as the example in Figure 7 shows. Also, Figure 2 illustrates that $\overrightarrow{K}$ does not preserve (weak) connectivity.

In [15], it was proved that the diameter of every connected graph G satisfies the inequality analogous to that in Theorem 13.4, that is, $\operatorname{diam}(G) - 1 \leq \operatorname{diam}(K(G)) \leq \operatorname{diam}(G) + 1$. Notice that the directed cycle $\overrightarrow{C}_n$ and $\overrightarrow{K}(\overrightarrow{C}_n)$ have the same diameter, since $\overrightarrow{C}_n \cong \overrightarrow{K}(\overrightarrow{C}_n)$. Also, Figure 3 shows a digraph D, where $\operatorname{diam}(D) = 4$ and $\operatorname{diam}(\overrightarrow{K}(D)) = 3$; Figure 5 shows a digraph with $\operatorname{diam}(D) = 3$ and $\operatorname{diam}(\overrightarrow{K}(D)) = 4$. Furthermore, Figure 7 shows a case in which $\operatorname{diam}(D) = \infty$ and $\operatorname{diam}(\overrightarrow{K}(D)) = 2$, while Figure 2 shows a case with $\operatorname{diam}(D) = \infty = \operatorname{diam}(\overrightarrow{K}(D))$ and $\operatorname{diam}(\overrightarrow{K}^2(D)) = 0$.

On the other hand, the girth is invariant under $\overrightarrow{K}$ (Theorem 13.5). However, large cycles can indeed disappear quite dramatically as the following theorem shows. We assume here

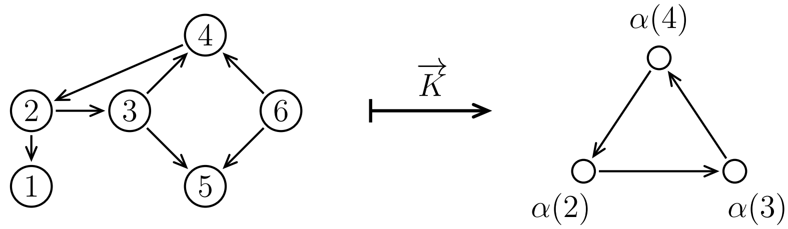

Figure 7: A non-strongly connected digraph with a strongly connected diclique digraph.

that the reader is familiar with basic group theory terminology. The *circulant* $\overrightarrow{C}_n(J)$ is the digraph whose vertices are $\mathbb{Z}_n$ and $i \to j$ if and only if $j - i \in J \subseteq \mathbb{Z}_n$. In our case, we only consider simple digraphs and hence we must have $0 \notin J$ to avoid loops, and $\left|J \cap \{i, -i\}\right| \leq 1$ for every $i \in \mathbb{Z}_n$ to avoid symmetric arcs.

We define $A + m = \{a + m \pmod n) : a \in A\}$ with $A \subseteq \mathbb{Z}_n$ and $m \in \mathbb{Z}_n$. Similarly, $A + B = \{a + b \pmod n) : a \in A, b \in B\}$ with $A, B \subseteq \mathbb{Z}_n$.

Theorem 14. *Let $n = rs$ with $r, s \geq 3$. Consider the subgroup $J_0 = r\mathbb{Z}_n$ of $\mathbb{Z}_n$ and a set $J = J_0 + m$. Then the diclique digraph of the circulant $\overrightarrow{C}_n(J)$ is $\overrightarrow{K}(\overrightarrow{C}_n(J)) = \overrightarrow{C}_r(\{m\})$. Furthermore, if $(m, r) = 1$, we have $\overrightarrow{K}(\overrightarrow{C}_n(J)) \cong \overrightarrow{C}_r$.*

Proof. Without loss of generality, we can take $m \in \{0, 1, \ldots, r-1\}$. Note that, since J_0 is a group, we have $J_0 \pm J_0 = J_0$, $J - J = J_0$ and $J + J_0 = J$.

Let (X, Y) be a diclique of $\overrightarrow{C}_n(J)$. For any $i \in X$ we have $Y \subseteq N^+(i) = i + J$. Also, for any $j \in Y \subseteq i + J$ we have $X \subseteq N^-(j) = j - J \subseteq (i + J) - J = i + J_0$. Since $(i + J_0) + J = i + J$ we have that $(i + J_0, i + J)$ is a disimplex. By the maximality of (X, Y) we have that $(X, Y) = (i + J_0, i + J)$ for some i. Hence, all the dicliques are of this form and i can be taken to be $i \in \{0, 1, \ldots, r-1\}$.

Now if we have an arc $(i + J_0, i + J) \to (i' + J_0, i' + J)$, then $(i + J) \cap (i' + J_0) \neq \varnothing$. It follows that $i' \in i + J_0 + m$ and $i' \equiv i + m \pmod r$. Hence $\overrightarrow{K}(C_n(J)) \cong \overrightarrow{C}_r(m)$. When $(m, r) = 1$, m is a unit of $\mathbb{Z}_r$ and we have $\overrightarrow{K}(\overrightarrow{C}_n(J)) \cong \overrightarrow{C}_r$. □

We remark that the previous theorem states that there exists an infinite family of $\overrightarrow{K}$-convergent circulant digraphs (which are strongly connected). Another infinite family of convergent non strongly connected digraphs are showed in Theorem 18. Particularly, they are self-diclique digraphs (those digraphs D such that $D \cong \overrightarrow{K}(D)$).

In the case of the biclique operator, almost every graph G is KB- divergent [8], while the number of KB-convergent graphs with no false twins vertices is finite (recall that two vertices $u, v \in V(G)$ are *false twins* if $N(u) = N(v)$, for more details, see [9]). For the clique operator, it is an open problem whether the convergence to the trivial graph is decidable. By contrast, our recent theorems allow us to decide when a digraph D converges to the empty digraph (i.e. D is $\overrightarrow{K}$-mortal) under the diclique operator.

Theorem 15. *A digraph D converges to $\varnothing$ if and only if it is acyclic.*

Proof. If D is acyclic, Theorem 13.6 says that for some n, $h(\overrightarrow{K}^n(D)) = 0$. Hence, $\overrightarrow{K}^n(D)$ does not contain arcs and $\overrightarrow{K}^{n+1}(D) = \varnothing$. If D contains cycles, then $g(D) < \infty = g(\varnothing)$, and the girth is an invariant thanks to Theorem 13.5. $\square$

5 Diclique divergence

Clique divergence in graphs has been widely studied in the last few decades from the seminal paper [19] by Neumann Lara through many others, see for instance [16–18]. This problem is very difficult and far from being solved. On the other hand, when concerns to the biclique operator, it was proven that a graph G with at least 7 bicliques is KB-divergent [10] (Theorem 4.3).

Recall that a digraph is $\overrightarrow{K}$*-divergent* if $\lim_{n\to\infty} |\overrightarrow{K}^n(D)| = \infty$. Also, recall that $TT_0 = \varnothing$. For $n \geq 1$, we say 1 is the *first* vertex of TT_n and that n is the *last* vertex of TT_n.

Let D be a digraph with vertex set $\{v_i : i \in \{1, \dots, n\}\}$ and consider a family of digraphs $\{F_i : i \in \{1, \dots, n\}\}$, which we assume to be vertex-disjoint (i.e. renaming vertices if necessary). The *lexicographic product* (or *X-join*) $D[F_1, F_2, \dots, F_n]$ is a digraph with vertex set $\bigcup_{i=1}^n V(F_i)$ and arc set

$$\left(\bigcup_{i=1}^n A(F_i)\right) \cup \{(f_i, f_j) : f_i \in V(F_i), f_j \in V(F_j), (v_i, v_j) \in A(D)\}.$$

Note that no ambiguities arise in this definition in the case when some of the F_i are the empty digraph.

Let $r \geq 1$, and let $s_0, s_1, \dots, s_r$ and $n_1, n_2, \dots, n_r$ be two sequences of integers satisfying $s_i \geq 0$ and $n_i \geq 3$ for all i. We define $T = T(s_0, n_1, s_1, \dots, n_r, s_r)$ as the following lexicographic product (see Fig. 8):

$$T(s_0, n_1, s_1, \dots, n_r, s_r) = TT_{2r+1}[TT_{s_0}, \overrightarrow{C}_{n_1}, TT_{s_1}, \dots, \overrightarrow{C}_{n_r}, TT_{s_r}].$$

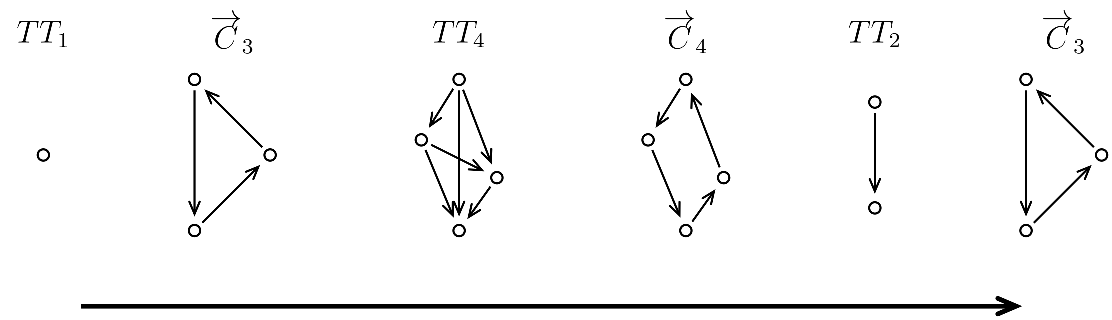

Figure 8: $T = T(1, 3, 4, 4, 2, 3, 0)$. All arcs between any two vertices from different components go from left to right as suggested by the bold arc.

For $i \in \{0, 1, \dots, 2r\}$, we denote by X_i the vertex sets of the digraphs involved in the definition of $T(s_0, n_1, s_1, \dots, n_r, s_r)$, that is $X_0 = V(TT_{s_0}), X_1 = V(\overrightarrow{C}_{n_1}), X_2 = V(TT_{s_1}), X_3 =$

$V(\overrightarrow{C}_{n_2}), \ldots, X_{2r-1} = V(\overrightarrow{C}_{n_r}), X_{2r} = V(TT_{s_r})$. Note that the X_i corresponding to the cycles are exactly the odd ones: X_{2i-1} for $i \in \{1, 2, \ldots, r\}$. Then we characterize the dicliques of T (see also Fig. 9):

Theorem 16. *The dicliques of* $T = T(s_0, n_1, s_1, \ldots, n_r, s_r)$ *are:*

$$\mathcal{D}(T) = \{\alpha(x) : x \in V(T), N^+(x) \neq \varnothing\} \cup \{\alpha(X_{2i-1}) : N^+(X_{2i-1}) \neq \varnothing, i \in \{1, 2, \ldots, r\}\}.$$

Proof. By Theorem 3, all the dicliques of T are of the form $\alpha(X)$ for some $X \subseteq V(T)$ with $X \neq \varnothing \neq N^+(X)$. We show that these sets X can be taken to be either singletons $X = \{x\}$ for some $x \in V(T)$ or $X = X_{2i-1} = V(\overrightarrow{C}_{n_i})$ for some $i \in \{1, 2, \ldots, r\}$.

We define a partial order on the vertices of T as follows:

$$x \preccurlyeq y \iff N^+(x) \subseteq N^+(y).$$

Let $X \subseteq V(T)$, and assume there exists a minimum element $m = \min_{\preccurlyeq}(X)$ of X under $\preccurlyeq$. Then we have $N^+(X) = \cap_{x \in X} N^+(x) = N^+(m)$, and hence $\alpha(X) = \alpha(m)$. It follows that X can be replaced by the singleton $\{m\}$ in this case. Two of these kind of dicliques are represented in Fig. 9 (a) and (b).

Assume next, that $X \subseteq V(T)$, $X \neq \varnothing$, does not have a minimum element. Then it has at least two minimal elements x, y. By construction, this may only happen in T when both x and y belong to the same cycle, i.e. when $x, y \in X_{2i-1} = V(\overrightarrow{C}_{n_i})$ for some i. Also by construction, in this case we must have $N^+(X) = N^+(\{x, y\}) = N^+(X_{2i-1})$. It follows that X may be replaced by X_{2i-1} in this case. A diclique of this type is represented in Fig. 9 (c). □

Theorem 17. $\overrightarrow{K}(T(s_0, n_1, s_1, \ldots, n_r, s_r)) \cong T(s_0, n_1, s_1{+}1, n_2, s_2{+}1, \ldots, n_{r-1}, s_{r-1}{+}1, n_r, s_r)$.

Proof. By Theorem 16, the dicliques of T are (see also Fig. 9):

$$\mathcal{D}(T) = \{\alpha(x) : x \in V(T), N^+(x) \neq \varnothing\} \cup \{\alpha(X_{2i-1}) : N^+(X_{2i-1}) \neq \varnothing, i \in \{1, 2, \ldots, r\}\}.$$

It should be clear by the definition of T that for any two different $X, Y \in \{\{x\} : x \in V(T)\} \cup \{X_1, X_3, \ldots, X_{2r-1}\}$, we have $N^+(X) \neq N^+(Y)$. Also, exactly one of those sets has the property $N^+(X) = \varnothing$, and this one does not generate a diclique, since, for such X, $\alpha(X) = (V, \varnothing) \notin \mathcal{D}(T)$. It follows that $|\mathcal{D}(T)| = |T| + r - 1$.

Now, if we take the subdigraph W of $\overrightarrow{K}(T)$ induced by $\{\alpha(x) : x \in V(T)\}$, we claim that

$$W \cong \begin{cases} T(s_0, n_1, s_1, \ldots, n_r, s_r) & \text{if} \quad s_r = 0 \\ T(s_0, n_1, s_1, \ldots, n_r, s_r - 1) & \text{if} \quad s_r \neq 0 \end{cases}$$

Indeed, every $x \in V(T)$ generates a unique diclique $\alpha(x)$, except that, in the case $s_r \neq 0$, the last vertex w of the last TT_{s_r}, has $N^+(w) = \varnothing$ and hence it does not generate a diclique due to Remark 5. By Theorem 6, it follows that $x \to y$ implies $\alpha(x) \to \alpha(y)$. The converse is not true in general, but in T a new edge $\alpha(x) \to \alpha(y)$ with $x \not\to y$, could only happen when $y \not\to x$ (otherwise, we would have $\alpha(x) \leftrightarrow \alpha(y)$, a contradiction) and this implies that x, y belong to the same cycle $x, y \in X_i$ for some i. But a direct inspection of such dicliques

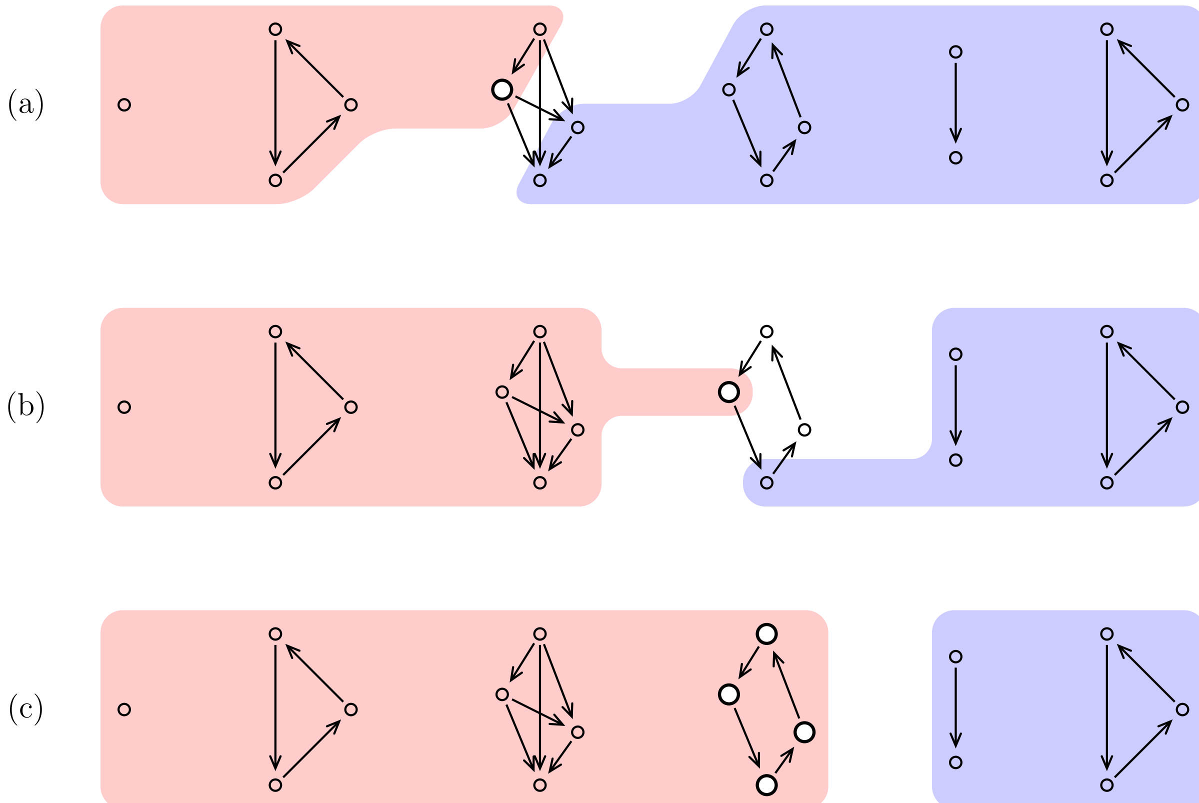

Figure 9: Some dicliques, $\alpha(X)$, of $T = T(1,3,4,4,2,3,0)$. The vertices in X, which generate the diclique, are highlighted. The dicliques (A,B) are shown by grouping the vertices of A and B in red and blue respectively.

show that $\alpha(x) \to \alpha(y)$ if and only if $x \to y$. It follows that W is isomorphic to an induced subgraph of T, namely $W \cong T$ when $s_r = 0$ or $W \cong T - \{w\} = T(s_0, n_1, s_1, \ldots, n_r, s_r - 1)$ when $s_r \neq 0$ as claimed.

There are some additional dicliques of the form $\alpha(X_{2i-1})$ for $i \in \{1, 2, \ldots, r\}$. These dicliques are $\alpha(X_{2i-1}) = (A_i, B_i)$, with $A_i = \cup_{j=0}^{2i-1} X_j$ and $B_i = \cup_{j=2i}^{2r} X_j$. Note that in the case $s_r = 0$, $N^+(X_{2r-1}) = \varnothing$, and hence $\alpha(X_{2r-1}) = (V, \varnothing)$ is not a diclique. It follows that we have either r or $r-1$ additional dicliques, depending on whether $s_r \neq 0$ or $s_r = 0$.

Now take one of these additional dicliques $\alpha(X_{2i-1})$ and note that in $\overrightarrow{K}(T)$, we have $\alpha(X_{2i-1}) \to \alpha(X_{2j-1})$ for all $j > i$, $\alpha(X_{2j-1}) \to \alpha(X_{2i-1})$ for all $j < i$, $\alpha(X_{2i-1}) \to \alpha(x)$ for all $x \in X_j$, $j > 2i-1$, and $\alpha(x) \to \alpha(X_{2i-1})$ for all $x \in X_j$, $j \leq 2i-1$.

It follows that, in the case $s_r \neq 0$, each $X_{2i-1} = V(\overrightarrow{C}_{n_i})$ generates one vertex $\alpha(X_{2i-1})$ in $\overrightarrow{K}(T)$ which extends the TT_{s_i} induced by $\{\alpha(x) : x \in X_{2i} = V(TT_{s_i})\}$ in $\overrightarrow{K}(T)$ by adding a new source to it, and thus producing a TT_{s_i+1}. Hence, when adding these additional vertices to W, we transform it from $W \cong T(s_0, n_1, s_1, n_2, s_2, \ldots, n_{r-1}, s_{r-1}, n_r, s_r - 1)$ to $\overrightarrow{K}(T) \cong T(s_0, n_1, (s_1+1), n_2, (s_2+1), \ldots, n_{r-1}, (s_{r-1}+1), n_r, s_r)$.

The only difference in the case $s_r = 0$ is that $\alpha(X_{2r-1})$ is not a diclique, thus the corresponding additional vertex is not present and hence the corresponding $TT_{s_r} = TT_0$ is not extended. But in this case, we have $W \cong T(s_0, n_1, s_1, n_2, s_2, \ldots, n_{r-1}, s_{r-1}, n_r, s_r)$, so the additional vertices transform W into $\overrightarrow{K}(T) \cong T(s_0, n_1, (s_1+1), n_2, (s_2+1), \ldots, n_{r-1}, (s_{r-1}+1), n_r, s_r)$, exactly as before. $\square$

Theorem 18. *$T = T(s_0, n_1, s_1, \ldots, n_r, s_r)$ is self-diclique for $r = 1$ and $\overrightarrow{K}$-divergent for $r \geq 2$. Its growth rate is linear and given by $|\overrightarrow{K}^n(T)| = |T| + n(r-1)$.*

Proof. Immediate from Theorem 17. $\square$

We already mentioned in the introduction that this family of $\overrightarrow{K}$-divergent digraphs, is novel in the way that its digraphs are simple, not strongly connected and grow linearly. They are also NL-digraphs, contrary to Prisner's family. Whenever $\alpha(X_{2i-1})$ is a diclique, $X_{2i} = V(TT_{s_i})$ is not empty and hence, it has a first element $w_i = \max_{\preccurlyeq}(X_{2i})$. It follows that $\alpha(X_{2i-1}) = \beta(w_i)$ and that $T(s_0, n_1, s_1, \ldots, n_r, s_r)$ is a NL-digraph. Hence, the digraphs in this family are the first known divergent ones which are NL-digraphs.

6 Categorical product and polynomial growth

Polynomial growth under the clique operator was studied in [16]. In this section, we present an infinite family of digraphs with a polynomial growth under the diclique operator. We need to introduce a new tool for this goal.

The *tensor product* (or *categorical product*) $D \times H$ of two digraphs D and H is defined by $V(D \times H) = V(D) \times V(H)$ and $(d, h) \to (d', h')$ is an arc of $D \times H$ if and only if $d \to d'$ in D and $h \to h'$ in H (Figure 10). We point out that both projections $\pi_1(d, h) = d$ and $\pi_2(d, h) = h$ are morphisms. Figure 10 illustrates this definition for $\overrightarrow{P}_3 \times \overrightarrow{P}_3$ and $\overrightarrow{P}_2 \times \overrightarrow{P}_2$ (here $\overrightarrow{P}_n$ denotes a directed path on n vertices). Note that $\overrightarrow{K}(\overrightarrow{P}_3 \times \overrightarrow{P}_3) \cong \overrightarrow{K}(\overrightarrow{P}_3) \times \overrightarrow{K}(\overrightarrow{P}_3)$. This is not just an isolated coincidence as the following theorem shows.

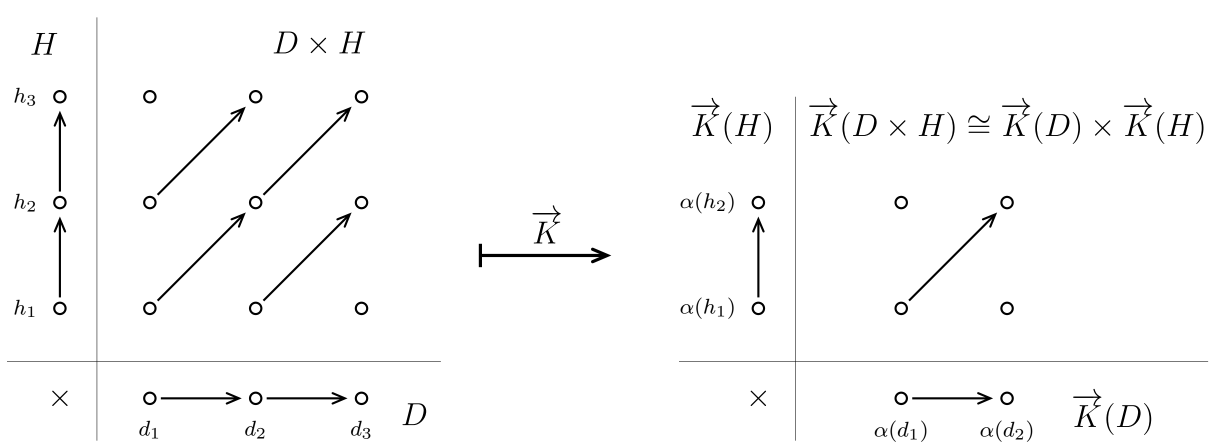

Figure 10: Tensor product and the diclique operator.

Theorem 19. $\overrightarrow{K}(D \times H) \cong \overrightarrow{K}(D) \times \overrightarrow{K}(H)$.

Proof. Notice that all the vertices of $\overrightarrow{K}(D) \times \overrightarrow{K}(H)$ are of the form $(\,(X_D, Y_D), (X_H, Y_H)\,)$ with $(X_D, Y_D) \in \overrightarrow{K}(D)$ and $(X_H, Y_H) \in \overrightarrow{K}(H)$. We claim that the vertices of $\overrightarrow{K}(D \times H)$ are exactly the pairs $(X_D \times X_H, Y_D \times Y_H)$ with (X_D, Y_D) and (X_H, Y_H) as defined before (see Figure 11).

In order to prove the claim, let us show first that all the pairs of the form $(X_D \times X_H, Y_D \times Y_H)$ with $(X_D, Y_D) \in \overrightarrow{K}(D)$ and $(X_H, Y_H) \in \overrightarrow{K}(H)$ are dicliques of $D \times H$ (see Figure 10). Consider $(x_D, x_H) \in X_D \times X_H$ and $(y_D, y_H) \in Y_D \times Y_H$. These are vertices of $D \times H$. The respective projections of these vertices induce arcs of D and H, namely $x_D y_D \in A(D)$ and $x_H y_H \in A(H)$ (since (X_D, Y_D) and (X_H, Y_H) are dicliques of D and H, respectively). It follows that $((x_D, x_H), (y_D, y_H)) \in A(D \times H)$ and hence $(X_D \times X_H, Y_D \times Y_H)$ is a disimplex of $D \times H$. Now suppose that there is a vertex $(z_D, z_H) \in D \times H$ that could extend the disimplex $(X_D \times X_H, Y_D \times Y_H)$. Without loss of generality (by duality), we may assume that the extended disimplex is $(X_D \times X_H, Y_D \times Y_H \cup \{(z_D, z_H)\})$. Since $(z_D, z_H) \notin Y_D \times Y_H$, it follows that either $z_D \notin Y_D$ or $z_H \notin Y_H$, let us assume the former without loss of generality again. Therefore, by projection, we have that $(x_D, z_D) \in A(D)$ for all $x_D \in X_D$ and the diclique (X_D, Y_D) could be extended to $(X_D, Y_D \cup \{z_D\})$, a contradiction. We conclude that $(X_D \times X_H, Y_D \times Y_H)$ is maximal and hence a diclique of $D \times H$.

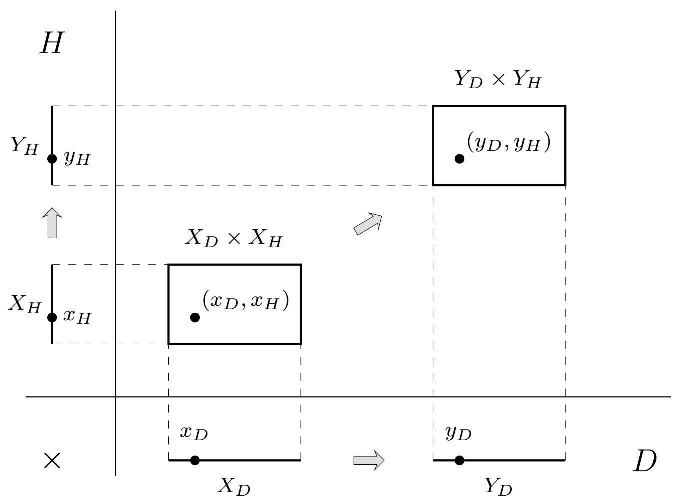

Figure 11: Dicliques of a tensor product.

Conversely, we show now that every clique is of the form $(X_D \times X_H, Y_D \times Y_H)$ with $(X_D, Y_D) \in \overrightarrow{K}(D)$ and $(X_H, Y_H) \in \overrightarrow{K}(H)$. Consider a diclique (X, Y) of $D \times H$. The respective projections $(X_D, Y_D) := (\pi_1(X), \pi_1(Y))$ and $(X_H, Y_H) := (\pi_2(X), \pi_2(Y))$ must be disimplices of D and H respectively (since morphisms preserve arcs). In particular, we must have $X_D \cap Y_D \neq \varnothing$ and $X_H \cap Y_H \neq \varnothing$. Now $(X, Y) \preceq (X_D \times X_H, Y_D \times Y_H)$ implies $(X, Y) = (X_D \times X_H, Y_D \times Y_H)$ since (X, Y) is maximal. Also observe that if any of these disimplices could be extended in any way (say $(X_D, Y_D \cup \{d\})$), then the original diclique could also be extended (i.e. $(X_D \times X_H, (Y_D \cup \{d\}) \times Y_H)$) contrary to the maximality of (X, Y). It follows that (X_D, Y_D) and (X_H, Y_H) are dicliques of D and H respectively.

We conclude that the vertices of $\overrightarrow{K}(D \times H)$ are exactly all the pairs $(X_D \times X_H, Y_D \times Y_H)$ with $(X_D, Y_D) \in \overrightarrow{K}(D)$ and $(X_H, Y_H) \in \overrightarrow{K}(H)$.

Now, the required isomorphism $\varphi : \overrightarrow{K}(D) \times \overrightarrow{K}(H) \to \overrightarrow{K}(D \times H)$ is given by:

$$\varphi(((X_D, Y_D), (X_H, Y_H))) = (X_D \times X_H, Y_D \times Y_H).$$

Indeed, we have that:

$$\begin{aligned}
&((X_D, Y_D), (X_H, Y_H)) \to ((X'_D, Y'_D), (X'_H, Y'_H)) \\
&\Leftrightarrow (X_D, Y_D) \to (X'_D, Y'_D) \text{ and } (X_H, Y_H) \to (X'_H, Y'_H) \\
&\Leftrightarrow Y_D \cap X'_D \neq \varnothing \text{ and } Y_H \cap X'_H \neq \varnothing \\
&\Leftrightarrow (Y_D \times Y_H) \cap (X'_D \times X'_H) \neq \varnothing \\
&\Leftrightarrow (X_D \times X_H, Y_D \times Y_H) \to (X'_D \times X'_H, Y'_D \times Y'_H) \\
&\Leftrightarrow \varphi(((X_D, Y_D), (X_H, Y_H))) \to \varphi(((X'_D, Y'_D), (X'_H, Y'_H))).
\end{aligned}$$

The morphism φ is clearly bijective and hence an isomorphism. □

The existence of digraphs with linear growth rate under the diclique operator was established in Theorem 18 where a parameterized family of examples $T(s_0, n_1, s_1, \ldots, n_r, s_r)$ was given. We provide here another (minimal) example of a digraph of this type.

For $m \geq 0$, let us define W_m as the digraph obtained from TT_{m+6} by simply reversing the arcs $(1, 3)$ and $(m + 4, m + 6)$, that is:

$$A(W_m) = \{(3, 1), (m+6, m+4)\} \cup \{(x, y) : x < y, (x, y) \neq (1, 3) \text{ and } (x, y) \neq (m+4, m+6)\}.$$

We claim that W_0 has a linear growth rate under $\overrightarrow{K}$:

Theorem 20. *$\overrightarrow{K}(W_m) \cong W_{m+1}$. In particular, $|\overrightarrow{K}^n(W_0)| = |W_n| = n + 6 = \Theta(n)$.*

Proof. It is straightforward to verify that the dicliques of W_m are exactly $\alpha(\{1, 2, 3\}) = \beta(4)$ and $\alpha(x)$ for $x \in W_m$ and that the required isomorphism $\varphi : W_{m+1} \to \overrightarrow{K}(W_m)$ is given by:

$$\varphi(x) = \begin{cases} \alpha_{W_m}(x) & \text{for } x < 4, \\ \beta_{W_m}(4) & \text{for } x = 4, \\ \alpha_{W_m}(x - 1) & \text{for } x > 4. \end{cases}$$

□

Theorem 21. *For any integer $m \geq 0$ there is a digraph D with m-th degree polynomial growth rate under the diclique operator, that is, $|\overrightarrow{K}^n(D)| = \Theta(n^m)$.*

Proof. Consider:

$$W_0^m = \underbrace{W_0 \times W_0 \times \cdots \times W_0}_{m \text{ times}}.$$

By Theorems 19 and 20, we have that

$$|\overrightarrow{K}^n(W_0^m)| = |(\overrightarrow{K}^n(W_0))^m| = |\overrightarrow{K}^n(W_0)|^m = (n + 6)^m = \Theta(n^m).$$

□

7 Open problems

There are only two known families of $\overrightarrow{K}$-divergent digraphs, our family that has a linear growth rate, and Prisner's family that has a super-exponential growth rate. Also, all known periodic digraphs are self-diclique. Several natural open questions arise here. First we recall open problem 39 posed in [21]:

Problem 1. *Are there periodic digraphs with period greater than one? That is, is there a digraph D with $\overrightarrow{K}^p(D) \cong D \ncong \overrightarrow{K}(D)$?*

Concerning the study of periodic graphs under the clique operator, F. Escalante determined the existence of p-periodic graphs under K for every $p \geq 1$ (more details are found in [16]). On the other hand, M. Groshaus and L. Montero proved that a graph G either diverges or converges under the biclique operator to the complete graphs K_1 or K_3 in at most three steps [10]. The stated problem for periodic digraphs is even open for $p = 2$.

Problem 2. *Are there digraphs with exponential growth rate? That is, is there a digraph D with $|\overrightarrow{K}^m(D)| = \Theta(a^m)$ for some $a > 1$?*

Problem 3. *Can self-diclique digraphs be characterized?*

Problem 4. *Is $\overrightarrow{K}$-divergence decidable?*

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