Dismantlings and Iterated Clique Graphs

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Abstract

Given a graph G and two vertices $x, y \in V(G)$, we say that x is dominated by y if the closed neighbourhood of x is contained in that of y. Here we prove that if x is a dominated vertex, then G and $G - \{x\}$ have the same dynamical behaviour under the iteration of the clique operator.

Key words: clique graphs, iterated clique graphs, dismantlings, clique behaviour

1 Introduction and Terminology

All our graphs are finite, simple and loopless. We shall identify induced subgraphs with their vertex sets, in particular, we shall write $x \in G$ instead of $x \in V(G)$. Given $x \in G$, the *closed neighbourhood* $N_G[x]$ of x is the set consisting of x and all its neighbours. Given $x, y \in G$ we say that *x is dominated by y* (in G) if $N_G[x] \subseteq N_G[y]$. Note that every vertex is dominated by itself, however we say that *x is dominated* (without specifying who is y) only when x

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is dominated by a different vertex. Given two graphs G and H we say that G is *dismantleable* to H if there is a sequence of graphs $G_0, G_1, \ldots, G_r$ satisfying $G = G_0$, $H \cong G_r$ and $G_{i+1} = G_i - \{x_i\}$ where x_i is a dominated vertex of G_i.

A *clique* of G is a maximal complete subgraph. The *clique graph* $k(G)$ of G is the intersection graph of all cliques of G: every clique is a vertex, two of them being adjacent iff they share at least one vertex. Similarly, $c(G)$ is the intersection graph of all complete subgraphs of G. Clearly, $k(G)$ is an induced subgraph of $c(G)$. We define inductively the *iterated clique graphs* by the formulas $k^0(G) = G$ and $k^{n+1}(G) = k(k^n(G))$. Iterated clique graphs have been studied in several papers, for a large bibliography see [10,11]. It is known (and easy to prove) that a graph G is either $k-$*divergent* (i.e. $\lim_{n\to\infty} |k^n(G)| = \infty$) or $k-$*stationary* (i.e. $k^n(G) \cong k^m(G)$ for some $n < m$). A special case of a $k-$stationary graph is a $k-$*null* graph: for some n, $k^n(G)$ is isomorphic to the one vertex graph K_1. We say that two graphs G and H have the same $k-$*behaviour* if both are $k-$divergent or both are $k-$stationary and both are $k-$null or both are not $k-$null.

Given two graphs G and H, we say that H is a *retract* of G if there are two weak morphism of graphs (images of adjacent vertices are adjacent or equal) $\alpha : H \longrightarrow G$ and $\beta : G \longrightarrow H$ such that $\beta \circ \alpha$ is the identity in H.

Since whenever G is dismantleable to H, we have that H is a retract of G, Neumann-Lara's retraction theorem [7,8] tells us that if H is $k-$divergent then so is G. Also, Prisner proved [9] that if G is dismantleable to K_1 then G is $k-$null. Our main Theorem (Thm. 5) states a stronger result: If G is dismantleable to H then G and H have the same $k-$behaviour.

A special kind of dismantlings will play a key rôle in what follows:

Definition 1 *Let G and H be graphs, we say that $G \xrightarrow{\#} H$ if H is isomorphic to an induced subgraph H_0 of G such that every vertex x in G is dominated by some (not necessarily different) vertex y in H_0.*

It is straightforward to verify that $G \xrightarrow{\#} H$ implies that G is dismantleable to H. Also G is dismantleable to H iff there is a sequence of graphs satisfying $G \xrightarrow{\#} G_0 \xrightarrow{\#} G_1 \xrightarrow{\#} \cdots \xrightarrow{\#} G_r = H$. Note that $c(G) \xrightarrow{\#} k(G)$ for every graph G.

2 Dismantlings and $k-$behaviour

Lemma 2 *Assume H_0 is an induced subgraph of G satisfying that every vertex in G is dominated by some vertex in H_0. Let $Q_1, Q_2 \in k(G)$ (not necessarily different), then $Q_1 \cap Q_2 \neq \varnothing$ iff $Q_1 \cap Q_2 \cap H_0 \neq \varnothing$.*

PROOF. Take $Q_1, Q_2 \in k(G)$ and $x \in Q_1 \cap Q_2$, as x is dominated by some $y \in H_0$ (possibly $y = x$) it follows that $Q_1 \cup Q_2 \subseteq N_G[x] \subseteq N_G[y]$, therefore $Q_1 \cap Q_2 \cap H_0 \supseteq \{y\} \neq \varnothing$. $\square$

Theorem 3 *If* $G \xrightarrow{\#} H$, *then* $k(G) \xrightarrow{\#} k(H)$.

PROOF. Let $H_0 \cong H$ be a induced subgraph of G such that every vertex in G is dominated by some vertex in H_0. For each clique $Q \in k(H_0)$ select a fixed clique $f(Q) \in k(G)$ satisfying $Q \subseteq f(Q)$. Obviously $Q = f(Q) \cap H_0$, so we know f to be injective. Now Lemma 2 tells us that $Q_1, Q_2 \in k(H_0)$ are adjacent iff $f(Q_1)$ and $f(Q_2)$ are adjacent (in $k(G)$). It follows that $k(H) \cong k(H_0) \cong f(k(H_0))$, where $f(k(H_0))$ is the subgraph of $k(G)$ induced by $\{f(Q) : Q \in k(H_0)\}$.

Finally, if $Q \in k(G)$ let $Q_0 \in k(H_0)$ satisfying $Q \cap H_0 \subseteq Q_0$. We claim that Q is dominated by $f(Q_0)$: By Lemma 2 for every $Q_1 \in k(G)$ we have $Q_1 \cap Q \neq \varnothing$ iff $Q_1 \cap Q \cap H_0 \neq \varnothing$, but $Q_1 \cap Q \cap H_0 \subseteq Q_1 \cap Q_0 \subseteq Q_1 \cap f(Q_0)$. $\square$

Theorem 4 *If* $G \xrightarrow{\#} H$ *then* $kc(H) \xrightarrow{\#} k^2(G)$.

PROOF. Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_r\} \in k^2(G)$. We know by Lemma 2 that $\{Q_1 \cap H_0, \ldots, Q_r \cap H_0\}$ is a set of pairwise intersecting completes of H_0. Then for every clique $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_r\} \in k^2(G)$ select a fixed clique $f(\mathcal{Q}) \in kc(H_0)$ satisfying $f(\mathcal{Q}) \supseteq \{Q_1 \cap H_0, \ldots, Q_r \cap H_0\}$. We claim that f is an isomorphism onto its image and that every vertex in $kc(H_0)$ is dominated by a vertex in $f(k^2(G))$.

Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_r\}, \mathcal{P} = \{P_1, P_2, \ldots, P_s\} \in k^2(G)$. If $f(\mathcal{Q}) = f(\mathcal{P})$ we have $Q_i \cap H_0 \in f(\mathcal{P})$ for all $i = 1, \ldots, r$, since $f(\mathcal{P})$ is a clique, we have $Q_i \cap H_0 \cap P_j \neq \varnothing$ for all i and j. Then $Q_i \cap P_j \neq \varnothing$ for all i and j. It follows that $\mathcal{Q} = \mathcal{P}$ and therefore f is injective.

Obviously f preserves adjacencies. If $f(\mathcal{Q})$ is adjacent to $f(\mathcal{P})$ for some $\mathcal{Q}, \mathcal{P} \in k^2(G)$, let $C_0 \in f(\mathcal{Q}) \cap f(\mathcal{P})$ and and let Q_0 be any clique in $k(G)$ containing C_0. Then $Q_0 \in \mathcal{Q} \cap \mathcal{P}$ and therefore $\mathcal{Q}$ and $\mathcal{P}$ are adjacent in $k^2(G)$. Thus f is an isomorphism onto its image.

Now take $\mathcal{Q} = \{C_1, \ldots, C_r\} \in kc(H_0)$. Let $\{Q_1, Q_2, \ldots, Q_r\}$ be a set of cliques of G such that $C_i \subseteq Q_i$. Let $\mathcal{Q}_0 \in k^2(G)$ such that $\{Q_1, \ldots, Q_r\} \subseteq \mathcal{Q}_0$. We claim that $f(\mathcal{Q}_0)$ dominates $\mathcal{Q}$: If $\mathcal{P} \in kc(H_0)$ is adjacent (or equal!) to $\mathcal{Q}$, without loss, assume $C_1 \in \mathcal{Q} \cap \mathcal{P}$. Now $Q_1 \cap H_0 \in \mathcal{Q} \cap \mathcal{P}$ since every complete of H_0 intersecting C_1 also intersects $Q_1 \cap H_0 \supseteq C_1$. It follows that $\mathcal{P}$ is also adjacent to $f(\mathcal{Q}_0)$. $\square$

Theorem 5 *If G is dismantleable to H, G and H have the same $k-$behaviour. In particular, if x is a dominated vertex of G, G and $G - \{x\}$ have the same $k-$behaviour.*

PROOF. Obviously, we only have to prove this in the case $G \overset{\#}{\to} H$.

If H is $k-$null we have $k^n(G) \overset{\#}{\to} k^n(H) \cong K_1$ for some n, but then $k^n(G)$ must be a cone (must have a universal vertex), then $k^{n+2}(G) \cong K_1$. On the other hand, if G is $k-$null we have $K_1 \cong k^n(G) \overset{\#}{\to} k^n(H)$ which implies $k^n(H) \cong K_1$.

If H is $k-$divergent, then $k^n(G) \overset{\#}{\to} k^n(H)$ implies $|k^n(G)| \geq |k^n(H)|$ and therefore G is also $k-$divergent. Now, let us assume H to be $k-$stationary, hence $k^n(H) \cong k^{n+m}(H)$ for some $n \geq 0, m \geq 1$. Using Theorem 3 we know that $k^{n+mj}(G) \overset{\#}{\to} k^{n+mj}(H) \cong k^n(H)$ for all j. Then Theorem 4 gives us $kck^n(H) \overset{\#}{\to} k^{n+mj+2}(G)$ for all j. Since any finite graph may only be dismantleable to a finite number of (non-isomorphic) graphs, it follows that $k^{n+mj+2}(G) \cong k^{n+mi+2}(G)$ for some $i < j$. Thus, G is also $k-$stationary. □

If $k^t(G) \cong k^{t+p}(G)$ for some minimum $p \geq 1$ and some $t \geq 0$, we say that p is the *period* of G (we set $p = \infty$ for $k-$divergent graphs). The previous theorem tells us that the finiteness of p is invariant under dismantlings, we shall show now that p itself is not. Consider the graph R obtained from Fig. 1 identifying the following pairs of vertices: $\{a, a'\}$, $\{b, b'\}$ and $\{c, c'\}$.

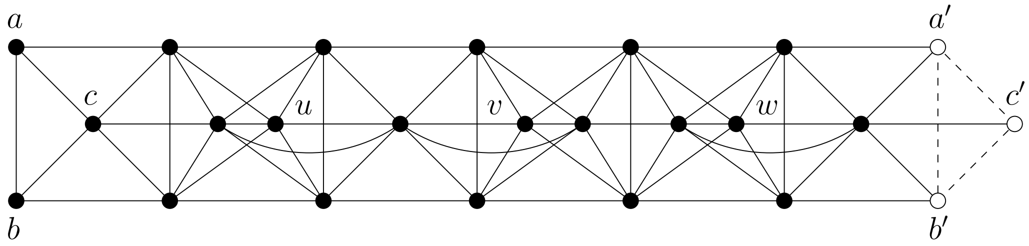

Fig. 1. A clockwork graph with period 3.

It has three dominated vertices: u, v and w. The period of R is 3, but $R - \{u\}$ and $R - \{v\}$ have periods 6 and 1 respectively. You may check this either by computer (we used GAP [2]) or by applying the theory of clockwork graphs developed in [4]. Clockwork graphs have been successfully used to construct examples in [5] (see also [6]) and others. Precursors of clockwork graphs were also used to construct examples in [1] and [3].

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