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ON STRONG GRAPH BUNDLES

F. LARRIÓN, M.A. PIZAÑA, AND R. VILLARROEL-FLORES

Abstract. We study *strong graph bundles*: a concept imported from topology which generalizes both covering graphs and product graphs. Roughly speaking, a strong graph bundle always involves three graphs E, B and F and a projection $p : E \to B$ with fiber F (i.e. $p^{-1}(x) \cong F$ for all $x \in V(B)$) such that the preimage of any edge xy of B is trivial (i.e. $p^{-1}(xy) \cong K_2 \boxtimes F$). Here we develop a framework to study which subgraphs S of B have trivial preimages (i.e. $p^{-1}(S) \cong S \boxtimes F$) and this allows us to compare and classify several variations of the concept of strong graph bundle. As an application, we show that the clique operator preserves *triangular graph bundles* (strong graph bundles where preimages of triangles are trivial) thus yielding a new technique for the study of clique divergence of graphs.

1. Introduction

In topology a *fiber bundle* is a space which is locally a product of spaces [41]. This concept has proved to be very important in many fields of mathematics including algebraic geometry, differential geometry and differential topology. Also, fiber bundles play a central role in general relativity. Thus, the importance of fiber bundles in mathematics and physics is difficult to overstate. The analogues of fiber bundles in graph theory, i.e. *graph bundles*, were introduced (as reported in [34]) by Pisanski and Vrabec in a 1982 unpublished preprint, and appeared for the first time (with Shawe-Taylor as an additional author) in [33]. Since there are several notions of a product in graph theory, there are also several notions of a graph bundle. Most works on graph bundles focus on *Cartesian graph bundles* [1–3, 5, 7, 8, 12, 13, 15–23, 30, 33–35, 40, 43–47], where graphs are locally a Cartesian product of graphs, but there is also research on *strong graph bundles* [18, 30, 43, 47], *tensor graph bundles* [14, 18] and *lexicographic graph bundles* [30]. The one just given is an exhaustive classification of all the papers on graph bundles that we could find. Here we shall focus on strong graph bundles, as the strong product $\boxtimes$ suits our purposes best.

More specifically, a strong graph bundle always involves three graphs E, B, and F and a projection $p : E \to B$. Saying that E is "locally a product" means then that the preimage $p^{-1}(x)$ of each vertex $x \in V(B)$ can be seen as $\{x\} \boxtimes F$ in such a way that the restriction of p is just the first projection, and also the preimage $p^{-1}(xy)$ of each edge $xy \in E(B)$ is isomorphic to $\{xy\} \boxtimes F$, again in a way equally compatible with the first projection. Mohar, Pisanski and Škoviera remarked in [30] that a more natural equivalent definition is obtained by asking that the preimage $p^{-1}(\mathrm{St}(x))$ of the star of each vertex $x \in V(B)$ can be seen as $\mathrm{St}(x) \boxtimes F$ in such a way that the restriction of p is just the first projection. As far as we know, this is the only notion of locality employed so far in the literature of graph bundles.

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But just as the concept of a product of graphs is not unique, neither is so that of locality in a graph, and hence each kind of locality may produce a variant version of graph bundles. We shall study strong graph bundles where these local subgraphs are indeed vertices and edges (or stars), but then we shall explore other types of locality by adding triangles, cliques and closed neighborhoods. It will turn out that all three of them are equivalent (Theorem 3.2), but not equivalent to the original one involving only vertices and edges (Figure 1a). The new kind of strong graph bundle introduced here will be called *triangular graph bundle*.

Our Theorem 2.8 (together with Lemma 2.1) provides a framework in which different versions of locality for strong graph bundles can be studied and compared. In Theorem 3.2 this result proves the equivalence of our three definitions of a triangular graph bundle, and in Corollary 2.9 it also yields the above-mentioned equivalence [30] of the two definitions of the original strong graph bundles. An interesting and useful tool behind Theorem 2.8 is the concept of *agreement* of graph morphisms at a vertex in Definition 2.3.

We also give an application to clique graphs: Theorem 4.1 states that triangular graph bundles are preserved by the clique operator. This yields, in Theorem 4.2, a new method for proving clique divergence or clique convergence which generalizes and unifies previously known results about strong products [24, 31] and triangular covering maps [25].

Let us quickly review the basic terminology now.

Our graphs are simple and finite. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, and $|G| = |V(G)|$ is the *order* of G. A graph H is a *subgraph* of G (denoted by $H \leq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* of G is a subgraph H of G such that whenever $x, y \in V(H)$ and $xy \in E(G)$, we also have $xy \in E(H)$. The *union* of the graphs G and H is given by $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$, and their *intersection* is given by $V(G \cap H) = V(G) \cap V(H)$ and $E(G \cap H) = E(G) \cap E(H)$. Two vertices x, y are *adjacent-or-equal* in G (denoted by $x \simeq y$), if $x = y$ or $xy \in E(G)$. The *closed neighborhood* of $x \in V(G)$ is the subgraph $N_G[x] \leq G$ induced by $\{y \in V(G) \mid x \simeq y\}$.

A *morphism* (or *map*) $f : G \to H$ is a function on the vertex sets $f : V(G) \to V(H)$ such that $x \simeq y \Rightarrow f(x) \simeq f(y)$. In this case we denote the *domain*, *codomain* and *image* of f by $\mathcal{D}_f$, $\mathcal{C}_f$ and $\mathcal{I}_f$ respectively, i.e. $\mathcal{D}_f = G$, $\mathcal{C}_f = H$, and $\mathcal{I}_f$ is given by $V(\mathcal{I}_f) = \{f(x) \mid x \in V(G)\}$ and $E(\mathcal{I}_f) = \{f(x)f(y) \mid xy \in E(G) \text{ and } f(x) \neq f(y)\}$. Note that $\mathcal{I}_f \leq \mathcal{C}_f$ may be non-induced. Given $f : G \to H$ and $S \leq H$, the *inverse image* of S is the subgraph $f^{-1}(S)$ of G given by $V(f^{-1}(S)) = f^{-1}(V(S))$ and $E(f^{-1}(S)) = \{xy \in E(G) \mid f(x) \simeq f(y) \text{ in } S\}$. Also $f^{-1}(S)$ could be non-induced in G. The *strong product* $G \boxtimes H$ of two graphs is determined by $V(G \boxtimes H) = V(G) \times V(H)$ and $E(G \boxtimes H) = \{(v, w)(v', w') \mid v \simeq v' \text{ in } G \text{ and } w \simeq w' \text{ in } H\}$.

As usual when studying clique graphs, a *complete* of G is a complete subgraph of G, and we reserve the word *clique* for maximal complete subgraphs. The *clique graph* $K(G)$ is the intersection graph of the cliques of G and the operator K is called the *clique operator*. Then the *iterated clique graphs* $K^n(G)$ are defined inductively by $K^0(G) = G$ and $K^n(G) = K(K^{n-1}(G))$. If the sequence $\{|K^n(G)|\}$ is bounded (equivalently, if $K^m(G) \cong K^n(G)$ for some $m > n$), we say that G is *K-convergent*. On the other hand, G is called *K-divergent* if the sequence $\{|K^n(G)|\}$ is unbounded. The *K-behavior* of G can be either K-convergent or K-divergent. This dichotomy is a major topic in the theory of clique graphs, and many papers have appeared providing techniques for determining the K-behavior (e.g. [4, 6, 9, 10, 25–29, 31, 32, 36, 42]). Applications of the theory of the clique operator include the fixed point property for posets [11] and loop quantum gravity [37–39].

2. Strong Graph Bundles

Given a graph B, hereinafter referred to as the *base* graph, a *projection* over B is a graph morphism $p : E \to B$ which is vertex-surjective and edge-surjective, i.e. $\mathcal{I}_p = \mathcal{C}_p$. The domain $E = \mathcal{D}_p$ will be called the *total* graph of the projection. The *fiber* of a vertex $x \in V(B)$ is the preimage $p^{-1}(x) \leq E$ of the one-vertex subgraph $\{x\} \leq B$. These *fibers of* p are non-empty induced subgraphs of E. In fact, if $S \leq B$ is induced, then $p^{-1}(S) \leq E$ is induced. Even if S is not induced the restriction of p, denoted also by $p : p^{-1}(S) \to S$, is again a projection.

Any projection $p : E \to B$ partitions $V(E)$ into the disjoint union of the vertex sets of its fibers, so each $v \in V(E)$ lies in a unique fiber of p, namely $v \in V(p^{-1}(x))$ for $x = p(v)$. The projection p being understood, we say that v *lies over* x, or that v *is a vertex over* x. The formula $v = \tilde{x}$ means both that $v \in V(E)$ and $p(v) = x$, so in what follows $\tilde{x}$ shall always denote a vertex of the total graph E lying over the vertex x of the base graph B.

If $p' : E' \to B$ is another projection over the same base graph B, a *morphism* from p to p' is a graph morphism $\varphi : E \to E'$ such that $p' \circ \varphi = p$, i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B. & \end{tikzcd}$$

In this case φ clearly sends (distinct) fibers of p into (distinct) fibers of p'. If φ is a graph isomorphism then φ^{-1} is also a morphism from p' to p, and in this case one says that p and p' are *isomorphic*. Notice that if this is the case φ restricts to an isomorphism from $p^{-1}(x)$ to $(p')^{-1}(x)$ over each vertex x of B: isomorphic projections are fiberwise isomorphic.

When all the fibers of a projection $p : E \to B$ are isomorphic to some graph F we say that p is a projection *with fiber* F. There is always a *trivial* projection with any given base B and fiber F: the total space is the strong product $B \boxtimes F$ and the projection is just the first canonical projection $\pi_1 : B \boxtimes F \to B$ given by $\pi_1(v, w) = v$. Any projection isomorphic to a trivial one is also called trivial. Thus, a projection $p : E \to B$ is *trivial* (with fiber F) if there is an isomorphism $\varphi : E \to B \boxtimes F$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & B \boxtimes F \arrow[dl, "\pi_1"] \\ & B. & \end{tikzcd}$$

In this situation, the isomorphism φ is called a *trivialization* of p. Being morphisms to a product, trivializations are determined by their *components* as in the following diagram:

$$\begin{tikzcd} & & F \\ E \arrow[r, "\varphi"] \arrow[urr, bend left, "\tau"] \arrow[drr, bend right, "p"'] & B \boxtimes F \arrow[ur, "\pi_2"] \arrow[dr, "\pi_1"'] & \\ & & B \end{tikzcd} \qquad \begin{array}{c} \varphi = (p, \tau) = (\pi_1 \circ \varphi, \pi_2 \circ \varphi) \\ \\ \varphi(v) = (p(v), \tau(v)). \end{array}$$

As the first component has to be p, a candidate for a trivialization $\varphi : E \to B \boxtimes F$ is always given as $\varphi = (p, \tau)$ where $\tau : E \to F$ can be any morphism, and then $\varphi(v) = (p(v), \tau(v))$ for any $v \in E$, or rather $\varphi(\tilde{x}) = (x, \tau(\tilde{x}))$ for any $\tilde{x}$ over x as explained above. This $\varphi = (p, \tau)$ is called the *product map* of its components p and τ. As φ needs to restrict to isomorphisms from the fibers of p to those of π_1, the restrictions of τ to the fibers of p should be isomorphisms onto F. This would already imply that φ is vertex-bijective, and it would even be part of the other condition (that is, to be edge-surjective) that φ still needs to satisfy in order to be indeed a trivialization.

Graph bundles won't be forced to be trivial, they will only have to be trivial over certain subgraphs of the base graph. Let $p : E \to B$ be a projection and take a subgraph $S \leq B$. We say that p is *trivial over* S (with fiber F), or that S is (p, F)*-trivial*, if the restricted projection $p : p^{-1}(S) \to S$ is trivial with fiber F. In other words, S is (p, F)-trivial if there exists some isomorphism $\varphi_S : p^{-1}(S) \to S \boxtimes F$ for which the following diagram commutes:

$$\begin{tikzcd} p^{-1}(S) \arrow[rr, "\varphi_S"] \arrow[dr, "p"'] & & S \boxtimes F \arrow[dl, "\pi_1"] \\ & S. & \end{tikzcd}$$

In this case we say that φ_S is a *trivialization* (of p) *over* S. The following is immediate:

Lemma 2.1. *If $T \leq S \leq B$ and S is (p, F)-trivial, then T is also (p, F)-trivial.* □

Note that a projection $p : E \to B$ has fiber F if and only if all one-vertex subgraphs of B are (p, F)-trivial. Indeed, if $\tau_x : p^{-1}(x) \to F$ is any morphism, the product map $\varphi_x = (p, \tau_x) : p^{-1}(x) \to \{x\} \boxtimes F$ is an isomorphism (and hence trivializes p over x) if and only if $\tau_x = \pi_2 \circ \varphi_x$ is an isomorphism, as $\pi_2 : \{x\} \boxtimes F \to F$ is an isomorphism in this case.

Definition 2.2. [30]: *Let $p : E \to B$ be a projection with fiber F. Then (E, B, F, p) is a* strong graph bundle *if every edge $xy \in E(B)$ (as a subgraph $\{xy\} \leq B$) is (p, F)-trivial.*

Since we will only consider strong graph bundles, we will call them just *graph bundles*. Notice that (E, B, F, p) is a graph bundle if and only if $p : E \to B$ is a projection and every complete subgraph $C \leq B$ of order 1 or 2 is (p, F)-trivial.

For the rest of the section we shall need the notions of twinship and agreement.

Two vertices x, y are *twins in* G (denoted by $x \approx y$) if $V(N_G[x]) = V(N_G[y])$. Neighborhoods are induced subgraphs, so this is the same as asking that $N_G[x] = N_G[y]$. With this definition every vertex is a twin of itself, and twinship is an equivalence relation.

Definition 2.3. *Two morphisms f, g will be said to* agree *at a vertex $x \in V(\mathcal{D}_f \cap \mathcal{D}_g)$ if $f(x)$ and $g(x)$ are twins (i.e. $f(x) \approx g(x)$) in $\mathcal{I}_f \cup \mathcal{I}_g$.*

We also say that two trivializations φ_S and φ_T of our projection $p : E \to B$ *agree over* $x \in V(S \cap T)$ if they agree at every vertex $\tilde{x}$ over x.

We use 'twinship' instead of 'equality' in our definition of agreement since this way our statements and proofs are easier and smoother. Now we prove that agreement over one vertex can always be achieved (the proof actually establishes strict agreement here):

Theorem 2.4. *If two subgraphs $S, T \leq B$ are (p, F)-trivial and $x \in V(S \cap T)$, then for any trivialization φ_S over S there is a trivialization φ_T over T which agrees with φ_S over x.*

Proof. Let $\varphi_S : p^{-1}(S) \to S \boxtimes F$ and $\varphi_T^0 : p^{-1}(T) \to T \boxtimes F$ be any two trivializations. We know already that in the following diagram the restrictions of τ_S and τ_T^0 to $p^{-1}(x)$ are isomorphisms, because so are π_2 and both φ's. Then there is an automorphism $\alpha \in \mathrm{Aut}(F)$ which makes the whole diagram commute, namely, with the restricted τ's, $\alpha = \tau_S \circ (\tau_T^0)^{-1}$:

$$\begin{tikzcd}
\{x\} \boxtimes F \arrow[dr, "\pi_2"'] & p^{-1}(x) \arrow[l, "\varphi_T^0"'] \arrow[r, "\varphi_S"] \arrow[d, "\tau_T^0"'] \arrow[dr, "\tau_S"] & \{x\} \boxtimes F \arrow[d, "\pi_2"] \\
& F \arrow[r, dashed, "\alpha"] & F.
\end{tikzcd}$$

Define $\tau_T : p^{-1}(T) \to F$ by $\tau_T = \alpha \circ \tau_T^0$. Then, restricting to $p^{-1}(x)$, $\tau_T = \alpha \circ \tau_T^0 = \tau_S$ over x.

Now define $\varphi_T : p^{-1}(T) \to T \boxtimes F$ by $\varphi_T = (p, \tau_T)$. Then $\varphi_T = (p, \tau_T)$ and $\varphi_S = (p, \tau_S)$ agree over x since $\tau_T(\tilde{x}) = \tau_S(\tilde{x})$ for all $\tilde{x}$ over x. Notice that $\varphi_T = (1_T \boxtimes \alpha) \circ \varphi_T^0$ is a composition of isomorphisms:

$$\begin{tikzcd}
p^{-1}(T) \arrow[r, "\varphi_T^0"] \arrow[rr, bend right, "\varphi_T"'] & T \boxtimes F \arrow[r, "1_T \boxtimes \alpha"] & T \boxtimes F.
\end{tikzcd}$$

Therefore φ_T is an isomorphism, and hence it is a trivialization of p over T. $\square$

Now, consider two vertices $(x, v), (x, w)$ of $B \boxtimes F$, both in the same fiber of $\pi_1 : B \boxtimes F \to B$: as $N_{B\boxtimes F}[(x, v)] = N_B[x] \boxtimes N_F[v]$ and $N_{B\boxtimes F}[(x, w)] = N_B[x] \boxtimes N_F[w]$, we see that $(x, v) \approx (x, w)$ in $B \boxtimes F$ if and only if $v \approx w$ in F. Hence we can characterize agreement of the trivializations φ_S, φ_T in terms of their second components τ_S and τ_T:

Theorem 2.5. *Assume that $S, T \leq B$ and that φ_S and φ_T are trivializations over S and T. Assume further that $x \in V(S \cap T)$. Then φ_S and φ_T agree over x if and only if $\tau_S(\tilde{x}) \approx \tau_T(\tilde{x})$ in F for all $\tilde{x} \in V(p^{-1}(x))$.*

Proof. Let $\tilde{x}$ be a vertex over x. Note that $\mathcal{I}_{\varphi_S} \cup \mathcal{I}_{\varphi_T} = (S \boxtimes F) \cup (T \boxtimes F) = (S \cup T) \boxtimes F$, and that both $\varphi_S(\tilde{x}), \varphi_T(\tilde{x}) \in (S \cup T) \boxtimes F$ are in the same fiber $\{x\} \boxtimes F$ of the first projection $\pi_1 : (S \cup T) \boxtimes F \to S \cup T$. Using that $\varphi_S(\tilde{x}) = (x, \tau_S(\tilde{x}))$ and $\varphi_T(\tilde{x}) = (x, \tau_T(\tilde{x}))$ we get, as observed above, that $\varphi_S(\tilde{x}) \approx \varphi_T(\tilde{x})$ in $\mathcal{I}_{\varphi_S} \cup \mathcal{I}_{\varphi_T}$ if and only if $\tau_S(\tilde{x}) \approx \tau_T(\tilde{x})$ in F. $\square$

A key property is that twins of adjacent-or-equal vertices are also so: $x \approx y \simeq z \approx w$ implies $x \simeq w$. We next prove that agreement over one vertex implies agreement over its neighbors:

Theorem 2.6. *Assume that $S, T \leq B$ and that φ_S and φ_T are trivializations over S and T. Assume further that $x, y \in V(S \cap T)$ with $x \simeq y$ in $S \cap T$. Then, if φ_S and φ_T agree over x, they also agree over y.*

Proof. Since φ_S and φ_T agree over x, $\tau_S(\tilde{x}) \approx \tau_T(\tilde{x})$ in F for all $\tilde{x}$ over x by Theorem 2.5. By the same result it suffices to show, given a vertex $\tilde{y}$ over y, that $\tau_S(\tilde{y}) \approx \tau_T(\tilde{y})$ in F.

Let $z \in N_F[\tau_S(\tilde{y})]$, so $z \in F$ and $z \simeq \tau_S(\tilde{y})$. The restriction $\tau_S : p^{-1}(x) \to F$ is onto, so $z = \tau_S(\tilde{x})$ for some $\tilde{x}$ over x. Then $\tau_S(\tilde{x}) \simeq \tau_S(\tilde{y}) \implies \varphi_S(\tilde{x}) \simeq \varphi_S(\tilde{y}) \implies \tilde{x} \simeq \tilde{y}$ in $p^{-1}(S)$. Hence $\tilde{x} \simeq \tilde{y}$ also in $p^{-1}(T)$ and $\tau_T(\tilde{x}) \simeq \tau_T(\tilde{y})$. It follows that $z = \tau_S(\tilde{x}) \approx \tau_T(\tilde{x}) \simeq \tau_T(\tilde{y})$, and hence $z \simeq \tau_T(\tilde{y})$. Therefore $N_F[\tau_S(\tilde{y})] \subseteq N_F[\tau_T(\tilde{y})]$. By symmetry we also have $N_F[\tau_S(\tilde{y})] \supseteq N_F[\tau_T(\tilde{y})]$, and thus $\tau_S(\tilde{y}) \approx \tau_T(\tilde{y})$ in F. □

Given two morphisms f, g we define the *glued* function $f \bigsqcup g : V(\mathcal{D}_f \cup \mathcal{D}_g) \to V(\mathcal{C}_f \cup \mathcal{C}_g)$ by

$$\left(f \bigsqcup g\right)(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{D}_f \\ g(x) & \text{otherwise.} \end{cases}$$

Notice that the gluing operation is not commutative, but it is associative. Therefore when we have an ordered list of morphisms $f_1, f_2, \ldots, f_r$ we can define:

$$\bigsqcup_{i=1}^{r} f_i = f_1 \bigsqcup f_2 \bigsqcup \cdots \bigsqcup f_r \ .$$

In general the glued function $f \bigsqcup g$ is not a morphism, but we have the following theorem:

Theorem 2.7. *If the graph morphisms f and g agree at every $x \in V(\mathcal{D}_f \cap \mathcal{D}_g)$, then the glued function $f \bigsqcup g : \mathcal{D}_f \cup \mathcal{D}_g \to \mathcal{C}_f \cup \mathcal{C}_g$ is a morphism of graphs.*

Proof. Let xy be an edge of $\mathcal{D}_f \cup \mathcal{D}_g$. We have to prove that $(f \bigsqcup g)(x) \simeq (f \bigsqcup g)(y)$, and we shall consider four cases:

Case 1: $xy \in E(\mathcal{D}_f)$.
Then $(f \bigsqcup g)(x) = f(x) \simeq f(y) = (f \bigsqcup g)(y)$.

Case 2: $xy \notin E(\mathcal{D}_f)$, $xy \in E(\mathcal{D}_g)$ and $x, y \in V(\mathcal{D}_f \cap \mathcal{D}_g)$.
Then $(f \bigsqcup g)(x) = f(x) \approx g(x) \simeq g(y) \approx f(y) = (f \bigsqcup g)(y)$.

Case 3: $xy \notin E(\mathcal{D}_f)$, $xy \in E(\mathcal{D}_g)$ and (say) $x \in V(\mathcal{D}_f \cap \mathcal{D}_g) \not\ni y$.
Then $(f \bigsqcup g)(x) = f(x) \approx g(x) \simeq g(y) = (f \bigsqcup g)(y)$.

Case 4: $xy \notin E(\mathcal{D}_f)$, $xy \in E(\mathcal{D}_g)$ and $x, y \notin V(\mathcal{D}_f \cap \mathcal{D}_g)$.
Then $(f \bigsqcup g)(x) = g(x) \simeq g(y) = (f \bigsqcup g)(y)$. □

We can now show that triviality extends over unions of connectedly intersecting regions:

Theorem 2.8. *Assume that $S, T \leq B$ are both (p, F)-trivial and that $S \cap T$ is non-empty and connected. Then $S \cup T$ is (p, F)-trivial.*

Proof. By Theorem 2.4 we can choose trivializations φ_S and φ_T over S and T which agree over some $x \in V(S \cap T)$. Then φ_S and φ_T agree over all $y \in V(S \cap T)$ by Theorem 2.6. Hence the glued function $\varphi = \varphi_S \bigsqcup \varphi_T : p^{-1}(S \cup T) \to (S \cup T) \boxtimes F$ is a morphism of graphs by Theorem 2.7. We shall only have to keep in mind that, for each $\tilde{x} \in V(p^{-1}(S \cup T))$, the definition of φ is that $\varphi(\tilde{x}) = \varphi_S(\tilde{x})$ if $x \in V(S)$, and $\varphi(\tilde{x}) = \varphi_T(\tilde{x})$ if $x \notin V(S)$. Let us now prove that φ is indeed a trivialization of p over $S \cup T$.

Just as $\varphi_S : p^{-1}(S) \to S \boxtimes F$ and $\varphi_T : p^{-1}(T) \to T \boxtimes F$ are the product maps of their components $\varphi_S = (p, \tau_S)$ and $\varphi_T = (p, \tau_T)$, it is quite clear that $\varphi = (p, \tau)$ with $\tau = \tau_S \bigsqcup \tau_T$, so the following diagram certainly commutes:

$$\begin{tikzcd} p^{-1}(S \cup T) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & (S \cup T) \boxtimes F \arrow[dl, "\pi_1"] \\ & S \cup T. & \end{tikzcd}$$

As both τ_S and τ_T restricted to fibers of p are isomorphisms to F, and τ acts as τ_S over the vertices of S and as τ_T over those of $T \setminus S$, we already know that φ is a vertex-bijective morphism and hence, it only remains to show that it is edge-surjective.

Let $(x, v), (y, w) \in (S \cup T) \boxtimes F$ with $(x, v) \simeq (y, w)$. Then $x \simeq y$ in $S \cup T$ and $v \simeq w$ in F. Take two vertices $\tilde{x}$ and $\tilde{y}$ in $p^{-1}(S \cup T)$ such that $\varphi(\tilde{x}) = (x, v)$ and $\varphi(\tilde{y}) = (y, w)$. Thus we have $\varphi(\tilde{x}) \simeq \varphi(\tilde{y})$ in $(S \cup T) \boxtimes F$. We shall prove that $\tilde{x} \simeq \tilde{y}$ in $p^{-1}(S \cup T)$.

As a first case we assume that $x \simeq y$ in S. Then $\varphi(\tilde{x}) \simeq \varphi(\tilde{y})$ in $S \boxtimes F$ because $v \simeq w$ in F, and also $\varphi(\tilde{x}) = \varphi_S(\tilde{x})$ and $\varphi(\tilde{y}) = \varphi_S(\tilde{y})$ because $x, y \in V(S)$. Now, as $\varphi_S(\tilde{x}) \simeq \varphi_S(\tilde{y})$ in $S \boxtimes F$ and $\varphi_S : p^{-1}(S) \to S \boxtimes F$ is an isomorphism, $\tilde{x} \simeq \tilde{y}$ in $p^{-1}(S) \leq p^{-1}(S \cup T)$.

Assume then that $x \not\simeq y$ in S, so $x \simeq y$ in T and $\varphi(\tilde{x}) \simeq \varphi(\tilde{y})$ in $T \boxtimes F \leq (S \cup T) \boxtimes F$ because $v \simeq w$ in F. We claim now that $\varphi_T(\tilde{x}) \simeq \varphi_T(\tilde{y})$ in $T \boxtimes F$. Let $\tilde{z}$ be one of $\tilde{x}, \tilde{y}$. If $z \notin S$, $\varphi(\tilde{z}) = \varphi_T(\tilde{z})$ and, if $z \in S$, $\varphi(\tilde{z}) = \varphi_S(\tilde{z}) \approx \varphi_T(\tilde{z})$ in $(S \cup T) \boxtimes F$. In both cases, $\varphi(\tilde{z}) \approx \varphi_T(\tilde{z})$ in $(S \cup T) \boxtimes F$. Then in $(S \cup T) \boxtimes F$ we have $\varphi_T(\tilde{x}) \approx \varphi(\tilde{x}) \simeq \varphi(\tilde{y}) \approx \varphi_T(\tilde{y})$ and hence $\varphi_T(\tilde{x}) \simeq \varphi_T(\tilde{y})$ in $(S \cup T) \boxtimes F$. Since $x \simeq y$ in T, it follows that $\varphi_T(\tilde{x}) \simeq \varphi_T(\tilde{y})$ also in $T \boxtimes F$ as claimed. Once again, since $\varphi_T : p^{-1}(T) \to T \boxtimes F$ is an isomorphism, $\tilde{x} \simeq \tilde{y}$ in $p^{-1}(T) \leq p^{-1}(S \cup T)$. □

The *star* of a vertex $x \in B$ is the subgraph of B consisting of all edges incident to x (and their vertices). The following is now immediate:

Corollary 2.9. [30]: *Let $p : E \to B$ be a projection with fiber F. Then (E, B, F, p) is a graph bundle if and only the star of each vertex of B is (p, F)-trivial.* □

3. Triangular Graph Bundles

We observed previously that (E, B, F, p) is a graph bundle if and only if $p : E \to B$ is a projection and every complete subgraph $C \leq B$ of order 1 or 2 is (p, F)-trivial. Thus, enlarging the notion of "locality" by substituting "order 1, 2 or 3" for "order 1 or 2" is just taking this one step further:

Definition 3.1. *A* triangular graph bundle *is a strong graph bundle (E, B, F, p) for which every triangle, considered as a subgraph $xyz \leq B$ is (p, F)-trivial.*

Trivial bundles are always triangular because every subgraph of B is (π_1, F)-trivial for the strong product projection $\pi_1 : B \boxtimes F \to B$. Triangular covering maps were characterized in [25] as being just the *local isomorphisms*, i.e. those graph morphisms $p : \tilde{G} \to G$ for which all the restrictions $p_| : N_{\tilde{G}}[\tilde{x}] \to N_G[p(\tilde{x})]$ are isomorphisms. It is easy to see that triangular covering maps with connected base graph are the same as triangular graph bundles with connected base and discrete (i.e. without edges) fiber.

Therefore, triangular graph bundles are a common generalization of both strong products and triangular covering maps, and we shall see in Section 4 that they are just as well suited to the study of clique graphs as those particular cases.

We now prove that once triangles are admitted into the notion of locality, complete subgraphs of all sizes get in, and even closed neighborhoods. Indeed, triangular graph bundles can be defined by means of several different notions of locality.

Theorem 3.2. *For a projection $p : E \to B$ with fiber F the following are equivalent:*

(1) (E, B, F, p) is a triangular graph bundle.
(2) Every complete subgraph of B is (p, F)-trivial.
(3) Every clique of B is (p, F)-trivial.
(4) Every closed neighborhood in B is (p, F)-trivial.

Proof. That (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) follows from Lemma 2.1. Let us prove (1) $\Rightarrow$ (4). Let (E, B, F, p) be a triangular graph bundle, so that any vertex, edge or triangle of B is (p, F)-trivial. Taking $x \in V(B)$, we shall prove that $N[x]$ is (p, F)-trivial. Let $T_1 = \{x\}$ and let $T_2, T_3, \ldots, T_r$ be all the edges and triangles of B which contain x. Clearly $N[x] = \cup_{i=1}^{r} T_i$. If $r = 1$, $N[x] = \{x\}$ is indeed (p, F)-trivial. Otherwise, note that for all $s \in \{2, 3, \ldots, r\}$ $(\cup_{i=1}^{s-1} T_i) \cap T_s$ is non-empty and connected. It follows by reiterated application of Theorem 2.8 that $\cup_{i=1}^{s} T_i$ is (p, F)-trivial and, in particular, $N[x] = \cup_{i=1}^{r} T_i$ is (p, F)-trivial. $\square$

Not all graph bundles are triangular, see Figure 1(a). Some further notions of locality lead still to triangular graph bundles, but there are others that do not, as for example adding 4-cycles to vertices, edges and triangles, see Figure 1(b).

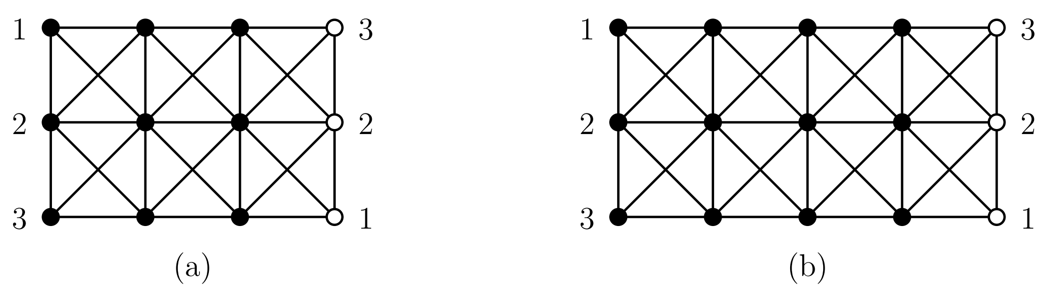

FIGURE 1. Two examples. (Identify vertices with equal labels in each graph). (a) A non-triangular graph bundle with base C_3 and fiber P_3. (b) A triangular graph bundle with base C_4 and fiber P_3 where $B = C_4$ is not (p, F)-trivial

As another example of the use of Theorem 2.8 to produce new kinds of locality leading to triangular graph bundles, we present the following result which will be used in Section 4.

Theorem 3.3. *If (E, B, F, p) is a triangular graph bundle, then the union of any three pairwise intersecting cliques q_1, q_2 and q_3 of B is (p, F)-trivial.*

Proof. If $q_1 \cap q_2 \cap q_3 \neq \varnothing$, then $q_1 \cup q_2 \cup q_3 \leq N[x]$ for some x, and $q_1 \cup q_2 \cup q_3$ is (p, F)-trivial by Theorem 3.2. Otherwise, $q_1 \cap q_2 \cap q_3 = \varnothing$. Note that in this case $(q_1 \cup q_2) \cap q_3$ is not connected and hence we can not apply Theorem 2.8 directly. Let $x \in V(q_1 \cap q_2)$, $y \in V(q_2 \cap q_3)$ and $z \in V(q_1 \cap q_3)$ and let T be the triangle induced by $\{x, y, z\}$. Then $q_1 \cup q_2 \cup q_3 = T \cup q_1 \cup q_2 \cup q_3$ is (p, F)-trivial by Theorem 2.8, since all the required intersections $T \cap q_1$, $(T \cup q_1) \cap q_2$ and $(T \cup q_1 \cup q_2) \cap q_3$ are connected. □

4. Clique Graphs

If $p : E \to B$ is any morphism and q is a clique of E, $p(V(q)) = \{p(x) \mid x \in V(q)\}$ induces a complete subgraph in B. Then we can define a morphism $K(p) : K(E) \to K(B)$ by selecting for every clique $q \in V(K(E))$ a fixed clique $K(p)(q) \in V(K(B))$ containing $p(V(q))$.

Furthermore, when p is, for instance, a local isomorphism no choosing is necessary as $p(V(q))$ already induces a clique in B. The same will happen for the projection of a triangular graph bundle. It is well known, and easy to see, that the cliques of a product $B \boxtimes F$ are precisely the products of cliques: $V(K(B \boxtimes F)) = \{q_1 \boxtimes q_2 \mid q_1 \in V(K(B)), q_2 \in V(K(F))\}$. Therefore, for the trivial graph bundle with projection $\pi_1 : B \boxtimes F \to B$ the image of a clique is already a clique. But then the same is true for any triangular graph bundle, as it is trivial over the image of any clique of the total graph.

It is known that the clique operator preserves triangular covering maps [25] and strong products [24, 31]. The next theorem extends and unifies both results, and the theorem following it does the same for the corresponding techniques for proving clique convergence or divergence.

Theorem 4.1. *If (E, B, F, p) is a triangular graph bundle, then $(K(E), K(B), K(F), K(p))$ is also a triangular graph bundle.*

Proof. Let q be any clique of B, i.e. $q \in V(K(B))$. Let S be any (p, F)-trivial subgraph of B containing q (which exists by Theorem 3.2) and let $\varphi_S : p^{-1}(S) \to S \boxtimes F$ be a trivialization. Then any clique q_E of E satisfying $K(p)(q_E) = q$ must be a subgraph of $p^{-1}(q)$. Note that the restriction of φ_S to $p^{-1}(q)$ is an isomorphism onto its image $\varphi_S : p^{-1}(q) \to q \boxtimes F$. Then the set of cliques of $p^{-1}(q)$ is precisely $\{\varphi_S^{-1}(q \boxtimes q_F) \mid q_F \in V(K(F))\}$. All these cliques of $p^{-1}(q)$ are also cliques of E. Therefore

$$V(K(p)^{-1}(q)) = \{\varphi_S^{-1}(q \boxtimes q_F) \mid q_F \in V(K(F))\}.$$

Given $\varphi_S^{-1}(q \boxtimes q_F), \varphi_S^{-1}(q \boxtimes q'_F) \in V(K(p)^{-1}(q))$, we have $\varphi_S^{-1}(q \boxtimes q_F) \simeq \varphi_S^{-1}(q \boxtimes q'_F) \iff \varphi_S^{-1}(q \boxtimes q_F) \cap \varphi_S^{-1}(q \boxtimes q'_F) \neq \varnothing \iff (q \boxtimes q_F) \cap (q \boxtimes q'_F) \neq \varnothing \iff q_F \cap q'_F \neq \varnothing \iff q_F \simeq q'_F$ in $K(F)$. Hence $K(p)^{-1}(q) \cong \{q\} \boxtimes K(F)$ and therefore $\{q\}$ is $(K(p), K(F))$-trivial. Note in particular that $K(p)$ is vertex-surjective and that the required trivialization $\varphi_{\{q\}} : K(p)^{-1}(\{q\}) \to \{q\} \boxtimes K(F)$ is $\varphi_{\{q\}} = K(\varphi_S)$.

Now let $q_1, q_2, q_3 \in V(K(B))$ with $q_i \cap q_j \neq \varnothing$ for all i, j, so $\{q_1, q_2, q_3\}$ induces a triangle Δ in $K(B)$. Let $S = q_1 \cup q_2 \cup q_3$, which is (p, F)-trivial by Theorem 3.3. By the previous argument we know, for $i = 1, 2, 3$, that:

$$V(K(p)^{-1}(q_i)) = \{\varphi_S^{-1}(q_i \boxtimes q_F) \mid q_F \in V(K(F))\}$$

Given $\varphi_S^{-1}(q_i \boxtimes q_F), \varphi_S^{-1}(q_j \boxtimes q'_F) \in V(K(p)^{-1}(\Delta))$, we have $\varphi_S^{-1}(q_i \boxtimes q_F) \simeq \varphi_S^{-1}(q_j \boxtimes q'_F) \iff \varphi_S^{-1}(q_i \boxtimes q_F) \cap \varphi_S^{-1}(q_j \boxtimes q'_F) \neq \varnothing \iff (q_i \boxtimes q_F) \cap (q_j \boxtimes q'_F) \neq \varnothing \iff q_F \cap q'_F \neq \varnothing \iff q_F \simeq q'_F$ in $K(F)$. It follows that $K(p)^{-1}(\Delta) \cong \Delta \boxtimes K(F)$. Hence any triangle $\Delta \leq K(B)$ is $(K(p), K(F))$-trivial. We point out, as before, that the required trivialization $\varphi_\Delta : K(p)^{-1}(\Delta) \to \Delta \boxtimes K(F)$ is $\varphi_\Delta = K(\varphi_S)$.

The case of an edge $q_1q_2 \in E(K(B))$ is entirely analogous to the previous one. Here is where we readily see that $K(p) : K(E) \to K(B)$ is edge-surjective. The commutativity property $K(p) = \pi_1 \circ \varphi_T$ in all three cases (when T is a vertex, and edge or a triangle of $K(B)$) is trivially true. □

Theorem 4.2. *If (E, B, F, p) is a triangular graph bundle, E is K-divergent if and only if at least one of B and F is K-divergent.*

Proof. As $p^{-1}(x) \cong F$ for each $x \in V(B)$, it follows that $|E| = |B| \cdot |F|$. By Theorem 4.1 we also have $|K^n(E)| = |K^n(B)| \cdot |K^n(F)|$ for all n, and the result follows. □

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F. Larrión (paco@math.unam.mx)
Instituto de Matemáticas, Universidad Nacional Autónoma de México.
México 04510 D.F. MEXICO

M.A. Pizaña (map@xanum.uam.mx, http://xamanek.izt.uam.mx/map)
Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica.
Av. San Rafael Atlixco 186. Col Vicentina. Del. Iztapalapa.
México 09340 D.F. MEXICO

R. Villarroel-Flores (rafaelv@uaeh.edu.mx)
Área Académica de Matemáticas y Física,
Universidad Autónoma del Estado de Hidalgo,
Carr. Pachuca-Tulancingo km. 4.5,
Pachuca 42184 Hgo. MEXICO