ON EXPANSIVE GRAPHS

F. LARRIÓN, V. NEUMANN-LARA, AND M.A. PIZAÑA

1. Introduction

This article presents results on expansive graphs that Víctor Neumann-Lara obtained in the late 1970's and reported mostly without proofs in [8] and [9]. The untimely death of Víctor left us to complete this project, which had started in [7]. We had at hand [8, 9] and the manuscript [10] that, using his old notebooks, Víctor wrote with one of us in 1995. The original material has been thoroughly rewritten and recast in the setting of [7]. This allowed for a clearer and more natural presentation. Some shortcuts were found, and applications added.

Our graphs are finite, simple and non-empty. We identify induced subgraphs and vertex sets. The *clique graph* $K(G)$ of a graph G is the intersection graph of its *cliques* (maximal complete subgraphs, or just maximal *completes*). The *iterated clique graphs* $K^n(G)$ are defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. We refer to [12, 4, 13] for the literature on iterated clique graphs.

In the study of the dynamics of the clique operator K, two types of *K-behaviour* stand out: G is *clique convergent* if $K^n(G) \cong K^m(G)$ for some pair $n < m$, and G is *clique divergent* if $|V(K^n(G))|$ tends to infinity with n (iff this sequence is unbounded). A graph is clique divergent if and only if it is not clique convergent.

In this paper we study *expansivity*, a stronger notion than clique divergence. Expansivity works for *coaffine graphs*. These are graphs with a special kind of symmetry: a fixed automorphism (a *coaffination*) that maps each vertex out of its closed neighbourhood, as for instance the antipodal maps of the octahedron and the icosahedron. If A and B are coaffine graphs, adding to their disjoint union all possible edges from A to B we obtain their *Zykov sum* $A + B$, which is coaffine with the union of the coaffinations of A and B.

We will show that, from the additive viewpoint, the great majority of coaffine graphs are expansive: any Zykov sum of at least 3 coaffine summands is expansive. A *coaffine subgraph* of a coaffine graph is any subgraph (induced or not) which is invariant under the coaffination. If A is an expansive coaffine subgraph of B, then B is expansive. Thus, a coaffine graph does not need to be a Zykov sum to be expansive: it is enough that it has a complete tripartite coaffine subgraph. Furthermore, a complete bipartite coaffine subgraph will suffice if one of the parts induces a connected subgraph: In fact, if G and H are coaffine and H is connected, then $G + H$ is expansive. Moreover, it is not even necessary to contain Zykov sums in order to be expansive: interesting examples include complements and powers of cycles; indeed, the K-behaviour of these complements and powers is completely characterized in this work. A further interesting consequence of the theory is that, save possibly for one, every connected graph each of whose neighbourhoods is either a square or a pentagon is K-divergent. We also show that every graph is an induced subgraph of some expansive graph.

2. Preliminaries

We review here some results and terminology from [7]. An *automorphic graph* is a pair $\mathbb{A} = (A, \alpha)$ where A is a graph and $\alpha \in \mathrm{Aut}(A)$. We say that $\mathbb{A}$ is *r-coaffine*, and that α is an *r-coaffination* of A, if $d(x, \alpha(x)) \geq r$ for all $x \in A$.

Given graphs A, B we say that $f : A \to B$ is a *graph relation* if f is a vertex relation $f \subseteq V(A) \times V(B)$ and the image of any complete of A is a complete of B. Equivalently, $f \subseteq V(A) \times V(B)$ is a graph relation if images of vertices and edges are always complete (non-empty in particular). Note that *graph morphisms* (vertex functions where images of adjacent vertices are adjacent or equal) are particular cases of graph relations.

For any graph relation $f : A \to B$ there is a graph relation $f_K : K(A) \to K(B)$ given by $f_K(Q) = \{Q' \in K(B) : f(Q) \subseteq Q'\}$ for all $Q \in K(A)$. In general f_K is not a graph morphism even if f is so, but when f is an automorphism f_K is also an automorphism. We define the clique operator for automorphic graphs by $K(\mathbb{A}) = K(A, \alpha) = (K(A), \alpha_K)$. If $\mathbb{A}$ is r-coaffine, then so is $K(\mathbb{A})$.

An *admissible relation* between two automorphic graphs $f : \mathbb{A} \to \mathbb{B}$ is a graph relation $f : A \to B$ satisfying $f \circ \alpha = \beta \circ f$. In particular, if $\mathbb{A}$ is an r-coaffine subgraph (induced or not) of $\mathbb{B}$, the inclusion map is an admissible morphism. Any composition of admissible relations is admissible. The clique operator does not preserve compositions, but it preserves admissibility: if $f : \mathbb{A} \to \mathbb{B}$ is admissible, then $f_K : K(\mathbb{A}) \to K(\mathbb{B})$ is also admissible.

From now on, all our automorphic graphs will be assumed to be r-coaffine for some fixed $r \geq 2$. We will be often interested in the existence of an admissible relation between two r-coaffine graphs, and seldom in the specific relation or its name. Thus, for the sake of brevity, we shall write "$\mathbb{B} \leftarrow \mathbb{A}$" instead of "there is an admissible relation $f : \mathbb{A} \to \mathbb{B}$". For instance, we have already mentioned that:

2.1. Lemma. [7] $\mathbb{C} \leftarrow \mathbb{B}$ *and* $\mathbb{B} \leftarrow \mathbb{A}$ *imply* $\mathbb{C} \leftarrow \mathbb{A}$.

2.2. Lemma. [7] $\mathbb{B} \leftarrow \mathbb{A}$ *implies* $K(\mathbb{B}) \leftarrow K(\mathbb{A})$.

The *rank* of $\mathbb{A}$ is the greatest integer n such that there exist non-empty, pairwise disjoint, α-invariant sets $A_1, \ldots, A_n \subseteq V(A)$ such that $d_A(a_i, a_j) < r$ whenever $a_i \in A_i$, $a_j \in A_j$ and $i \neq j$. The same concept is obtained if we ask the sets A_i to be α-orbits. Note that $\mathrm{rank}(\mathbb{A})$ depends on both r and $\mathbb{A}$.

2.3. Theorem. [7] $\mathbb{B} \leftarrow \mathbb{A}$ *implies* $\mathrm{rank}(\mathbb{B}) \geq \mathrm{rank}(\mathbb{A})$.

We say that $\mathbb{A}$ is *rank divergent* if the sequence $\{\mathrm{rank}(K^n(\mathbb{A}))\}$ is not bounded. Since $\mathrm{rank}(\mathbb{A}) < |V(A)|$, we have that every rank divergent graph is clique divergent. Note that if some $K^m(\mathbb{A})$ is rank divergent, then $\mathbb{A}$ itself is rank divergent.

The following result, which immediately follows from 2.2 and 2.3, will play an important role in this work:

2.4. Theorem. [7] *If* $\mathbb{A}$ *is rank divergent and* $\mathbb{B} \leftarrow \mathbb{A}$*, then* $\mathbb{B}$ *is rank divergent.*

From now on we shall focus on the case $r = 2$, so we can simplify our terms: a *coaffination* is just a 2-coaffination, and a *coaffine* graph is a 2-coaffine graph. When $r = 2$ we say that a rank divergent graph is *expansive*. A *coaffinable graph* is one that admits some coaffination. We will assume that all our automorphic graphs $\mathbb{A}, \mathbb{B}, \ldots$ are coaffine. Of course, $\mathbb{A}$ and $\mathbb{B}$ are *isomorphic* if there is an admissible isomorphism $f : \mathbb{A} \to \mathbb{B}$.

3. Zykov Sum and Circle Product

The *Zykov sum* $A+B$ is the disjoint union $A \cup B$ plus all edges between A and B. The *circle product* $A \circ B$ is defined by $V(A \circ B) = V(A) \times V(B)$ and $(a,b) \simeq (a',b')$ iff $a \simeq a'$ or $b \simeq b'$. Here, "$x \simeq y$" means "x is adjacent or equal to y". For coaffine graphs, define $\mathbb{A} + \mathbb{B} = (A+B, \alpha \cup \beta)$ and $\mathbb{A} \circ \mathbb{B} = (A \circ B, \alpha \times \beta)$, which are also coaffine. Up to isomorphism, these operations are commutative, associative and satisfy the distributive law: $\mathbb{A} \circ (\mathbb{B} + \mathbb{C}) \cong \mathbb{A} \circ \mathbb{B} + \mathbb{A} \circ \mathbb{C}$. It is easily seen that:

3.1. Lemma. $\operatorname{rank}(\mathbb{A}_1 + \cdots + \mathbb{A}_s) = \operatorname{rank}(\mathbb{A}_1) + \cdots + \operatorname{rank}(\mathbb{A}_s) \geq s$. □

It is immediate that if $f : \mathbb{A} \to \mathbb{X}$ and $g : \mathbb{B} \to \mathbb{Y}$ are admissible relations, then $f \cup g : \mathbb{A} + \mathbb{B} \to \mathbb{X} + \mathbb{Y}$ and $f \times g : \mathbb{A} \circ \mathbb{B} \to \mathbb{X} \circ \mathbb{Y}$ are also admissible. Hence:

3.2. Lemma. $\mathbb{X} \leftarrow \mathbb{A}$ *and* $\mathbb{Y} \leftarrow \mathbb{B}$ *imply* $\begin{cases} (1) & \mathbb{X} + \mathbb{Y} \leftarrow \mathbb{A} + \mathbb{B}. \\ (2) & \mathbb{X} \circ \mathbb{Y} \leftarrow \mathbb{A} \circ \mathbb{B}. \end{cases}$ □

3.3. Proposition. (1) $K(\mathbb{A} + \mathbb{B}) \leftarrow K(\mathbb{A}) \circ K(\mathbb{B})$.
(2) $K(\mathbb{A} \circ \mathbb{B}) \leftarrow K(\mathbb{A}) + K(\mathbb{B})$.

Proof: Even admissible morphisms exist: the first is $(Q_A, Q_B) \mapsto Q_A \cup Q_B$, the second $Q_A \mapsto Q_A \times V(B)$ and $Q_B \mapsto V(A) \times Q_B$. If $Q_A \times V(B)$ were not a clique of $A \circ B$, then B would be a cone ($N_B[v] = B$ for some $v \in B$) and $\mathbb{B}$ would not be coaffine. The rest of the proof is straightforward. □

In fact, $K(\mathbb{A}+\mathbb{B}) \cong K(\mathbb{A}) \circ K(\mathbb{B})$. The morphism given above is an isomorphism, but we shall not use this fact. The other morphism is usually not surjective.

By a *polynomial* we shall mean a nonzero polynomial $P = P(x_1, \ldots, x_m)$ with non-negative integer coefficients and zero constant term. Any polynomial P can be written as a sum $P = \Sigma_{i=1}^{s} m_i$ of a positive number $s = s(P)$ of possibly repeated *monomials* m_i, each being just a product of variables. This essentially unique expression we call the *normal form*. For coaffine $\mathbb{A}_1, \ldots, \mathbb{A}_m$ we obtain the coaffine graph $P(\mathbb{A}_1, \ldots, \mathbb{A}_m)$ by *evaluation*, i.e. replacing x_i with $\mathbb{A}_i$ in P and interpreting all sums as Zykov sums and all products as circle products. Up to isomorphism, $P(\mathbb{A}_1, \ldots, \mathbb{A}_m)$ depends only on P and the $\mathbb{A}_i$'s, not on the way P is written.

If $P(x_1, \ldots, x_m)$ is a polynomial, $P^*(x_1, \ldots, x_m)$ will denote the polynomial resulting from the normal form of P upon interchange of $\cdot$ and $+$. When iterating the star operator we can put $P^{*1} = P^*$, $P^{*2} = P^{**}$, $P^{*3} = P^{***}$, etc.

3.4. Proposition. *Let $P(x_1, \ldots, x_m)$ be a polynomial and $\mathbb{A}_1, \ldots, \mathbb{A}_m$ coaffine graphs. Then $K(P(\mathbb{A}_1, \ldots, \mathbb{A}_m)) \leftarrow P^*(K(\mathbb{A}_1), \ldots, K(\mathbb{A}_m))$. In general, for $n \geq 1$, we have that $K^n(P(\mathbb{A}_1, \ldots, \mathbb{A}_m)) \leftarrow P^{*n}(K^n(\mathbb{A}_1), \ldots, K^n(\mathbb{A}_m))$.*

Proof: For $n = 1$, use 3.3.2 for monomials, and then 3.3.1 and 3.2.2. Inductive step: Apply 2.2 to the induction hypothesis, then use the base case and 2.1. □

If P is a polynomial with $s_1 = s(P)$ monomials, each of degree s_0, the normal form of P^* is a polynomial with $s_2 = s_0^{s_1}$ monomials, each of degree s_1. Likewise, the normal form of P^{**} has $s_3 = s_1^{s_2}$ monomials of degree s_2. It follows that the normal form of P^{*n} has $s_{n+1} = s_{n-1}^{s_n}$ monomials of degree s_n. Therefore, if $s_0, s_1 \geq 2$, the sequence $\{s_n\}$ grows exponentially at each step. In this case, the growth rate of the sequence is not only superexponential, but even non-elementary: Let $f_1(n) = 2^n$ and $f_{k+1}(n) = 2^{f_k(n)}$; we say that a function $s : \mathbb{Z}^+ \to \mathbb{Z}^+$ grows *non-elementarily* if s grows faster than every f_k.

If P is not homogeneous, let $d_1, d_2, \ldots, d_s$ be the degrees of its monomials. The normal form of P^* is already a homogeneous polynomial with $d_1 d_2 \cdots d_s$ monomials of degree s. Therefore, if $s \geq 2$ and some $d_i \geq 2$, the number of monomials $s(P^{*n})$ also grows non-elementarily in this case.

As $K^n(P(\mathbb{A}_1, \ldots, \mathbb{A}_m)) \leftarrow P^{*n}(K^n(\mathbb{A}_1), \ldots, K^n(\mathbb{A}_m))$ by 3.4, using 2.3 and 3.1, $\operatorname{rank}(K^n(P(\mathbb{A}_1, \ldots, \mathbb{A}_m))) \geq \operatorname{rank}(P^{*n}(K^n(\mathbb{A}_1), \ldots, K^n(\mathbb{A}_m))) \geq s(P^{*n})$. Thus:

3.5. Theorem. *If P is a polynomial of degree at least two and having at least two monomials, then $P(\mathbb{A}_1, \ldots, \mathbb{A}_m)$ is expansive. In particular, $\mathbb{A} + \mathbb{B} \circ \mathbb{C}$ is expansive. Every such evaluation of P grows non-elementarily under the clique operator.* □

Denote by $\mathbb{I}_n$ the coaffine graph with $n \geq 2$ vertices, no edges and any cyclic permutation of the vertices. Any α-orbit of a coaffine $\mathbb{A}$ gives a coaffine subgraph of $\mathbb{A}$ of the form $\mathbb{I}_n$ for some n. Since we are focused on the case $r = 2$, $\operatorname{rank}(\mathbb{A})$ is clearly the greatest p for which $\mathbb{A}$ has a coaffine subgraph of the form $\mathbb{I}_{n_1} + \cdots + \mathbb{I}_{n_p}$. Of course, as a graph, $I_{n_1} + \cdots + I_{n_p}$ is just the complete multipartite graph $K_{n_1, \ldots, n_p}$.

3.6. Neumann-Lara's Three Summands Theorem.
$\mathbb{A} + \mathbb{B} + \mathbb{C}$ and $\mathbb{A} \circ \mathbb{B} \circ \mathbb{C}$ are expansive and grow non-elementarily under K.

Proof: We show first that $\mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r$ and $\mathbb{I}_p + \mathbb{I}_q + \mathbb{I}_r$ are expansive.
For $x \in I_p$, $y \in I_q$, $z \in I_r$, the following are cliques of $I_p \circ I_q \circ I_r$:

$$\begin{array}{rclrcl} Q_x & = & \{x\} \times V(I_q) \times V(I_r) & Q_{xyz} & = & V(I_p) \times \{y\} \times \{z\} \cup \\ Q_y & = & V(I_p) \times \{y\} \times V(I_r) & & & \{x\} \times V(I_q) \times \{z\} \cup \\ Q_z & = & V(I_p) \times V(I_q) \times \{z\} & & & \{x\} \times \{y\} \times V(I_r). \end{array}$$

Now $x \mapsto Q_x$, $y \mapsto Q_y$, $z \mapsto Q_z$, $(x, y, z) \mapsto Q_{xyz}$ give an admissible morphism

$$f : \mathbb{I}_p + \mathbb{I}_q + \mathbb{I}_r + \mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r \to K(\mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r).$$

Therefore $K(\mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r)$, and hence $\mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r$, is expansive by 3.5 and 2.4.
Since $K(\mathbb{I}_n) \cong \mathbb{I}_n$, it follows from 3.3.1 that $K(\mathbb{I}_p + \mathbb{I}_q + \mathbb{I}_r) \leftarrow \mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r$.
Hence $K(\mathbb{I}_p + \mathbb{I}_q + \mathbb{I}_r)$, and therefore $\mathbb{I}_p + \mathbb{I}_q + \mathbb{I}_r$, is also expansive by 2.4.
Now recall that for any coaffine $\mathbb{X}$ we have that $\mathbb{X} \leftarrow \mathbb{I}_n$ for some $n \geq 2$.
Therefore, by 3.2, $\mathbb{A} + \mathbb{B} + \mathbb{C} \leftarrow \mathbb{I}_p + \mathbb{I}_q + \mathbb{I}_r$ and $\mathbb{A} \circ \mathbb{B} \circ \mathbb{C} \leftarrow \mathbb{I}_p \circ \mathbb{I}_q \circ \mathbb{I}_r$.
Finally, using 2.4 again, $\mathbb{A} + \mathbb{B} + \mathbb{C}$ and $\mathbb{A} \circ \mathbb{B} \circ \mathbb{C}$ are expansive. □

The following result is immediate from 3.6 and 2.4:

3.7. Theorem. *$\mathbb{X}$ is expansive if and only if $\operatorname{rank}(K^n(\mathbb{X})) \geq 3$ for some $n \geq 0$. Every expansive graph grows non-elementarily under the clique operator.* □

3.8. Neumann-Lara's Chipote Theorem.
If $K^n(\mathbb{A}) \leftarrow \mathbb{A} + \mathbb{X}$ for some $n \geq 1$, then $\mathbb{A}$ is expansive.

Proof: First use n times 2.2 to get $K^{2n}(\mathbb{A}) \leftarrow K^n(\mathbb{A} + \mathbb{X})$. By 3.3 and 2.2, $K^n(\mathbb{A} + \mathbb{X}) \leftarrow K^n(\mathbb{A}) + K^n(\mathbb{X})$ or $K^n(\mathbb{A} + \mathbb{X}) \leftarrow K^n(\mathbb{A}) \circ K^n(\mathbb{X})$ (use induction). By 3.2, $K^n(\mathbb{A}) + K^n(\mathbb{X}) \leftarrow \mathbb{A} + \mathbb{X} + K^n(\mathbb{X})$ and $K^n(\mathbb{A}) \circ K^n(\mathbb{X}) \leftarrow (\mathbb{A} + \mathbb{X}) \circ K^n(\mathbb{X})$. Therefore, by 2.1, $K^{2n}(\mathbb{A}) \leftarrow \mathbb{A} + \mathbb{X} + K^n(\mathbb{X})$ or $K^{2n}(\mathbb{A}) \leftarrow (\mathbb{A} + \mathbb{X}) \circ K^n(\mathbb{X})$. Thus $\mathbb{A}$ is expansive by 2.4 and 3.6 or 3.5. □

4. One connected summand

As said before, the morphism $K(\mathbb{A})+K(\mathbb{B}) \to K(\mathbb{A}\circ\mathbb{B})$ is usually not surjective. In fact, $K(\mathbb{A}\circ\mathbb{B})$ is as a rule quite complex. A spanning subgraph $\mathbb{A}\ltimes\mathbb{B}$ of $\mathbb{A}\circ\mathbb{B}$ affords better control. For graphs A, B define $A \ltimes B$ as the graph with vertex set $V(A)\times V(B)$ where $(a,b)\simeq(a',b')$ iff $a=a'$ and $d_B(b,b')\le 2$ OR $b\simeq b'$. We will write $A_1\ltimes A_2\ltimes\cdots\ltimes A_n$ instead of $A_1\ltimes(A_2\ltimes(\cdots(A_{n-1}\ltimes A_n)\cdots))$. For coaffine graphs we define $\mathbb{A}\ltimes\mathbb{B}=(A\ltimes B,\alpha\times\beta)$, which is also coaffine.

Let $C(G)$ be the intersection graph of all completes of G. If $\gamma\in\mathrm{Aut}(G)$, define $\gamma_C : C(G)\to C(G)$ by $\gamma_C(X)=\gamma(X)=\{\gamma(x) : x\in X\}$. If $\mathbb{G}=(G,\gamma)$ is coaffine, then $C(\mathbb{G})=(C(G),\gamma_C)$ is coaffine: Indeed, if $X\simeq\gamma_C(X)$ there is an $x\in X\cap\gamma_C(X)$, but then $x,\gamma(x)\in\gamma(X)$ and $\mathbb{G}$ is not coaffine or X is not complete.

4.1. Proposition. *$K(\mathbb{G})\leftarrow C(\mathbb{G})$.*

Proof: Define $f : C(G)\to K(G)$ by $f(X)=\{Q\in K(G) : X\subseteq Q\}$. Since $f(\gamma_C(X))=\{Q\in K(G) : \gamma(X)\subseteq Q\}=\{\gamma(Q)\in K(G) : X\subseteq Q\}=\gamma_K(f(X))$, we have that the graph relation f is admissible. $\square$

4.2. Lemma. *$K(\mathbb{I}_n\ltimes\mathbb{H})\leftarrow\mathbb{I}_n\ltimes\mathbb{H}$.*

Proof: Define $f : \mathbb{I}_n\ltimes\mathbb{H}\to C(\mathbb{I}_n\ltimes\mathbb{H})$ by $f(i,h)=(V(I_n)\times\{h\})\cup(\{i\}\times N[h])$. Then f is an admissible morphism. By 4.1 and 2.1, $K(\mathbb{I}_n\ltimes\mathbb{H})\leftarrow\mathbb{I}_n\ltimes\mathbb{H}$. $\square$

4.3. Lemma. *$K(\mathbb{I}_n\circ\mathbb{H})\leftarrow\mathbb{I}_n+\mathbb{I}_n\ltimes\mathbb{H}$.*

Proof: Define an admissible morphism $f : \mathbb{I}_n+\mathbb{I}_n\ltimes\mathbb{H}\to C(\mathbb{I}_n\circ\mathbb{H})$ putting $f(i)=\{i\}\times V(H)$ and $f(i,h)=(V(I_n)\times\{h\})\cup(\{i\}\times N[h])$. Finish as in 4.2. $\square$

For short, put $\mathbb{H}_1=\mathbb{I}_n\ltimes\mathbb{H}$, $\mathbb{H}_2=\mathbb{I}_n\ltimes\mathbb{I}_n\ltimes\mathbb{H}$, $\mathbb{H}_3=\mathbb{I}_n\ltimes\mathbb{I}_n\ltimes\mathbb{I}_n\ltimes\mathbb{H}$, etc.

4.4. Proposition. *Let $m\ge 1$. Then $K^{2m}(\mathbb{I}_n\circ\mathbb{H})\leftarrow\mathbb{I}_n\circ\mathbb{H}_m$.*

Proof: Induction on m. For the base case use first 4.3+2.2, 3.3.1, and 4.2+3.2.2: $K(K(\mathbb{I}_n\circ\mathbb{H}))\leftarrow K(\mathbb{I}_n+\mathbb{I}_n\ltimes\mathbb{H})\leftarrow\mathbb{I}_n\circ K(\mathbb{I}_n\ltimes\mathbb{H})\leftarrow\mathbb{I}_n\circ(\mathbb{I}_n\ltimes\mathbb{H})$. Then, by 2.1, $K^2(\mathbb{I}_n\circ\mathbb{H})\leftarrow\mathbb{I}_n\circ\mathbb{H}_1$. Now apply $2m$ times 2.2: $K^{2m+2}(\mathbb{I}_n\circ\mathbb{H})\leftarrow K^{2m}(\mathbb{I}_n\circ\mathbb{H}_1)$. By the inductive hypothesis, $K^{2m}(\mathbb{I}_n\circ\mathbb{H}_1)\leftarrow\mathbb{I}_n\circ(\mathbb{H}_1)_m=\mathbb{I}_n\circ\mathbb{H}_{m+1}$, and we end by applying 2.1 again. $\square$

4.5. Proposition. *If $\mathrm{diam}(\mathbb{H})\le 2^m$ then $\mathrm{rank}(K(\mathbb{H}_m))\ge 2$.*

Proof: For each $h\in H$, let $X_h=V(I_n)\times\cdots\times V(I_n)\times\{h\}\subseteq H_m$. Thus, $X_h\in C(H_m)$. Now put $\mathcal{X}=\{X_h : h\in V(H)\}\subseteq C(H_m)$. Since $\mathcal{X}$ is an invariant subset of $C(\mathbb{H}_m)$, it induces a coaffine subgraph $\mathbb{X}$ of $C(\mathbb{H}_m)$.

For $i=(i_1,\ldots,i_m)\in(V(I_n))^m$, let $Y_i=\{(i_1,\ldots,i_m)\}\times V(H)\subseteq V(H_m)$. Since $(i_s,i_{s+1},\ldots,i_m,h)\simeq(i_s,i_{s+1},\ldots,i_m,h')$ in H_{m-s+1} if $d(h,h')\le 2^{m-s+1}$, we get $Y_i\in C(H_m)$ as $\mathrm{diam}(\mathbb{H})\le 2^m$. Put $\mathcal{Y}=\{Y_i : i\in V(I_n)\times\cdots\times V(I_n)\}\subseteq C(H_m)$. Again, $\mathcal{Y}$ induces a coaffine subgraph $\mathbb{Y}$ of $C(\mathbb{H}_m)$.

Since $\mathcal{X}\cap\mathcal{Y}=\varnothing$ & $X_h\cap Y_i\ne\varnothing$ $\forall h,i$, $\mathbb{X}+\mathbb{Y}$ is a coaffine subgraph of $C(\mathbb{H}_m)$. Then $\mathrm{rank}(K(\mathbb{H}_m))\ge 2$ by 4.1, 2.1 and 3.1: $K(\mathbb{H}_m)\leftarrow C(\mathbb{H}_m)\leftarrow\mathbb{X}+\mathbb{Y}$. $\square$

4.6. Neumann-Lara's Connected Summand Theorem.
If $\mathbb{H}$ is connected, then $\mathbb{G} \circ \mathbb{H}$ and $\mathbb{G} + \mathbb{H}$ are expansive.

Proof: If $2^m \geq \mathrm{diam}(\mathbb{H})$, $K(\mathbb{H}_m) \leftarrow \mathbb{X} + \mathbb{Y}$ by 4.5. By 4.4+2.2, 3.3 and 3.2 we get $K^{2m+1}(\mathbb{I}_n \circ \mathbb{H}) \leftarrow K(\mathbb{I}_n \circ \mathbb{H}_m) \leftarrow \mathbb{I}_n + K(\mathbb{H}_m) \leftarrow \mathbb{I}_n + \mathbb{X} + \mathbb{Y}$, so $\mathbb{I}_n \circ \mathbb{H}$ is expansive by 3.6 and 2.4. By 3.3, $K(\mathbb{I}_n + \mathbb{H}) \leftarrow \mathbb{I}_n \circ K(\mathbb{H})$, so $\mathbb{I}_n + \mathbb{H}$ is expansive ($K(\mathbb{H})$ is connected). Since $\mathbb{G} \circ \mathbb{H} \leftarrow \mathbb{I}_n \circ \mathbb{H}$ & $\mathbb{G} + \mathbb{H} \leftarrow \mathbb{I}_n + \mathbb{H}$ for some n, we are done. □

4.7. Theorem. *$\mathbb{G} + \mathbb{H}$ is expansive if and only if one of $\mathbb{G}$ and $\mathbb{H}$ contains a connected coaffine subgraph.*

Proof: If $\mathbb{H}_0 \leq \mathbb{H}$ is connected, we use 3.2.1, 4.6 and 2.4: $\mathbb{G} + \mathbb{H} \leftarrow \mathbb{G} + \mathbb{H}_0$. If $\mathbb{G}$ and $\mathbb{H}$ lack connected coaffine subgraphs, their coaffinations permute their connected components leaving no fixed component. Shrinking components we get coaffine discrete quotients $\bar{\mathbb{G}}$, $\bar{\mathbb{H}}$ with $\bar{\mathbb{G}} + \bar{\mathbb{H}} \leftarrow \mathbb{G} + \mathbb{H}$. Since $\bar{\mathbb{G}} + \bar{\mathbb{H}}$ is triangleless, it is K-convergent [2] and thus not expansive. Then $\mathbb{G} + \mathbb{H}$ is not expansive. □

5. Some applications

Expanding slightly the meaning of the term, we can say that a graph G is *expansive* when there is a coaffination γ of G such that $\mathbb{G} = (G, \gamma)$ is expansive.

5.1. Theorem. *Every connected graph G of order $n > 1$ is an induced subgraph of some expansive graph A of order $2n + 2$.*

Proof: Let x, y be distinct vertices of G, and let x', y' be the the corresponding vertices in a disjoint copy G' of G. Define H as the disjoint union $G \cup G'$ plus the two edges xy', yx'. The coaffination η in H interchanges corresponding vertices of the two copies. Now let $\mathbb{A} = \mathbb{I}_2 + (H, \eta)$ and apply 4.6. □

5.2. Theorem. *$G = K_{n_1,\ldots,n_p}$ is K-divergent iff $p \geq 3$ and all $n_i \geq 2$.*

Proof: If $p = 1$, G is complete and $K(G) = K_1$. If $p = 2$, G is triangleless and so K-convergent by [2]. If some $n_i = 1$, G is a cone and $K^2(G) = K_1$. In the remaining cases there is a coaffination γ of G such that $(G, \gamma) \cong \mathbb{I}_{n_1} + \cdots + \mathbb{I}_{n_p}$ and therefore G is expansive by 3.1 and 3.7. □

5.3. Theorem. *A power of a cycle C_n^p is K-divergent iff $n/3 \leq p < \lfloor n/2 \rfloor$.*

Proof: Let $A = C_n^p$. If $0 \leq p < n/3$, $K(A) \cong A$ by [3, Lemma 1]. If $p \geq \lfloor n/2 \rfloor$, $A \cong K_n$ is K-convergent too. For $n/3 \leq p < \lfloor n/2 \rfloor$, $\mathbb{A} = (A, \alpha)$ is coaffine with $\alpha(i) = i + p + 1$. Put $P_i = i + \{0, p, 2p\}$ and $Q_i = i + \{0, 1, \ldots, p\}$. Let $\mathbb{G}$ and $\mathbb{H}$ be the coaffine subgraphs of $C(\mathbb{A})$ induced by $\{P_i\}_{i \in \mathbb{Z}_n}$ and $\{Q_i\}_{i \in \mathbb{Z}_n}$ respectively. Then $\mathbb{G} + \mathbb{H}$ is a coaffine subgraph of $C(\mathbb{A})$. Since $\mathbb{H}$ is connected, $\mathbb{G} + \mathbb{H}$ is expansive by 4.6. Then $C(\mathbb{A})$ is expansive by 2.4, and $K(\mathbb{A})$ is expansive by 4.1 and 2.4. □

5.4. Theorem. *Let $n \geq 3$. Then $\overline{C}_n$ is K-divergent if and only if $n \geq 8$.*

Proof: Let $A = \overline{C}_n$. A direct inspection shows that A is K-convergent for $n \leq 7$. Assume that $n \geq 8$ and define $\alpha : A \to A$ by $\alpha(i) = i + 1$. Then $\mathbb{A} = (A, \alpha)$ is coaffine. If $n = 2m+1$, $\overline{C}_n \cong C_n^{m-1}$ and $\mathbb{A}$ is expansive by 5.3. If n is even, proceed as in 5.3, using $P_i = i + \{0, 2, 4, 6, \ldots, n-2\}$ and $Q_i = i + \{0, 3, 5\}$. □

Our last application will be to locally cyclic graphs. If $v \in G$, let $N_G(v)$ be the subgraph of G induced by the neighbours of v. If G and H are graphs, G is said to be *locally* H if $N_G(v) \cong H$ for all $v \in G$. For a family $\mathcal{F}$ of graphs we say that G is *locally* $\mathcal{F}$ if each $N_G(v)$ is isomorphic to some graph in $\mathcal{F}$. A *locally cyclic graph* is just a locally $\{C_n : n \geq 3\}$ graph. Save for the tetrahedron K_4 (the only locally cyclic graph with a vertex of degree 3) locally cyclic graphs are precisely the 1-skeletons of the Whitney triangulations of closed surfaces: a simplicial complex is *Whitney* if its simplexes are precisely the completes of its 1-skeleton.

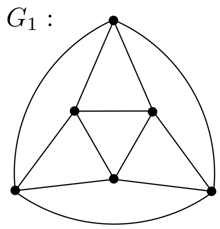
The K-behaviour of regular locally cyclic graphs is known: Given a $t \geq 4$, a locally C_t graph is K-convergent if and only if $t \geq 7$, if and only if the corresponding triangulation has negative Euler characteristic [4, 5, 6, 11]. In the non-regular case, it is also known that all locally $\{C_n : n \geq 7\}$ graphs are K-convergent [6].

Consider connected locally $\{C_4, C_5\}$ graphs G. If n is the order of such a G, clearly $n > 5$, and $n > 6$ if some vertex has degree five. Let k be the number of vertices of degree four in G, so $0 \leq k \leq n$. The Euler characteristic of the associated triangulation is $\chi = \frac{n+k}{6} > 0$, so $\chi \in \{1, 2\}$. If $\chi = 1$ we get $n + k = 6$, which is absurd. Therefore $\chi = 2$, our G triangulates the sphere, and the possible pairs (n, k) are $(6, 6), (7, 5), \ldots, (11, 1), (12, 0)$. There is exactly one locally $\{C_4, C_5\}$ graph corresponding to each of these pairs, save for the penultimate, since there is no such graph on 11 vertices. Indeed, the six easy cases to consider (in each case assume previous cases to be false) are:

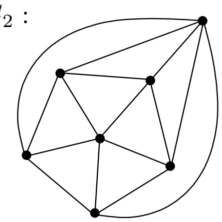
(1) There is triangle of vertices of degree 4.
(2) There is a path of length 2 of vertices of degree 4.
(3) There is an edge formed by vertices of degree 4.
(4) There are two vertices of degree 4 at distance 2.
(5) There is at least one vertex of degree 4.
(6) There are no vertices of degree 4.

Each of these cases leads naturally and uniquely to one of the following graphs:

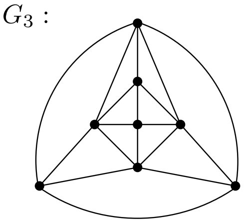
G_1 :



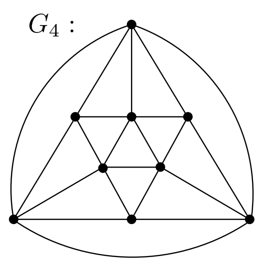
G_2 :



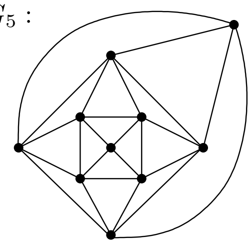
G_3 :



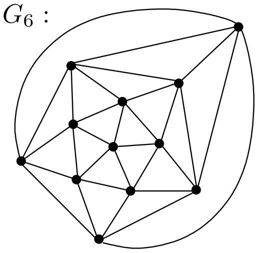
G_4 :



G_5 :



G_6 :



We devote the rest of the section to prove the following:

5.5. Theorem. *Save possibly for the snub disphenoid, each locally $\{C_4, C_5\}$ graph is K-divergent.*

The octahedron $G_1 = 3I_2 = C_4 + I_2$ is expansive by 3.6 or 4.6. The suspension of the pentagon $G_2 = C_5 + I_2$ is expansive by 4.6. The graph G_3 is the *snub disphenoid*: it is not coaffinable, but we conjecture that it is K-divergent [7]. The icosahedron G_6 is not expansive but it is a 3-coaffinable rank divergent graph, so it is K-divergent [11, 7]. Both G_4 and G_5 will turn out to be also expansive.

The graph G_4 is a triangular prism with the (vertical) rectangular faces replaced by 4-wheels. Let γ be a one-third turn about the vertical axis followed by the up-down reflection. Then $\mathbb{G}_4 = (G_4, \gamma)$ is coaffine, has two γ-orbits, and rank one. Now consider $K(\mathbb{G}_4) = (K(G_4), \gamma_K)$. Let G be the subgraph of $K(G_4)$ induced by the two horizontal triangles, and let H be induced by the six vertical triangles that meet both the top and the bottom. Then $K(\mathbb{G}_4)$ is expansive by 4.6

Finally, take C_8^2 with $V(C_8^2) = \mathbb{Z}_8$ and $i \sim j$ iff $j - i \in \{\pm1, \pm2\}$. We get G_5 adding two vertices τ, β to C_8^2 and joining τ to $0, 2, 4, 6$ and β to $1, 3, 5, 7$. Put $\gamma(\tau) = \beta$, $\gamma(\beta) = \tau$ and $\gamma(i) = i + 3$, so $\mathbb{G}_5 = (G_5, \gamma)$ is coaffine. It has two orbits and rank one, but now also $K(\mathbb{G}_5)$ has two orbits (the 8 triangles containing either τ or β and the other 8) and rank one. The cliques of $K(G_5)$ are of two kinds: for each vertex $v \in G_5$ the *star* of v is $v^* = \{Q \in K(G_5) : v \in Q\}$, and for each triangle T of G_5 the *necktie* of T is the set $\widehat{T}$ of the 4 triangles of G_5 which share at least an edge with T (the proof of [6, Prop.10] works for all locally cyclic graphs save for the tetrahedron and the octahedron). It follows easily that $K^2(\mathbb{G}_5)$ has order 26, four orbits, and rank one. A computer verification (we used GAP [1]) shows that $K^3(\mathbb{G}_5)$ has 72 vertices, 9 orbits and rank one. At the next step, $K^4(\mathbb{G}_5)$ has 450 vertices, 57 orbits and rank two. There is a coaffine subgraph $\mathbb{G} + \mathbb{H}$ of $K^4(\mathbb{G}_5)$ where $G \cong K_4 \cup K_4$ and $H \cong C_8^2$, so $K^4(\mathbb{G}_5)$ is expansive by 4.6 and 2.4. This was rather fortunate, since $K^5(G_5)$ has 265,944 vertices and would have been more difficult to analyze.

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