On the Homotopy Type of the Clique Graph*

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Abstract

If G is a graph, its clique graph $K(G)$ is the intersection graph of all its (maximal) cliques. The complex $G^\uparrow$ of a graph G is the simplicial complex whose simplexes are the vertex sets of the complete subgraphs of G.

Here we study a sufficient condition for $G^\uparrow$ and $K(G)^\uparrow$ to be homotopic. Applying this result to Whitney triangulations of surfaces, we construct an infinite family of examples which solve in the affirmative Prisner's open problem 1 in *Graph Dynamics* (Longman, Harlow, 1995): Are there finite connected graphs G that are periodic under K and where the second modulo 2 Betti number is greater than 0?

Keywords: clique graphs, clique convergence, Whitney triangulations, clean triangulations, simplicial complexes, modulo 2 Betti numbers.

1 Introduction and Terminology

All our graphs are simple. If G is a graph, a *complete* of G is a complete subgraph of G and a *clique* is a maximal complete of G. The clique number $\omega(G)$ is the maximum order of a clique of G. We shall often identify induced subgraphs with their vertex sets. In particular, we shall often write $x \in G$ instead of $x \in V(G)$.

We say that G is *locally* H if the subgraph $N_G(x)$ induced in G by the (open) neighbourhood of any vertex $x \in G$ is isomorphic to H. We say that G is *locally* $\mathcal{H} = \{H_1, H_2, \dots\}$ if for every $x \in G$, $N_G(x) \cong H_i$ for some $H_i \in \mathcal{H}$. C_n and P_n are, respectively, the cyclic and path graphs on n vertices. We say that G is *locally cyclic* if it is locally $\{C_n : n \geq 3\}$.

The *clique graph* $K(G)$ of G has all cliques of G as vertices, two of them being adjacent iff they (are different and) share some vertex of G. We call K the *clique operator*. Iterated clique graphs are inductively defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. G is K*-periodic* if $G \cong K^n(G)$ for some $n \geq 1$. Extensive bibliography on clique graphs can be found in [14].

A graph G is clique-Helly if whenever $X = \{q_1, \dots, q_n\} \subseteq V(K(G))$ is a family of pairwise intersecting cliques, then $\bigcap X \neq \varnothing$. We say that $Q = \{q_1, \dots, q_n\} \in V(K^2(G))$ is a *star* of G if $\bigcap Q \neq \varnothing$, otherwise it is a *necktie* of G. Obviously, a graph is clique-Helly iff it has no necktie.

If G is a graph, $G^\uparrow$ is the simplicial complex whose simplexes are the completes of G. We say that two simplicial complexes are homotopic ($\mathbb{K} \simeq \mathbb{L}$) when their geometric realizations are homotopic ($|\mathbb{K}| \simeq |\mathbb{L}|$). The behaviour of topological invariants of $G^\uparrow$ under several graph operators (including the clique operator) has been studied in [9, 10, 11]. In particular, Prisner proved in [10] that if G is clique-Helly, $G^\uparrow \simeq K(G)^\uparrow$. Our main result (Theorem 2.4) states that this is also true for many non-clique-Helly graphs. As an application of this, we will show (Theorem 2.5) that if G is free of tetrahedra and induced octahedra, then $G^\uparrow \simeq K(G)^\uparrow$.

An interesting particular case is when the realization $|\mathbb{K}|$ is a compact surface (with or without border), i.e. $\mathbb{K}$ is a triangulation of a compact surface. If G is the underlying graph (or 1-skeleton) of a surface triangulation $\mathbb{K}$, every face of $\mathbb{K}$ is a triangle of G but the converse may not be true. We shall be interested

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in surface triangulations where every triangle of G is a face of $\mathbb{K}$: such a triangulation is a *Whitney triangulation* [17]. Thus, if $\mathbb{K}$ is Whitney it is determined by G, and we tend to identify G with $\mathbb{K}$, and sometimes even with $|\mathbb{K}|$. If $\mathbb{K}$ is Whitney, (except for the tetrahedron K_4) the cliques of G are precisely the faces of the triangulation. Whitney triangulations have other names and have been studied before [2, 5, 8, 16, 17]. In particular, the description of the dynamical behaviour under the clique operator of the regular Whitney triangulations has been completed in [8]. As a corollary to our Theorem 2.5, we will have that the only Whitney triangulation of a compact surface which is not homotopic to its clique graph is the octahedron. We shall use the following two theorems:

Theorem 1.1 [8] *G is the underlying graph of a Whitney triangulation of a closed surface (resp. compact surface) if and only if G is locally cyclic (resp. G is locally $\{C_n, P_m : n \geq 3, m \geq 2\}$).* □

Theorem 1.2 [8] *For every Whitney triangulation G of a closed surface with minimum degree at least 7 we have $K(G) \cong K^3(G)$.* □

We refer to [1], [10] and [13] for undefined concepts.

2 Homotopy

If $\mathcal{H}$ is a hypergraph, $\mathcal{H}^*$ denotes its dual hypergraph, and $\mathcal{H}^{\downarrow}$ is the smallest simplicial complex containing the hyperedges of $\mathcal{H}$ as simplexes. The following reformulation is due to Prisner [9, 10]:

Theorem 2.1 (Dowker, [3]) *For every hypergraph $\mathcal{H}$, $\mathcal{H}^{\downarrow}$ and $\mathcal{H}^{*\downarrow}$ are homotopic.* □

If G is a graph, $\mathcal{K}(G)$ is its *clique hypergraph*: $\mathcal{K}(G)$ has the same vertex set as G and its hyperedges are the cliques of G. It follows immediately from the definitions that $G^{\uparrow} = \mathcal{K}(G)^{\downarrow}$.

The *star hypergraph* $\mathcal{S}(G)$ of G has the same vertex set as $K(G)$ and the hyperedges are the cliques $Q = \{q_1, q_2, \ldots, q_r\}$ of $K(G)$ satisfying $\bigcap Q \neq \varnothing$. It follows that $\mathcal{K}(G)^{*\downarrow} = \mathcal{S}(G)^{\downarrow}$ and that $\mathcal{S}(G)^{\downarrow} \subseteq K(G)^{\uparrow}$. The equality $\mathcal{S}(G)^{\downarrow} = K(G)^{\uparrow}$ holds precisely when G is clique-Helly.

Then, as pointed out by Prisner [10, Proposition 2.2], it follows from Dowker's theorem that $G^{\uparrow}$ and $K(G)^{\uparrow}$ are homotopic for every clique-Helly graph G. A reformulation of this result will be useful to us:

Theorem 2.2 (Prisner, [10]) *For every graph G, we have $G^{\uparrow} = \mathcal{K}(G)^{\downarrow} \simeq \mathcal{K}(G)^{*\downarrow} = \mathcal{S}(G)^{\downarrow} \subseteq K(G)^{\uparrow}$. In particular, if G is clique-Helly, then $G^{\uparrow} \simeq K(G)^{\uparrow}$.* □

Prisner provided examples of graphs G (namely the n-dimensional octahedra, for $n \geq 3$) such that $G^{\uparrow}$ and $K(G)^{\uparrow}$ are not homotopic. As we shall see shortly, this property of the octahedra is tightly connected to the fact that octahedra contain neckties without a center.

Definition 2.3 *If X is a complete of $K(G)$ satisfying $\bigcap X = \varnothing$, then $q_0 \in K(G)$ is called a* center *of X if:*

$$Y \subseteq X \text{ and } \bigcap Y \neq \varnothing \text{ imply } \bigcap (Y \cup \{q_0\}) \neq \varnothing.$$

Note that $X \cup \{q_0\}$ is always a complete of $K(G)$. Also, when such an X is a clique of $K(G)$, X must contain all its centers.

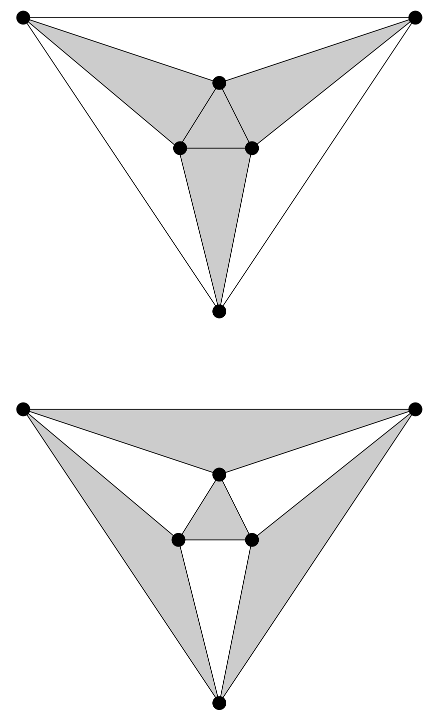

Figure 1: Two neckties of the octahedron, with center (above) and without center (below).

Many non-Helly graphs G satisfy $G^{\uparrow} \simeq K(G)^{\uparrow}$, Indeed we shall show that for many non-Helly graphs G, $\mathcal{S}(G)^{\downarrow}$ is a strong deformation retract of $K(G)^{\uparrow}$.

Let's rename $\mathbb{S} = \mathcal{S}(G)^{\downarrow}$ and $\mathbb{K} = K(G)^{\uparrow}$. We know that $\mathbb{S} \subseteq \mathbb{K}$. Note that the 0-simplexes of $\mathbb{S}$ and $\mathbb{K}$ are the same. In order to easily define the required mappings, we take the barycentric subdivision $\mathbb{K}'$ of $\mathbb{K}$ relative to $\mathbb{S}$ as used in [12, page 19].

Equivalently, we define the complex $\mathbb{K}'$ whose vertices are those of $\mathbb{K}$ (denoted by q_i) plus a (formal) barycenter $b(s)$ for each $s \in \mathbb{K} - \mathbb{S}$, and whose simplexes are of the form $\{q_1, \dots, q_n, b(s_1), \dots, b(s_m)\}$ and satisfy:

1. $\{q_1, \dots, q_n\} \in \mathbb{S}$.
2. $s_j \in \mathbb{K} - \mathbb{S}$ for all j.
3. $q_i \in s_1$ for all i.
4. $s_j \subseteq s_{j+1}$ for all j.

and then we may prove that this is indeed a subdivision of $\mathbb{K}$ using Theorem 3.3.4 in [13]. Of course, we still have $\mathbb{S} \subseteq \mathbb{K}'$.

The idea behind this is to grab the offending simplexes (those in $\mathbb{K} - \mathbb{S}$) by its barycenters and retract them into $\mathbb{S}$. Now we can prove our main result:

Theorem 2.4 *Let G be a graph. Assume that any complete X of $K(G)$ with $\bigcap X = \varnothing$ has a center which belongs to every necktie containing X. Then $\mathcal{S}(G)^{\downarrow}$ is a strong deformation retract of $K(G)^{\uparrow}$. In particular, $G^{\uparrow} \simeq K(G)^{\uparrow}$.*

Proof. For every simplex s in $\mathbb{K} - \mathbb{S}$ select, using the hypothesis, a fixed center $q(s)$ of s belonging to every maximal simplex (i.e. necktie) that contains s. Also, for each $s \in \mathbb{K} - \mathbb{S}$, define

$$\widehat{s} = \bigcap \{Q \in K^2(G) : s \subseteq Q\}.$$

Note: $s \subseteq \widehat{s} \in \mathbb{K} - \mathbb{S}$, and $s \subseteq s'$ implies $q(\widehat{s}) \in \widehat{s} \subseteq \widehat{s'}$.

Now define the map $\varphi_1 : \mathbb{K}' \to \mathbb{K}$ by $\varphi_1(q_i) = q_i$ and $\varphi_1(b(s_j)) = q(\widehat{s_j})$. Then for any simplex of $\mathbb{K}'$ we have that $\varphi_1(\{q_1, \dots, q_n, b(s_1), \dots, b(s_m)\}) = \{q_1, \dots, q_n, q(\widehat{s_1}), \dots, q(\widehat{s_m})\}$. This is a simplex of $\mathbb{K}$ because there is a clique Q of $K(G)$ such that $q_i \in Q$ and $s_j \subseteq Q$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$ (take a Q with $s_m \subseteq Q$). Therefore $q_i, q(\widehat{s_j}) \in Q$ for all i and j. It follows that $\varphi_1 : \mathbb{K}' \to \mathbb{K}$ is a simplicial map, so $|\varphi_1| : |\mathbb{K}'| \to |\mathbb{K}|$ is continuous.

We claim now that $\mathrm{Im}(\varphi_1) = \mathbb{S}$: As $q_1 \cap \cdots \cap q_n \neq \varnothing$ and $\{q_1, \cdots, q_n\} \subseteq s_1 \subseteq \widehat{s_1}$, we obtain that $q_1 \cap \cdots \cap q_n \cap q(\widehat{s_1}) \neq \varnothing$. Using that $\widehat{s_1} \subseteq \widehat{s_2} \subseteq \cdots \subseteq \widehat{s_m}$ and $q(\widehat{s_j}) \in \widehat{s_j}$ for all j, it follows by induction that $\{q_1, \dots, q_n, q(\widehat{s_1}), \dots, q(\widehat{s_m})\}$ is a simplex of $\mathbb{S}$. Now we know that $\mathrm{Im}(|\varphi_1|) = |\mathbb{S}|$ and that the restriction of $|\varphi_1|$ to $|\mathbb{S}|$ is the identity in $|\mathbb{S}|$.

On the other hand, consider the canonical homeomorphism $\varphi_0 : |\mathbb{K}| \to |\mathbb{K}'|$. Let $\varphi = |\varphi_1| \circ \varphi_0$. Note that for all $x \in |\mathbb{K}|$ there is a simplex $s \in \mathbb{K}$ such that $x, \varphi(x) \in |s|$ (any maximal simplex $s \in \mathbb{K}$ satisfying $x \in |s|$ will do). Then it follows that $\varphi \simeq 1_{|\mathbb{K}|}$ via the homotopy $H(x,t) = tx + (1-t)\varphi(x)$ (see, for example [6, Prop. 1.7.5]). Since $\varphi \big|_{|\mathbb{S}|} = 1_{|\mathbb{S}|}$, we have that $H(x,t) = x$ for all $x \in |\mathbb{S}|$. Therefore $|\mathbb{S}|$ is and strong deformation retract of $|\mathbb{K}|$. □

An interesting consequence is the following:

Theorem 2.5 *If G is a graph free of induced octahedra and $\omega(G) \leq 3$, then $G^{\uparrow} \simeq K(G)^{\uparrow}$*

Proof. Without loss of generality we assume G to be connected and non-trivial. Then we observe that every clique of G is a triangle or an edge.

Let X be a complete of $K(G)$ satisfying $\bigcap X = \varnothing$, and let $Z = \{q_1, \dots, q_r\}$ be a minimal subset of X also satisfying $\bigcap Z = \varnothing$.

Since Z is minimal and necessarily $r \geq 3$, we may take $x_{23} \in \bigcap(Z - \{q_1\})$, $x_{13} \in \bigcap(Z - \{q_2\})$ and $x_{12} \in \bigcap(Z - \{q_3\})$. Hence, $q_0 = \{x_{12}, x_{13}, x_{23}\}$ is a clique of G. This very construction was used by J. L. Szwarcfiter in his celebrated characterization of clique-Helly graphs [15].

It follows that $q_1 = \{x_{12}, x_{13}, a\}$, $q_2 = \{x_{12}, x_{23}, b\}$ and $q_3 = \{x_{13}, x_{23}, c\}$ for some three (different) vertices $a, b, c \in G$. Since $q_1 \cap q_2 \cap q_3 = \varnothing$ it follows that $Z = \{q_1, q_2, q_3\}$.

Let $Q \in K^2(G)$ be a necktie containing Z, and let $q \in Q$. If $q \cap q_0 = \varnothing$, then $q = \{a, b, c\}$ and the set of vertices $\{x_{12}, x_{13}, x_{23}, a, b, c\}$ induces an octahedron in G, contradicting our hypotheses. If $|q \cap q_0| = 1$, say $q \cap q_0 = \{x_{12}\}$, then $q \cap q_3 = \{c\}$ and $\{x_{12}, x_{13}, x_{23}, c\}$ would contradict $\omega(G) \leq 3$. Therefore $|q \cap q_0| \geq 2$ for every $q \in Q$.

Since the set $\{q \in K(G) : |q \cap q_0| \geq 2\}$ is a complete of $K(G)$ it follows that $Q = \{q \in K(G) : |q \cap q_0| \geq 2\}$. Now the condition on the clique number implies that q_0 is a center of Q. Then Q is the unique necktie containing Z, so it is also unique containing X. Therefore q_0 is a center of X which belongs to every necktie containing X, and we apply the previous theorem. □

The following result is an immediate consequence:

Corollary 2.6 *The only Whitney triangulation of a compact surface (with or without border) which is not homotopic to its clique graph is the octahedron.* □

Now let's denote the i-th modulo 2 Betti number of a complex $\mathbb{K}$ by $\hat{\beta}_i(\mathbb{K})$. Take any locally $\{C_t : t \geq 7\}$ graph H. By Theorem 1.1 H is a Whitney triangulation of a closed surface, so we have $\hat{\beta}_2(H^{\uparrow}) = 1$. Since

$H^{\uparrow} \simeq K(H)^{\uparrow}$, we have $\hat{\beta}_2(K(H)^{\uparrow}) = 1$. Then Theorem 1.2 tells us that $G := K(H)$ is K-periodic, thus solving Prisner's open problem 1 in [11].

As a concrete example, it is shown in [8] that $I \times K_3$ is a locally C_{10} graph (here I is the icosahedron and $\{(a,b),(a',b')\} \in E(A \times B)$ iff $\{a,a'\} \in E(A)$ and $\{b,b'\} \in E(B)$). In fact, Brown and Connelly [2] proved that for every t there is at least one finite locally C_t graph. Next, we shall construct an explicit infinite family of locally C_7 graphs.

3 Whitney triangulations

Let's start with an infinite graph T: $V(T) = \mathbb{Z} \oplus \mathbb{Z}$ and put $N = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$, then define $\{x,y\} \in E(T)$ if and only if $y - x \in N$.

Each vector $u \in \mathbb{Z} \oplus \mathbb{Z}$ gives rise to a translation $x \mapsto u + x$ which is an automorphism of T. Every finite locally C_6 graph triangulating the torus is a quotient T/Γ where Γ is the translation group generated by the translations given by two linearly independent vectors $u, v \in \mathbb{Z} \oplus \mathbb{Z}$. The group Γ must satisfy the following admissibility condition: for every $\gamma \in \Gamma$ and $v \in V(T)$, the distance in T from v to $\gamma(v)$ is at least 4 (otherwise, the resulting triangulation is not Whitney, see [7]).

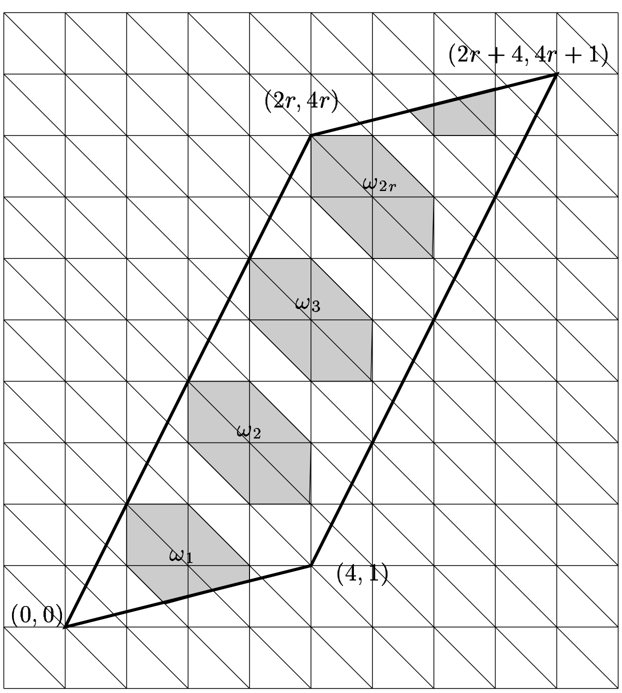

Figure 2: The parallelogram $\mathcal{P}_r$, for $r = 2$.

Let $u = (4,1)$, fix $r \geq 2$, and let $v_r = (2r, 4r)$. Let Γ_r be the translation group defined by u and v_r, and let $\mathcal{P}_r$ be the parallelogram defined by these two vectors. The locally C_6 graph $G_r = T/\Gamma_r$ defines a Whitney triangulation of the torus with $14r$ vertices: G_r is obtained by identifying the parallel edges of $\mathcal{P}_r$.

Now consider the $2r$ vertices $w_1, w_2, ..., w_{2r}$ of G_r which correspond to the vertices $(2,1), (3,3), ..., (2r+1, 4r-1)$ in $\mathcal{P}_r$, i.e. $w_i = (i+1, 2i-1)$. The vertices of G_r are the disjoint union of the closed neighbourhoods $N[w_i]$ of these vertices, and removing these vertices from G_r we obtain a locally P_5 graph G'_r of order $12r$. Let us call $\mathcal{S}_r$ the surface triangulated by G'_r, which is a torus with $2r$ open disks removed. All the vertices of G'_r lie in the border of $\mathcal{S}_r$. The connected components of the border of $\mathcal{S}_r$ are the hexagons $H_1, H_2, ..., H_{2r}$ which were the open neighbourhoods of the removed vertices $w_1, w_2, ..., w_{2r}$ of G_r.

Consider the locally P_4 graph C_{12}^2 in figure 3. This graph gives us a Whitney triangulation of a cylinder, all the vertices lie in the border whose connected components B_1 and B_2 are induced hexagons of C_{12}^2.

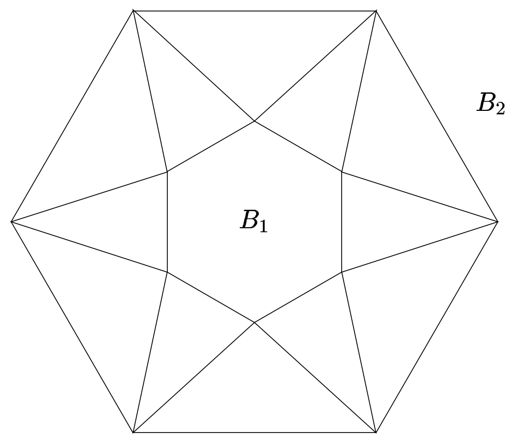

Figure 3: The graph C_{12}^2

Now, take the surface $\mathcal{S}_r$ (with its Whitney triangulation given by the graph G'_r) and r different copies of the cylinder (with the Whitney triangulation given by C_{12}^2). For the first copy, identify B_1 with H_1 and B_2 with H_{r+1} in an orientable manner, so a handle is glued to $\mathcal{S}_r$. For the second copy, identify B_1 with H_2 and B_2 with H_{r+2}, so a second handle is glued to $\mathcal{S}_r$. Continuing in this way, we obtain at the end a closed surface $\mathcal{S}'_r$ which is a sphere with $r+1$ handles. The graph $\overline{G}_r$ obtained from G'_r and the r copies of C_{12}^2 by the above method has $12r$ vertices and is the 1-skeleton of a triangulation of our surface $\mathcal{S}'_r$.

As we want $\overline{G}_r$ to be locally C_7 we have to take care so that the triangles in $\overline{G}_r$ are exactly the triangles already present ($16r$ in G'_r and 12 in each copy of C_{12}^2). This fails when two vertices $x \in H_i$ and $y \in H_{i+r}$ with $d(x,y) < 3$ in G'_r are identified with adjacent vertices in the i-th copy of C_{12}^2. Since $d(H_i, H_{r+i}) = r$, there is no problem for $r \geq 3$.

In case $r = 2$, there is an essentially unique way to glue the 2 copies of C_{12}^2 in such a way that no new triangles are created, and this produces a triangulation of the orientable closed surface of genus 3 (the "triple torus"). We verified this by computer using GAP [4]. It can be shown that the double torus does not admit a locally C_7 triangulation.

Notice that for $r \geq 3$ the construction allows more freedom at the time of gluing (so in principle more than one example may have been constructed at each genus $g > 3$) and that even non-orientable surfaces are obtained gluing one handle in a non-orientable manner. So we have proved:

Theorem 3.1 *Every orientable surface of genus at least* 3*, and every non-orientable surface with even Euler characteristic* $\chi \leq -6$ *admits a locally* C_7 *triangulation* G*. For any such* G*,* $K(G)$ *is a positive answer to Prisner's open problem* 1 *in* [11]. □

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