Posets, Clique Graphs and their Homotopy Type

F. Larrión [a] M.A. Pizaña [b] R. Villarroel–Flores [c,*]

[a] *Instituto de Matemáticas, U.N.A.M. C.U. México 04510 D.F. MEXICO.*

[b] *Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica. Av. San Rafael Atlixco 186. Col Vicentina. Del. Iztapalapa. México 09340 D.F. MEXICO.*

[c] *Instituto de Matemáticas U.N.A.M. (Unidad Cuernavaca) Av. Universidad s/n, Col. Lomas de Chamilpa C.P. 62210 Cuernavaca, Morelos MEXICO*

Abstract

To any finite poset P we associate two graphs which we denote by $\Omega(P)$ and $\mho(P)$. Several standard constructions can be seen as $\Omega(P)$ or $\mho(P)$ for suitable posets P, including the comparability graph of a poset, the clique graph of a graph and the 1–skeleton of a simplicial complex. We interpret graphs and posets as simplicial complexes using complete subgraphs and chains as simplices. Then we study and compare the homotopy types of $\Omega(P)$, $\mho(P)$ and P. As our main application we obtain a theorem, stronger than those previously known, giving sufficient conditions for a graph to be homotopy equivalent to its clique graph. We also introduce a new graph operator H that preserves clique–Hellyness and dismantlability and is such that $H(G)$ is homotopy equivalent to both its clique graph and the graph G.

Key words: clique graphs, graphs, posets, homotopy type

1 Introduction

For a finite poset P, denote by $\min(P)$ and $\max(P)$, respectively, the sets of minimal and maximal elements of P. We define $\Omega(P)$ as the graph with vertex set $\min(P)$ in which two distinct vertices x, y are adjacent if and only if there is $Z \in P$ such that $x \leq Z$ and $y \leq Z$. Dually we define the graph $\mho(P)$ with $V(\mho(P)) = \max(P)$ where $X \sim Y$ if they have a common lower bound. Note

* Corresponding author

Email addresses: `paco@math.unam.mx` (F. Larrión), `map@xanum.uam.mx` (M.A. Pizaña), `rafael@matem.unam.mx` (R. Villarroel–Flores).

URL: `http://xamanek.izt.uam.mx/map` (M.A. Pizaña).

that, in particular, $\Omega(P)$ is an induced subgraph of the upper bound graph of P introduced in [9].

Posets P and graphs G have associated simplicial complexes $\Delta(P)$ and $\Delta(G)$, whose vertices are respectively the points in P and the vertices of G, the simplices in $\Delta(P)$ are the totally ordered subsets, and the simplices in $\Delta(G)$ are the complete subgraphs. Since each simplicial complex Δ can be thought of as a topological space via its geometric realization $|\Delta|$, one can attach topological concepts to both posets and graphs. We will say, for instance, that P and G are homotopy equivalent, denoted $P \simeq G$, if $|\Delta(P)|$ and $|\Delta(G)|$ are so. All our graphs and simplices are nonempty. The *face poset* $\mathcal{P}(\Delta)$ of a complex Δ, has as points the faces of Δ and is ordered by inclusion. Since $\Delta(\mathcal{P}(\Delta))$ is the barycentric subdivision of Δ, $\mathcal{P}(\Delta)$ is homeomorphic to Δ. For a graph G, we denote $\mathcal{P}(\Delta(G))$ just as $\mathcal{P}(G)$.

The clique graph $K(G)$ of G is the intersection graph of its (maximal) cliques. It is known that G and $K(G)$ are not always homotopy equivalent [10]. The motivations for this work came from two fronts: Poset topology, as in [11,13,3], and homotopy type of clique graphs, as in [10,8]. In this work, we are interested in comparing the homotopy types of P and $\Omega(P)$, $\mho(P)$ and their clique graphs. For instance, under a mild condition on P, $\Omega(P)$ has the same homotopy type as the clique graph of $\mho(P)$, see Theorem 3.3. Furthermore, Theorem 5.7 generalizes the main result in [8], which was the strongest result asserting the homotopy equivalence of a graph and its clique graph. As we shall see, an interesting feature of Ω and $\mho$ is that, combined with standard constructions on posets, graphs and simplicial complexes, they yield several well known constructions, thus providing a unified approach to them. They can also be used to define new graph operators as our H in §7 which among other properties satisfies $G \simeq H(G) \simeq K(H(G))$ for any graph G.

2 Preliminaries

All our graphs, posets and complexes are finite. Our graphs are simple. Given a family of sets $\mathcal{F} = \{A_i\}_{i\in I}$, its *nerve* $\mathcal{N}(\mathcal{F})$ is the complex with vertex set I, and $\sigma \subseteq I$ is a simplex whenever $\cap_{i\in\sigma} A_i$ is not empty.

Proposition 2.1 ((10.6) from [3]) *Let Δ be a simplicial complex, and let $\mathcal{F} = \{\Delta_i\}_{i\in I}$ be a cover of Δ by subcomplexes. If $\cap_{i\in\sigma}\Delta_i$ is contractible for any $\sigma \in \mathcal{N}(\mathcal{F})$, then $\Delta \simeq \mathcal{N}(\mathcal{F})$.* □

Let P be a poset. Then $C \subseteq P$ is called a *crosscut* if we have (a) C is an antichain, (b) every maximal chain in P contains an element of C, and (c) if $A \subseteq C$ is bounded above or below in P, then either $\mathrm{lub}_P\, A$ or $\mathrm{glb}_P\, A$ is defined. For any $C \subseteq P$, the simplicial complex with vertex set C and with simplices the bounded subsets of C is denoted $\Gamma(P, C)$.

Proposition 2.2 ((10.8) from [3]) *If C is a crosscut of the poset P, then $\Gamma(P, C) \simeq P$.* □

Given posets P, Q, the *product poset* $P \times Q$ has as underlying set the Cartesian product of P and Q, and order relation given by $(x, y) \leq (z, w)$ if and only if $x \leq z$ and $y \leq w$. An *ideal* in a poset P is a subposet $I \subseteq P$ such that $i \in I$ and $x \leq i$ imply $x \in I$.

Proposition 2.3 ((10.10) from [3]) *Let P and Q be posets and $\mathcal{R}$ be an ideal in the product poset $P \times Q$. If $\mathcal{R}_x = \{ y \in Q \mid (x, y) \in \mathcal{R} \}$ is contractible for all $x \in P$ and $\mathcal{R}_y = \{ x \in P \mid (x, y) \in \mathcal{R} \}$ is contractible for all $y \in Q$, then P and Q are homotopy equivalent.* □

Proposition 2.4 ((10.12) from [3]) *Let P be a poset and $f \colon P \to P$ be an order–preserving map such that $f(x) \geq x$ for all $x \in P$. Then $P \simeq f(P)$.* □

A poset is *conically contractible* to $p \in P$ if there is an order preserving map $f \colon P \to P$ such that $x \leq f(x) \geq p$ for all $x \in P$. For example, suppose the poset P has a point p such that $f(x) = \mathrm{lub}_P\{p, x\}$ exists for all $x \in P$. In this case, we say that P is *join contractible* to p.

Theorem 2.5 ((1.6) from [11]) *Let $f \colon P \to Q$ be a map of posets, then $P \simeq Q$ whenever $f^{-1}(Q_{\leq x}) = \{ a \in P \mid f(a) \leq x \}$ is contractible for all $x \in Q$.* □

If P is an induced subposet of Q, and $x \in Q$, we define $P_{\leq x}$ as $\{ y \in P \mid y \leq x \}$. We define analogously $P_{<x}$, $P_{\geq x}$, $P_{>x}$. Given $x, y \in P$ we define the *closed interval* $[x, y] = P_{\geq x} \cap P_{\leq y}$.

Given a graph G, its set of vertices will be denoted by $V(G)$, and its set of edges by $E(G)$. We often refer to complete subgraphs just as *completes*. If $X \subseteq V$, the subgraph of G induced by X is denoted by $G[X]$. We usually identify induced subgraphs (hence completes) with their vertex sets. Let $N_G[x] = \{ y \in G \mid xy \in E(G) \} \cup \{x\}$ denote the *closed neighborhood* of x in G. The vertex x is *dominated by* $y \in G$ if $N_G[x] \subseteq N_G[y]$. The vertex x is *dominated* if it is dominated by some $y \neq x$. If $N_G[x] = N_G[y]$, then x and y are *twins*.

Proposition 2.6 ([10, proof of Proposition 3.2]) Let x be a dominated vertex in a graph G. Then $G - x \simeq G$. □

Following Harary ([7, p. 20]), we define a *clique* of a graph G as a maximal complete subgraph. The *clique graph* of G is the intersection graph $K(G)$ of the set of cliques of G. The *second clique graph* of G is $K^2(G) = K(K(G))$. A family $\mathcal{F}$ of subsets of a set $S \neq \varnothing$ is *Helly* if for every $\mathcal{F}' \subseteq \mathcal{F}$ such that all elements of $\mathcal{F}'$ intersect pairwise we have that $\cap\mathcal{F}' \neq \varnothing$. A graph G is *clique–Helly* if the collection of all cliques of G is Helly.

As in [6], we write $G \xrightarrow{\#} H$ if H is isomorphic to an induced subgraph H_0 of G such that every vertex $x \in G$ is dominated by some $y \in H_0$. By [6], H_0 can be obtained from G by removing one dominated vertex at a time so, by Proposition 2.6, $G \simeq H$ in this case. Note, however, that $G \xrightarrow{\#} H$ is a much stronger condition than $G \simeq H$, since it implies [6, Thm. 3] that $K(G) \xrightarrow{\#} K(H)$.

A graph G is *dismantlable* if there is an ordering $\{x_1, \ldots, x_n\}$ of its vertices where x_i is dominated in $G[x_i, x_{i+1}, \ldots, x_n]$ for $i = 1, \ldots, n-1$. Again by Proposition 2.6, dismantlable graphs are contractible.

3 Poset conditions

For $a \in P$, we will denote $a_* = \min(P) \cap P_{\leq a}$ and $a^* = \max(P) \cap P_{\geq a}$.

Definition 3.1 *We say that the poset P is:*

Up–Helly *if for any complete $\{X_1, \ldots, X_n\}$ of $\mho(P)$, there is $x \in \min(P)$ such that $x \leq X_i$ for all i, i. e., if the family $\{ X_* \mid X \in \max(P) \}$ is Helly.*
Down–Helly *if for any complete $\{x_1, \ldots, x_n\}$ of $\Omega(P)$, there is $X \in \max(P)$ such that $x_i \leq X$ for all i, i. e., if the family $\{ x^* \mid x \in \min(P) \}$ is Helly.*
Up–Sperner *if whenever $X, Y \in \max(P)$ and $X_* \subseteq Y_*$, then $X = Y$.*
Down–Sperner *if whenever $x, y \in \min(P)$ and $x^* \subseteq y^*$, then $x = y$.*
Atomized *if each subset of $\min(P)$ that has an upper bound in P has a least upper bound in P.*
Coatomized *if each subset of $\max(P)$ that has a lower bound in P has a greatest lower bound in P.*

Proposition 3.2 *Let P be any poset, and $x, y \in \min(P)$. Then:*

(1) if $x^* \subseteq y^*$ then x is dominated by y in $\Omega(P)$,
(2) if P is up–Helly and x is not dominated in $\Omega(P)$, x^* is a clique of $\mho(P)$,
(3) if P is up–Helly and down–Sperner, then x^* is a clique for all $x \in \min(P)$.

PROOF. (1): If $x^* \subseteq y^*$ and $z \in N_{\Omega(P)}[x]$, there is $X \in \max(P)$ such that $z, x \leq X$. Since $X \in x^* \subseteq y^*$ we have $y \leq X$. Hence $yz \in E(\Omega(P))$. For (2), suppose that P is up–Helly and x^* is not a clique in $\mho(P)$. Then there is $Z \in \max(P) - x^*$ with $x^* \cup \{Z\}$ complete. By the up–Helly condition, there is $z \in \min(P)$ less than all elements in $x^* \cup \{Z\}$. But then $x^* \subseteq z^*$ and so, by (1), x is dominated by z. For (3), we proceed as in the proof of (2) up to the point where we get $x^* \subseteq z^*$. By the down–Sperner condition, we would have $x = z$, a contradiction. Hence x^* is a clique. □

Theorem 3.3 *Let P be an up–Helly poset. Then $\Omega(P) \xrightarrow{\#} K(\mho(P))$.*

PROOF. Given $C = \{X_1, \ldots, X_n\}$ a clique of $\mho(P)$, by the up–Helly condition there is $x \in \min(P)$ such that $X_i \geq x$ for $i = 1, \ldots, n$. Define $\Phi(C)$ as one such x. It can be proven that Φ gives an embedding $\Phi\colon K(\mho(P)) \to \Omega(P)$, hence $\Omega(P)[\operatorname{im}\Phi] \cong K(\mho(P))$. Given $y \in \Omega(P)$, extend y^* to a clique C of $\mho(P)$. As $y^* \subseteq C = \Phi(C)^*$, y is dominated by $\Phi(C)$ by Proposition 3.2(1). □

If P is up–Helly and down–Sperner then, by Proposition 3.2(3), x^* is a clique for all $x \in \min(P)$, and the proof of Theorem 3.3 shows that Φ is an isomorphism. Therefore, we have:

Corollary 3.4 *Let P be a poset. If P is up–Helly and down–Sperner, then $K(\mho(P)) \cong \Omega(P)$. An isomorphism $\Omega(P) \to K(\mho(P))$ is given by $x \mapsto x^*$.*

4 The poset of complete subgraphs

In this section we fix a graph G, and $P = \mathcal{P}(G)$ is the poset of complete subgraphs of G, ordered by inclusion. Here $\min(P)$ can be identified with the vertices of G and $\max(P)$ is the set of cliques of G. Hence $\mho(P) = K(G)$ and $\Omega(P) = G$. This shows in particular that any graph is $\Omega(P)$ for some poset P. It is clear that for any G, P is down–Helly, up–Sperner, atomized and coatomized. The poset P is up–Helly if and only if G is clique–Helly, and P is down–Sperner whenever G has no dominated vertices. Thus, Theorem 3.3 and Corollary 3.4 give results of Escalante ([5]): if G is clique–Helly, then $G \xrightarrow{\#} K^2(G)$, and if in addition G has no dominated vertices, then it is K–periodic of period at most 2.

Definition 4.1 *We say that the poset P is:*

Join–increasing *if it is atomized and for bounded subsets $C \subsetneq D$ of $\min(P)$ we have $\operatorname{lub}_P C < \operatorname{lub}_P D$.*

Strongly up–Sperner *if it is atomized and $a = \operatorname{lub}_P a_*$ for all $a \in P$.*

The reader can prove that these properties characterize face posets:

Theorem 4.2 *A poset P is the face poset of a simplicial complex if and only if P is strongly up–Sperner and join–increasing.* □

Theorem 4.3 *Let P be a poset. Then the following statements are equivalent:*

(1) $P \cong \mathcal{P}(\Omega(P))$,
(2) $P \cong \mathcal{P}(\Omega(Q))$ for some poset Q,
(3) $P \cong \mathcal{P}(\mho(Q))$ for some poset Q,
(4) $P \cong \mathcal{P}(G)$ for some graph G,
(5) P is down–Helly, join–increasing and strongly up–Sperner.

PROOF. To prove (2) implies (3) we observe that $\mho(Q^{\mathrm{op}}) \cong \Omega(Q)$. The only nontrivial implication left to prove is that (5) implies (1). Let us assume that P is down–Helly, join–increasing and strongly up–Sperner. Define a poset map $P \to \mathcal{P}(\Omega(P))$ sending $x \mapsto x_*$ and, in the other direction, send $c = \{x_1, \ldots, x_n\}$ to $\mathrm{lub}_P\, c$. These maps are inverse to each other. □

We obtain as a corollary the result from [12, Prop. 6.3.11]: a face poset is the poset of completes of a graph if and only if it is down–Helly.

5 The posets of bounded complete subgraphs

We define the *poset of bounded complete subgraphs* $\mathcal{P}_b\mho(P)$ as the subposet of $\mathcal{P}(\mho(P))$ of completes $\{X_1, \ldots, X_r\}$ of $\mho(P)$ that are bounded below in P. Dually, we define $\mathcal{P}_b\Omega(P) = \mathcal{P}_b\mho(P^{\mathrm{op}})$. We have that $\mathcal{P}_b\mho(P) = \mathcal{P}(\Gamma(P, \max(P)))$, hence $\mathcal{P}_b\mho(P)$ is a face poset.

Proposition 5.1 *Let P be any finite poset. Then,*

(1) if P is up–Helly, then $\mathcal{P}_b\mho(P) = \mathcal{P}(\mho(P))$,
(2) if P is coatomized, then $P \simeq \mathcal{P}_b\mho(P)$,
(3) if P is up–Helly and coatomized, then $P \simeq \mho(P)$.

PROOF. (1): If P is up–Helly, all completes of $\mho(P)$ are bounded below in P, so $\mathcal{P}_b\mho(P) = \mathcal{P}(\mho(P))$. For (2), if P is coatomized then $\max(P)$ is a crosscut in P, so by Proposition 2.2 we have $\mathcal{P}_b\mho(P) = \mathcal{P}(\Gamma(P, \max(P))) \simeq P$. Then (3) follows from (1) and (2). □

For $P = \mathcal{P}(G)$ with G a graph, Proposition 5.1(3) gives a theorem of Prisner from [10]: if G is clique–Helly then $G \simeq K(G)$.

We now turn to homotopy properties of the posets of bounded complete subgraphs that are not consequences of the results in Section 3.

Proposition 5.2 *For any poset P, $\mathcal{P}_b\Omega(P) \simeq \mathcal{P}_b\mho(P)$.*

PROOF. Let $\mathcal{R} = \{\, (c, C) \in \mathcal{P}_b\Omega(P) \times \mathcal{P}_b\mho(P) \mid x \leq X$ for $x \in c,\ X \in C \,\}$. Then $\mathcal{R}$ is an ideal. If $c \in \mathcal{P}_b\Omega(P)$, then $\mathcal{R}_c$ has $C = \cap_{x \in c} x^*$ as maximum element, hence it is contractible. In a similar way, $\mathcal{R}_C$ is contractible for all $C \in \mathcal{P}_b\mho(P)$. The assertion now follows from Proposition 2.3. □

From this last proposition, Proposition 5.1(1) and its dual, we obtain:

Corollary 5.3 *If P is up–Helly and down–Helly, then $\Omega(P) \simeq \mho(P)$.* □

We shall give weaker conditions than up–Helly which instead of equality in Proposition 5.1(1) will yield homotopy equivalence, hence preserving the homotopy equivalence of Corollary 5.3.

Whenever P is coatomized, we define $h\colon P \to P$ by $h(a) = \mathrm{glb}_P\, a^*$. Then h is order preserving, and $a \leq h(a) = h(h(a))$ for all $a \in P$. We say that a subposet Q of a coatomized poset P is *nice* whenever $Q_{\leq h(x)}$ $(= \{\, y \in Q \mid y \leq h(x) \,\})$ is contractible for all $x \in P$.

Proposition 5.4 *If Q is a nice subposet of P then $P \simeq Q$.*

PROOF. Let $\mathcal{R} = \{\, (x, y) \in P^{\mathrm{op}} \times Q \mid y \leq h(x) \,\}$. Then $\mathcal{R}$ is an ideal, and given $x \in P$, we have $\mathcal{R}_x = Q_{\leq h(x)}$, which is contractible by hypothesis. Given $y \in Q$, we have $\mathcal{R}_y = \{\, x \mid y \leq h(x) \,\}$. Since for any $x \in \mathcal{R}_y$ we have that $h(x), y \in \mathcal{R}_y$ and $x \leq h(x) \geq y$, we obtain that for all y, $\mathcal{R}_y$ is conically contractible to y. Hence $P \simeq Q$ by Proposition 2.3. □

Corollary 5.5 *If $\mathcal{P}_b\mho(P)$ is nice in $\mathcal{P}(\mho(P))$ and $\mathcal{P}_b\Omega(P)$ is nice in $\mathcal{P}(\Omega(P))$, then $\Omega(P) \simeq \mho(P)$.* □

In the case that $P = \mathcal{P}(G)$ for some graph G, we obtain:

Theorem 5.6 *Let G be a graph and assume that $\mathcal{P}_b\mho(\mathcal{P}(G))_{\leq h(C)}$ is contractible for all completes C of $K(G)$. Then $G \simeq K(G)$.* □

For a family $C = \{q_1, \ldots, q_k\}$ of subsets of some set, we denote by $\Delta(C)$ the minimal simplicial complex with vertex set $\cup C$ such that each q_i is a simplex. In particular if C is a complete of $K(G)$, the vertices of $\Delta(C)$ are some vertices of G and C is the cover of maximal faces of $\Delta(C)$. We clearly have that $\mathcal{P}_b\mho(\mathcal{P}(G))_{\leq C} = \mathcal{P}(\mathcal{N}(C))$, and it follows by the Nerve Theorem 2.1 that $\mathcal{P}_b\mho(\mathcal{P}(G))_{\leq C} \simeq \Delta(C)$. Therefore the following is equivalent to Theorem 5.6:

Theorem 5.7 *Let G be a graph and assume that $\Delta(h(C))$ is contractible for all completes C of $K(G)$. Then $G \simeq K(G)$.* □

Note that in the particular case of theorems 5.6 and 5.7, $h(C)$ can be simply defined as the intersection of all the cliques $Q \in K^2(G)$ satisfying $C \subseteq Q$. It can be seen that the hypothesis in [8, Thm. 2.4] is equivalent to $\mathcal{P}_b\mho(\mathcal{P}(G))_{\leq h(C)}$ being *conically* contractible for all completes C of $K(G)$. As in [8], Theorem 5.7 implies that the only Whitney triangulation of a closed surface which is not homotopy equivalent to its clique graph is the octahedron. Here, a *Whitney triangulation* of a surface S is a graph G such that $|\Delta(G)| \cong S$.

6 The poset of atomic elements

For a poset P, we call an element $a \in P$ *atomic* if it is the least upper bound of the minimal elements below it. In this section we consider the subposet of all atomic elements of an atomized poset P.

Proposition 6.1 *Suppose that P is an atomized poset. Let P' be the subposet of P given by $P' = \{ a \in P \mid \mathrm{lub}_P\, a_* = a \}$. Then*

(1) $P \simeq P'$,
(2) if $a, b \in \max(P')$, $X \in \max(P)$ are such that $a, b \leq X$, then $a = b$. Hence $\{\{ X \in \max(P) \mid X \geq a \}\}_{a \in \max(P')}$ is a partition of the set $\{ X \in \max(P) \mid X \geq a \text{ for some } a \in \max(P') \}$.
(3) $\mho(P) \xrightarrow{\#} \mho(P')$. *A subgraph of $\mho(P)$ isomorphic to $\mho(P')$ can be obtained as the induced subgraph on a set of representatives of the partition of* (2).

PROOF. Define $f\colon P \to P'$ by $f(a) = \mathrm{lub}_P\, a_*$. Then f is surjective and order preserving, and $f(a) \leq a$ for all $a \in P$. This proves (1) by Proposition 2.4. With the hypothesis of (2), we obtain $a = f(a), b = f(b) \leq f(X) \in P'$. Since a, b are maximal in P', we get $a = f(X) = b$, proving (2). But note that the set at the end of (2) is not necessarily all of $\max(P)$. For (3), take a map $j\colon \mho(P') \to \mho(P)$ such that $j(a) \geq a$ for $a \in \max(P')$. This is injective by (2). If $ab \in E(\mho(P'))$ there is $x \in \min(P)$ with $x \leq a, b$, so $j(a)j(b) \in E(\mho(P))$. Clearly j is a graph isomorphism onto its image. We now show that each vertex in $\mho(P)$ is dominated by some vertex in $\mho(P)[\mathrm{im}\, j]$: Given $B \in \max(P)$, pick $b \in \max(P')$ with $f(B) \leq b$. Then $j(b)$ dominates B, since if $AB \in E(\mho(P))$, there is $x \in \min(P)$ with $x \leq A, B$, therefore $x \leq f(B) \leq b \leq j(b)$ and $Aj(b) \in E(\mho(P))$. □

We now focus on posets P of the form $P = \mathcal{P}(G)$ for a graph G. Since they are strongly up–Sperner, all elements of P are atomic, so in this case it is only interesting to consider *coatomic* elements, that is, those elements that are the greatest lower bound of the maximal elements above them. For a coatomized poset P define $P'' = h(P)$, where $h(a) = \mathrm{glb}_P\, a^*$ as before.

Lemma 6.2 *If P is coatomized, and $x, y \in \min(P)$ are such that $h(x) = h(y)$, then x, y are twins in $\Omega(P)$.* □

Lemma 6.3 *Let P be a down–Helly, up–Sperner and coatomized poset. Let $a \in P''$. Then:*
(1) If $x \in a_*$ and $y \in \min(P)$ dominates x in $\Omega(P)$, then $y \in a_*$.
(2) If $a \in \min(P'')$, then any pair of elements of a_* are twins in $\Omega(P)$. □

The *pared graph* $\mathrm{Pared}\, G$ of a graph G was defined by Prisner in [10]. Its vertices can be taken as the nonempty sets of vertices D in G such that any pair of vertices in D are twins and no vertex in D is dominated by a vertex outside D. For $D_1, D_2 \in V(\mathrm{Pared}\, G)$, we put $D_1 D_2 \in E(\mathrm{Pared}\, G)$ if there are $x \in D_1$, $y \in D_2$ such that $xy \in E(G)$. The reader can check that Theorem 6.4 follows from Lemmas 6.2 and 6.3.

Theorem 6.4 *For any graph G,* $\min(P(G)'') = V(\mathrm{Pared}\, G)$ *and, moreover,* $\Omega(P(G)'') \cong \mathrm{Pared}\, G$. □

From the dual of Proposition 6.1(3) we obtain that $G \xrightarrow{\#} \operatorname{Pared} G$ for any graph G, and so $G \simeq \operatorname{Pared} G$, which is Proposition 3.2 in [10].

7 The poset of intervals

Let P be any poset. Then $\operatorname{Int} P$ is the poset of all intervals $[a,b] \subseteq P$, ordered by inclusion. It is shown in [13] that $\operatorname{Int} P$ is homeomorphic to P.

The minimal elements in $\operatorname{Int} P$ are the intervals of the form $[a,a]$ and so $\min(\operatorname{Int} P)$ can be identified with P. We will write $[a,a]$ as $[a]$. We also have $\max(\operatorname{Int} P) = \{\, [x,X] \mid x \in \min(P), X \in \max(P), x \le X \,\}$. Hence $\Omega(\operatorname{Int} P)$ is a graph with vertices $\{\, [a] \mid a \in P \,\}$ with two vertices $[a], [b]$ adjacent if there is an interval $[x,y]$ of P containing both a and b. That is, $[a][b] \in E(\Omega(\operatorname{Int} P))$ if and only if $a \neq b$ and $\{a,b\}$ is both bounded above and below in P. In other words, $\Omega(\operatorname{Int} P)$ is $\operatorname{DB}(P)$, the double bound graph of P introduced in [9,4]. Two vertices $[x,X]$, $[y,Y]$ in $\mho(\operatorname{Int} P)$ are adjacent whenever there is $a \in P$ that is both an upper bound of $\{x,y\}$ and a lower bound of $\{X,Y\}$.

Since $\operatorname{Int} P^{\mathrm{op}} = \operatorname{Int} P$, we have that for any property α about P that implies or is implied by a property β about $\operatorname{Int} P$, the dual property of α also implies or is implied by β.

Proposition 7.1 *Let P be an atomized and coatomized poset and $a \in P$.*

(1) If $[a]$ is not dominated in $\Omega(\operatorname{Int} P)$, then $a \in P' \cap P''$.
(2) If $a \in P' \cap P''$ then $[a]^*$ is a clique in $\mho(\operatorname{Int} P)$.

PROOF. (1) follows from the fact that for all $a \in P$, the elements $[\operatorname{lub}_P a_*]$ and $[\operatorname{glb}_P a^*]$ dominate $[a]$ in $\Omega(\operatorname{Int} P)$. (2): Let $a \in P' \cap P''$ and suppose that $[a]^*$ is not a clique in $\mho(\operatorname{Int} P)$. Then there is an interval $[x,X]$ with $x \in \min(P)$, $X \in \max(P)$ that does not contain a, but intersects all maximal intervals that contain a. This implies that x is a lower bound of a^* and X is an upper bound of a_*. But then we have $x \le a \le X$, a contradiction. □

Proposition 7.2 *If P is a strongly up–Sperner poset, then:*

(1) $\Omega(\operatorname{Int} P)$ has no pair of distinct twins.
(2) If $[a]$ is dominated in $\Omega(\operatorname{Int} P)$, then $[a]^*$ is not a clique in $\mho(\operatorname{Int} P)$.

PROOF. Suppose $[a]$ is dominated by $[b]$ in $\Omega(\operatorname{Int} P) \cong \operatorname{DB}(P)$. If $x \in a_*$ then $[x][b] \in E(\Omega(\operatorname{Int} P))$, which implies $x \le b$, and so $a_* \subseteq b_*$. The strongly up–Sperner condition implies then that $a \le b$. From this, (1) follows. For (2), suppose that $[a]$ is dominated by $[b]$ in $\Omega(\operatorname{Int} P)$ with $a \neq b$. Then $a_* \subsetneq b_*$, so we can take $y \in b_* - a_*$. Let $Y \in \max(P)$ such that $a \le Y$. Then $a \notin [y,Y]$, but if $a \in [x,X]$ with $x \in \min(P)$, $X \in \max(P)$, then $b \in [x,X] \cap [y,Y]$. Hence $[a]^*$ is not a clique. □

From Propositions 7.1 and 7.2 we obtain:

Corollary 7.3 *If P is strongly up–Sperner and coatomized, and $a \in P$, the following are equivalent:*

(1) $[a]$ *is not dominated in* $\Omega(\operatorname{Int} P)$,
(2) $a \in P''$,
(3) $[a]^*$ *is a clique in* $\mho(\operatorname{Int} P)$. □

For the rest of the section, $P = \mathcal{P}(G)$ for G a graph. In this case, $\operatorname{Int} P$ is up–Helly, and strongly up–Sperner.

The *vertex–clique bipartite graph* of G is defined as the graph $BK(G)$ with $V(BK(G)) = V(G) \cup V(K(G))$, $E(BK(G)) = \{ xC \mid x \in G, C \in K(G), x \in C \}$. The *edge graph* $\mathcal{E}(G)$ (see [1]) has $V(\mathcal{E}(G)) = E(G)$ and two edges of G are adjacent vertices in $\mathcal{E}(G)$ if they intersect or are opposite edges of a 4–cycle in G. The vertex–clique bipartite graph was denoted in [1] as $I(G)$.

The graph $\Omega(\operatorname{Int} \mathcal{P}(G)) \cong \mathrm{DB}(\mathcal{P}(G))$ has as vertices the complete subgraphs of G and two distinct completes C, D are adjacent if $C \cap D \neq \varnothing$ and $C \cup D$ is complete. On the other hand $\mho(\operatorname{Int} \mathcal{P}(G))$ has as vertices the pairs (x, C), where x is a vertex of G and C is a clique of G with $x \in C$, and two distinct pairs (x, C), (y, D) are adjacent if $\{x, y\} \subseteq C \cap D$. That is, $\mho(\operatorname{Int} \mathcal{P}(G)) = \mathcal{E}(BK(G))$. We will denote $\mathcal{E}(BK(G))$ just as $H(G)$.

By Proposition 5.1(3), we have $\mathcal{P}(G) \simeq \operatorname{Int} \mathcal{P}(G) \simeq \mho(\operatorname{Int} \mathcal{P}(G)) = H(G)$, so:

Theorem 7.4 *For any graph G, $G \simeq H(G)$.* □

We now turn to the clique graph of $H(G)$.

Proposition 7.5 *For any graph G, $K(H(G)) \cong \mathrm{DB}(P(G)'')$.*

PROOF. Theorem 3.3 gives an embedding $\Phi \colon K(H(G)) \to \Omega(\operatorname{Int} \mathcal{P}(G))$ that sends a clique in $H(G)$, say $\mathcal{C} = \{[x_1, C_1], \ldots, [x_n, C_n]\}$ to a lower bound in $\min(\operatorname{Int}(\mathcal{P}(G)))$, which in this case must be $[\cap C_i]$. Conversely, if $C \in \mathcal{P}(G)''$, then $[C]^*$ is a clique in $H(G)$ by Corollary 7.3, so $[C] = \Phi([C]^*)$. □

Theorem 7.6 *For any graph G, $K(H(G)) \simeq G$.*

PROOF. There is a poset map $f \colon \mathcal{P}(\mathrm{DB}(\mathcal{P}(G))) \to \mathcal{P}(G)$ sending the complete $\{C_1, \ldots, C_n\}$ in $\mathrm{DB}(\mathcal{P}(G))$ to $\cup_{i=1}^n C_i$. For a fixed $D \in \mathcal{P}(G)$ we have that $F = f^{-1}(\mathcal{P}(G)_{\leq D}) = \{ \{C_1, \ldots, C_n\} \in \mathcal{P}(\mathrm{DB}(\mathcal{P}(G))) \mid \cup_{i=1}^n C_i \subseteq D \}$ is join contractible to $\{D\}$, since if $\{C_1, \ldots, C_n\} \in F$, then $\{C_1, \ldots, C_n\} \cup \{D\}$ is a complete subgraph in $\mathrm{DB}(\mathcal{P}(G))$ and an element of F. By Theorem 2.5, f is a homotopy equivalence, and the result then follows, since from Theorem 3.3, we obtain that $\mathrm{DB}(\mathcal{P}(G)) = \Omega(\operatorname{Int} \mathcal{P}(G)) \xrightarrow{\#} K(H(G))$. □

By theorems 3 and 4 of [1], a graph G is dismantlable if and only if $H(G)$ is dismantlable, and G is clique–Helly if and only if $H(G)$ is clique–Helly. The operator K preserves clique–Hellyness ([5]) and dismantlability ([2]). Hence we obtain a corollary of theorems 7.4 and 7.6:

Corollary 7.7 *Any composition of graph operators* $T = T_n \circ T_{n-1} \circ \cdots \circ T_1$ *such that* $T_i \in \{H, K \circ H\}$ *for all* $i = 1, \ldots, n$, *preserves clique–Hellyness, dismantlability, and homotopy type.* □

Acknowledgments. We thank the anonymous referees for many detailed and valuable suggestions that helped us to improve this paper.

References

[1] H.J. Bandelt, M. Farber and P. Hell. *Absolute reflexive retracts and absolute bipartite retracts.* Discrete Appl. Math. **44** (1993) 9–20.

[2] H.J. Bandelt and E. Prisner. *Clique graphs and Helly graphs.* J. Combin. Theory Ser. B **51** (1991) 34–45.

[3] A. Björner. Topological methods. In *Handbook of combinatorics, Vol. 1, 2*, pages 1819–1872. Elsevier, Amsterdam, 1995.

[4] D. Diny. *The double bound graph of a partially ordered set.* J. Combin. Inform. System Sci. **10** (1985) 52–56.

[5] F. Escalante. *Über iterierte Clique-Graphen.* Abh. Math. Sem. Univ. Hamburg **39** (1973) 59–68.

[6] M.E. Frías-Armenta, V. Neumann-Lara and M.A. Pizaña. *Dismantlings and iterated clique graphs.* Discrete Math. **282** (2004) 263–265.

[7] F. Harary. *Graph theory.* Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London, 1969.

[8] F. Larrión, V. Neumann-Lara and M.A. Pizaña. *On the homotopy type of the clique graph.* J. of the Brazilian Comp. Soc. **7** (2002) 69–73.

[9] F.R. McMorris and T. Zaslavsky. *Bound graphs of a partially ordered set.* J. Combin. Inform. System Sci. **7** (1982) 134–138.

[10] E. Prisner. *Convergence of iterated clique graphs.* Discrete Math. **103** (1992) 199–207.

[11] D. Quillen. *Homotopy properties of the poset of nontrivial p-subgroups of a group.* Adv. in Math. **28** (1978) 101–128.

[12] B.S.W. Schröder. *Ordered sets.* Birkhäuser Boston Inc., Boston, MA, 2003.

[13] J.W. Walker. *Canonical homeomorphisms of posets.* European J. Combin. **9** (1988) 97–107.