The Icosahedron is Clique Divergent

Miguel A. Pizaña

Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica. Av. Michoacán y Purísima s/n Col. Vicentina. México 09340 D.F. MÉXICO

Abstract

A clique of a graph G is a maximal complete subgraph. The clique graph $k(G)$ is the intersection graph of the set of all cliques of G. The iterated clique graphs are defined recursively by $k^0(G) = G$ and $k^{n+1}(G) = k(k^n(G))$. A graph G is said to be clique divergent (or $k-$divergent) if $\lim_{n\to\infty} |V(k^n(G))| = \infty$. The problem of deciding whether the icosahedron is clique divergent or not was (implicitly) stated in [13](1981) and cited in [14](1991) and [9](1999). This paper proves the clique divergence of the icosahedron among other results of general interest in clique divergence theory.

Key words: Clique graph, iterated clique graphs, clique divergence, icosahedron.

1 Introduction

Iterated clique graphs were introduced by Hedetniemi and Slater in [5](1972) and have been studied in several papers including: [1,3,6–9,11–15]. For a large bibliography see [16,17].

A graph G is said to be $k-$*convergent* if $k^n(G) \cong k^m(G)$ for some $n \neq m$; when $m = 0$, we say that G is k^n-*invariant*. G is $k-$*null* if $k^n(G)$ is trivial (one vertex) for some n (clearly a special case of a $k-$convergent graph). It is easy to see that every graph is either $k-$convergent or $k-$divergent; we call this attribute the $k-$*behaviour* of the graph.

Except for the icosahedron, the $k-$behaviour of every platonic graph has been long known: the tetrahedron is a complete graph and therefore $k-$null, the cube and the dodecahedron are k^2-invariant [5](1972), the octahedron was

Email address: map@xanum.uam.mx (Miguel A. Pizaña).
URL: http://xamanek.uam.mx/map (Miguel A. Pizaña).

the first known example of a $k-$divergent graph [3](1973), [12] (1978). Since then various families of graphs have been found to be $k-$divergent. A graph G is said to be *locally* H if the set of neighbors of any vertex of G induces a subgraph isomorphic to H. $\mathcal{C}_n$ is the cyclic graph with n vertices. Larrión and Neumann-Lara [9](1999) proved that every locally $\mathcal{C}_6$ graph is $k-$divergent but noted that while the $k-$behaviour of any locally $\mathcal{C}_n$ graph was well understood for $n \leq 4$, the $k-$behaviour of the only (connected) locally $\mathcal{C}_5$ graph, the icosahedron, was still unknown.

In section 2, we shall study the relation between distances on a graph and distances on its clique graph. Theorem 1, although simple, will be useful to shorten the proofs of theorem 5 and lemma 9 as well as it can be used to simplify proofs of theorems 1, 2 and 3 in [1]; theorem 3.1 in [7]; theorems 4.2 and 4.3 in [6] and theorem 2 in [15]. There are also some related results in [2].

A function between the vertex sets of two graphs is said to be a *reflexive morphism* if images of adjacent vertices are adjacent or equal; it is a *morphism* if images of adjacent vertices are adjacent. When studying iterated clique graphs it is natural to search for a class $\mathcal{C}$ of reflexive morphisms having the following property:

P: For every clique divergent graph A and every graph B, if there is a reflexive morphism $f : A \to B$ with $f \in \mathcal{C}$ then B is also clique divergent.

It is easy to see that the classes of injective morphisms, full embedding morphisms and vertex-surjective reflexive morphisms do not satisfy this property, as it is known from [5] that every *cone* (a graph where there is a vertex adjacent to any other vertex) is $k-$null and there is always a vertex-surjective reflexive morphism from any graph A to the complete graph with the same number of vertices, which is clearly also $k-$null. There are various studies involving such classes of reflexive morphisms. For example, [9] proves that the class of *triangular coverings* has this property as well as [12] shows that the class of *sections* (of a reflexive retraction) has property **P**. Also, [13] (see also [11]) proves that the class of *admissible reflexive morphisms* in the category of *coaffine graphs* has the property if we ask A to be *expansive*. In section 3, our main theorem (8) states that the class of admissible reflexive morphisms also has the property in certain categories of graphs as long as we ask A to be *absolutely saturated* and *absolutely free*. As absolute saturation and freedom constitute a partial generalization of expansiveness, our results in section 3 partially extend some theorems in [11].

Section 4 exhibits a family of $k-$divergent, absolutely saturated and absolutely free graphs and section 5 uses this family and theorem 8 to prove the clique divergence of the icosahedron.

Let G be a graph. We will represent paths as sequences of vertices $(v_0, \ldots, v_n)$. The *open neighbourhood* $N_G(v)$ of a vertex v in G is the set of all vertices of G wich are neighbours of v while $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighbourhood* of v in G. Recall that G is *regular of degree* r if every vertex has exactly r neighbours. We shall identify cliques with their corresponding vertex sets.

2 Distances On Clique Graphs

The *distance* between vertices $u, v \in V(G)$ will be written as $d_G(u, v)$ while the *distance set* between two vertex sets $A, B \subseteq V(G)$ will be denoted by $\mathcal{D}_G(A, B) = \{d_G(u, v) : u \in A, v \in B\}$. We will often omit subscripts. As noted in the introduction, the following theorem is very useful.

Theorem 1 *Let A be any graph and $Q, Q' \in V(k(A))$ with $Q \neq Q'$ then $d_{k(A)}(Q, Q') = \min \mathcal{D}_A(Q, Q') + 1$.*

PROOF. Call $r = \min \mathcal{D}(Q, Q')$ and $s = d(Q, Q')$. Let $(x_0, x_1, \ldots, x_r)$ be a path in A with $x_0 \in Q$ and $x_r \in Q'$. Take $Q_i \in V(k(A))$ such that $\{x_i, x_{i+1}\} \subseteq Q_i$ for $i = 0, 1, \ldots, r-1$. Then $(Q, Q_0, Q_1, \ldots, Q_{r-1}, Q')$ is a walk in $k(A)$ proving that $d(Q, Q') = s \leq r + 1$.

Now let $(Q_0, Q_1, \ldots, Q_s)$ be a minimal length path in $k(A)$ between $Q_0 = Q$ and $Q_s = Q'$. Then take $x_i \in Q_i \cap Q_{i+1}$ for $i = 0, 1, \ldots, s-1$ so $(x_0, x_1, \ldots, x_{s-1})$ is a walk in A, proving that $d(x_0, x_{s-1}) \leq s - 1$. As $x_0 \in Q$ and $x_{s-1} \in Q'$, it follows that $r + 1 = \min \mathcal{D}(Q, Q') + 1 \leq d(x_0, x_{s-1}) + 1 \leq s$. □

Note that for every graph G and any two cliques $Q, Q' \in V(k(G))$, we have $|\mathcal{D}(Q, Q')| \leq 3$ and $\mathcal{D}(Q, Q') \neq \{1\}$. If G has at least two vertices in every connected component, then $\mathcal{D}(Q, Q') \neq \{0\}$ and therefore $\mathcal{D}(Q, Q') = \{0, 1\}$ if and only if $d(Q, Q') = 0$ (i.e. $Q = Q'$).

3 Saturation And Freedom

Let Γ be a group which will remain fixed in what follows. A $\Gamma-$*graph* is a pair $\mathbb{A} = (A, \circ_A)$ where A is a (finite, simple, connected and non-empty) graph and $\circ_A$ is an action (i.e. a group homomorphism) $\circ_A : \Gamma \to Aut(A)$. Given such an action we denote by the same symbol $\circ_A$ the corresponding mapping $\circ_A : \Gamma \times V(A) \to V(A)$, so we write $\gamma \circ_A v$ instead of $\circ_A(\gamma)(v)$. Let $\mathbb{A} = (A, \circ_A)$ and $\mathbb{B} = (B, \circ_B)$ be two $\Gamma-$graphs, we shall say that $f : \mathbb{A} \to \mathbb{B}$

is an *admissible reflexive morphism* if and only if $f : A \to B$ is a *reflexive morphism* of graphs and $f(\gamma \circ_A u) = \gamma \circ_B f(u)$ for all $\gamma \in \Gamma$. We shall work in the category $\mathcal{G}_\Gamma$ of $\Gamma-$graphs with admissible reflexive morphisms. From now on, we shall use 'morphism', 'reflexive morphism' and 'admissible reflexive morphism' as synonymous. It is easily verifiable that $\mathcal{G}_\Gamma$ is closed under the clique operator (not a functor!) k_Γ defined as $k_\Gamma(A, \circ_A) = \left(k(A), \circ_{k(A)}\right)$ where $\gamma \circ_{k(A)} Q = \{\gamma \circ_A v : v \in Q\}$ for all $Q \in V(k(A))$ and $\gamma \in \Gamma$. It is clear how to define the iterated clique operators and the clique divergence in $\mathcal{G}_\Gamma$ and that $\mathbb{A}$ is $k_\Gamma-$divergent if and only if A is $k-$divergent. From now on, we shall omit the action's subscripts as the context will make them redundant and, for the same reason, use the same symbol for k and k_Γ.

The important concept of *coaffine graph* was introduced by Neumann-Lara in [12] (see also [13] and [11]) in the context of *automorphic graphs* also introduced by him in [11]. These concepts have proven to be very useful in the study of clique divergence. We shall use $\Gamma-$graphs instead of automorphic graphs and copy the notion of coaffine graph to this context.

Definition 2 *Let $\omega \in \Gamma$ and $r \geq 2$. A $\Gamma-$graph $\mathbb{A} = (A, \circ)$ is said to be* $(\omega, r)-$coaffine *if $d(v, \omega \circ v) \geq r$ for all $v \in V(A)$.*

We shall need the following definitions.

Definition 3 *Let $\omega \in \Gamma$ and $r \geq 2$. A $\Gamma-$graph $\mathbb{A} = (A, \circ)$ is said to be* $(\omega, r)-$saturated *if $d(u, v) + d(v, \omega \circ u) = r$ for all $u, v \in V(\mathbb{A})$. $\mathbb{A}$ is said to be* absolutely $(\omega, r)-$saturated *if $k^n(\mathbb{A})$ is $(\omega, r)-$ saturated for $n = 0, 1, 2, 3, \ldots$*

We shall write just $r-$*coaffine* (resp. $r-$ *saturated*) when it is clear or unimportant what ω is. Note that every $r-$saturated $\Gamma-$graph is $r-$coaffine. We chose the term 'saturated' because whenever an $(\omega, r)-$saturated graph A is a subgraph of an $(\omega, r)-$coaffine graph B with $V(A) = V(B)$ we have $A = B$.

Definition 4 *A $\Gamma-$graph $\mathbb{A} = (A, \circ)$ is said to be* free *if $|\Gamma \circ v| = |\Gamma|$ for all $v \in V(A)$. $\mathbb{A}$ is said to be* absolutely free *if $k^n(\mathbb{A})$ is free for $n = 0, 1, 2, 3, \ldots$*

Theorem 5 *(Neumann-Lara[10]) If a $\Gamma-$graph $\mathbb{A}$ is $(\omega, r)-$ coaffine then $k(\mathbb{A})$ is also $(\omega, r)-$coaffine.*

PROOF. Note that if $d(x, y) \leq 1$ then, by the triangle inequality,

$$d(x, \omega \circ y) \geq d(y, \omega \circ y) - d(x, y) \geq r - 1.$$

Clearly $Q \neq \omega \circ Q$ (by definition, $r \geq 2$). By theorem 1 $d(Q, \omega \circ Q) = \min \mathcal{D}(Q, \omega \circ Q) + 1 \geq (r-1) + 1 = r.$ $\square$

Thus, if $\Gamma = \langle\omega\rangle \cong \mathbb{Z}_2$, every $(\omega, r)-$coaffine $\Gamma-$graph is absolutely free.

The following three results constitute a first step to a generalization of the beautiful theory exposed in [11] (see also [12] and [13]). The results developed there, are mainly applicable to $2-$coaffine graphs which are not 3-coaffine. It is easily verifiable that the $2-$saturated graphs are precisely the octahedral graphs $\mathcal{O}_d$: regular graphs with $2d$ vertices and degree $2d-2$. It is known from [3] and [12] that $k(\mathcal{O}_d) \cong \mathcal{O}_{2^{d-1}}$ and therefore the octahedral graphs are $k-$divergent (for $d \geq 3$), absolutely $2-$saturated and for a suitable Γ, also absolutely free.

Theorem 6 *Let $\mathbb{A}, \mathbb{B} \in \mathcal{G}_\Gamma$. Assume that there is an admissible reflexive morphism $f : \mathbb{A} \to \mathbb{B}$ and $k(\mathbb{A})$ is free, then there is an admissible reflexive morphism $\widehat{f} : k(\mathbb{A}) \to k(\mathbb{B})$.*

PROOF. Denote $\mathcal{Q} = V(k(\mathbb{A}))$, and let $\mathcal{Q}_0$ be a complete set of representatives of $\mathcal{Q}$ modulo Γ. As $k(\mathbb{A})$ is free, we have that for all $\gamma, \gamma' \in \Gamma$ and $Q, Q' \in \mathcal{Q}_0$, $\gamma \circ Q = \gamma' \circ Q'$ if and only if $\gamma = \gamma'$ and $Q = Q'$. Also $\mathcal{Q} = \Gamma \circ \mathcal{Q}_0 = \{\gamma \circ Q : \gamma \in \Gamma, Q \in \mathcal{Q}_0\}$. On the other hand, since $f(Q)$ is a complete subgraph of $\mathbb{B}$ for all $Q \in \mathcal{Q}$, we can choose, for each $Q \in \mathcal{Q}_0$, a fixed clique $\widehat{Q}$ of $\mathbb{B}$ with $f(Q) \subseteq \widehat{Q}$. Now define $\widehat{f}(\gamma \circ Q) = \gamma \circ \widehat{Q}$ whenever $Q \in \mathcal{Q}_0$. Let us verify that $\widehat{f}$ is a morphism in $\mathcal{G}_\Gamma$:

First, as every $Q \in \mathcal{Q}$ has a unique representation $Q = \gamma \circ Q'$ with $\gamma \in \Gamma$ and $Q' \in \mathcal{Q}_0$, it follows that $\widehat{f}$ is well defined. Now, if $\gamma \circ Q \cap \gamma' \circ Q' \neq \phi$ then $\widehat{f}(\gamma \circ Q) \cap \widehat{f}(\gamma' \circ Q') = \gamma \circ \widehat{Q} \cap \gamma' \circ \widehat{Q'} \supseteq \gamma \circ f(Q) \cap \gamma' \circ f(Q') = f(\gamma \circ Q) \cap f(\gamma' \circ Q') \supseteq f(\gamma \circ Q \cap \gamma' \circ Q') \neq \phi$, so $\widehat{f}$ preserves adjacencies. Finally, $\widehat{f}$ is admissible: let $\gamma \circ Q \in \mathcal{Q}$ and $\gamma' \in \Gamma$, then $\widehat{f}(\gamma' \circ \gamma \circ Q) = \widehat{f}(\gamma' \cdot \gamma \circ Q) = \gamma' \cdot \gamma \circ \widehat{Q} = \gamma' \circ \gamma \circ \widehat{Q} = \gamma' \circ \widehat{f}(\gamma \circ Q)$. □

Note that the previous theorem is applicable to plain graphs just taking Γ as the trivial group.

Theorem 7 *Let $\mathbb{A}, \mathbb{B} \in \mathcal{G}_\Gamma$ and let $f : \mathbb{A} \to \mathbb{B}$ be a morphism. If $\mathbb{A}$ is $(\omega, r)-$saturated and $\mathbb{B}$ is $(\omega, r)-$coaffine then f is injective.*

PROOF. Let $x \neq y$ be vertices of $\mathbb{A}$, and let $p = (u_0, u_1, \ldots, u_r)$ be a path in $\mathbb{A}$, with $x = u_0$, $\omega \circ x = u_r$ and $y = u_a$ for some $a \in \{1, \ldots, r\}$ (As $\mathbb{A}$ is $r-$saturated, $d(x,y) + d(y, \omega \circ x) = r$). Now $f(p) = (f(u_0), f(u_1), \ldots, f(u_r))$ is a walk in $\mathbb{B}$ of length at most r, (f is a morphism), but as $f(u_0) = f(x)$, $f(u_r) = f(\omega \circ x) = \omega \circ f(x)$ and $\mathbb{B}$ is $r-$coaffine, it follows that $f(p)$ is of length at least r. Then $f(p)$ is a path of length r and therefore $f(x) \neq f(y)$. □

Now we have our main theorem:

Theorem 8 *If $\mathbb{A}$ is $k-$divergent, absolutely $(\omega, r)-$saturated and absolutely free, and there is a morphism $f : \mathbb{A} \to \mathbb{B}$ with $\mathbb{B}$ a $(\omega, r)-$coaffine $\Gamma-$graph, then $\mathbb{B}$ is $k-$divergent.*

PROOF. As $\mathbb{A}$ is absolutely free, by theorem 6, there is a morphism $f_n : k^n(\mathbb{A}) \to k^n(\mathbb{B})$ for every $n = 0, 1, 2, 3, \ldots$ Now $\mathbb{A}$ absolutely $(\omega, r)-$saturated and $\mathbb{B}$ $(\omega, r)-$coaffine imply by theorem 7 that every f_n is injective (recall also theorem 5) so $|V(k^n(\mathbb{A}))| \leq |V(k^n(\mathbb{B}))|$, which proves the $k-$divergence of $\mathbb{B}$. $\square$

It is easy to see that any $(\omega, 3)-$saturated $\Gamma-$graph $\mathbb{A}$ is regular of degree $\frac{1}{2}|V(\mathbb{A})| - 1$. The converse is not true, but we have the following:

Lemma 9 *If a $\Gamma-$graph $\mathbb{A}$ is $(\omega, 3)-$saturated and $k(\mathbb{A})$ is a regular graph of degree $\frac{1}{2}|V(k(\mathbb{A}))| - 1$, then $k(\mathbb{A})$ is also $(\omega, 3)-$saturated.*

PROOF. Recall that, for every two cliques Q and Q', $\mathcal{D}(Q, Q') = \{0, 1\}$ if and only if $d(Q, Q') = 0$, $|\mathcal{D}(Q, Q')| \leq 3$, and that the cases $\mathcal{D}(Q, Q') = \{0\}$ and $\mathcal{D}(Q, Q') = \{1\}$ are impossible. Keep in mind that the diameter of any $(\omega, r)-$saturated $\Gamma-$graph is r. Also it should be clear that $\mathcal{D}(Q', \omega \circ Q) = \{r - a : a \in \mathcal{D}(Q, Q')\}$. We shall use theorem 1 without explicit reference.

We shall prove that $d(Q, Q') + d(Q', \omega \circ Q) = 3$ for all $Q, Q' \in V(k(\mathbb{A}))$ by cases, depending on what $d(Q, Q')$ is:

First assume $d(Q, Q') = 0$, then, $\mathcal{D}(Q, Q') = \{0, 1\}$, so $\mathcal{D}(Q', \omega \circ Q) = \{2, 3\}$ and $d(Q', \omega \circ Q) = 3$.

Now suppose $d(Q, Q') = 1$, then $\mathcal{D}(Q, Q') = \{0, 1, 2\}$, so $\mathcal{D}(Q', \omega \circ Q) = \{1, 2, 3\}$ and $d(Q', \omega \circ Q) = 2$.

Let's assume $d(Q, Q') = 2$. We already know, by the first case in this proof, that $d(Q, \omega \circ Q) = 3$, so $N_{k(\mathbb{A})}[Q] \cap N_{k(\mathbb{A})}[\omega \circ Q] = \phi$. But now, as $k(\mathbb{A})$ is regular of degree $\frac{1}{2}|V(k(\mathbb{A}))| - 1$, we have $N_{k(\mathbb{A})}[Q] \cup N_{k(\mathbb{A})}[\omega \circ Q] = V(k(\mathbb{A}))$. Then $d(Q, Q') = 2$ implies $d(Q', \omega \circ Q) = 1$.

Consider the case $d(Q, Q') = 3$, then $\mathcal{D}(Q, Q') = \{2, 3\}$ or $\mathcal{D}(Q, Q') = \{2\}$. In the first case, $\mathcal{D}(Q', \omega \circ Q) = \{0, 1\}$ and $d(Q', \omega \circ Q) = 0$, as required. In the second case $\mathcal{D}(Q', \omega \circ Q) = \{1\}$, which is impossible.

Finally, if $d(Q, Q') \geq 4$ then $\mathcal{D}(Q, Q') = \{3\}$ so $\mathcal{D}(Q', \omega \circ Q) = \{0\}$ which is impossible. $\square$

4 An Example

Let $\mathcal{C}(\Gamma, r)$ be the class of all $k-$divergent, absolutely $r-$saturated and absolutely free $\Gamma-$graphs. The usability of our main theorem (8) is, of course, bound to the abundance of $\Gamma-$graphs in $\mathcal{C}(\Gamma, r)$. In general, it is difficult to prove a graph to be $k-$divergent. Also, if a $\Gamma-$graph is $k-$divergent, it is hard to prove it to be absolutely saturated. However there are many $\Gamma-$graphs that *seem* to be in $\mathcal{C}(\Gamma, r)$. For example, computer says that the icosahedron itself (with the Γ and the action that we will describe later) and its iterated clique graphs are $3-$saturated and free as far as it can check, that is, up to the fourth iterated clique graph. I am still not able to prove or disprove that the icosahedron is in $\mathcal{C}(\Gamma, 3)$.

For every $r \geq 2$, I know one family of $\Gamma-$graphs in $\mathcal{C}(\Gamma, r)$. When $r = 2$, as mentioned earlier, it is easy to verify that the $\Gamma-$graphs in $\mathcal{C}(\Gamma, r)$ are precisely the octahedral graphs. For $r \geq 3$, every $\Gamma-$graph in $\mathcal{C}(\Gamma, r)$ that I know is a *clockwork graph*. Clockwork graphs were introduced by Larrión and Neumann-Lara in [8]. However, for $r \geq 4$, in order to prove that these $\Gamma-$graphs are in $\mathcal{C}(\Gamma, r)$, we require some techniques beyond the scope of this paper.

Next we shall exhibit a family of $k-$divergent, absolutely $3-$saturated, and absolutely free $\Gamma-$graphs.

From now on we take $\Gamma = \langle \omega \rangle \cong \mathbb{Z}_2$ and also we assume every reference to any index named i as being taken modulo 4.

For every $n \geq 0$, we define H^n as the graph having as vertex set:

$$\begin{aligned} V(H^n) = & \{a_0, a_1, a_2, a_3\} \cup \{b_0, b_1, b_2, b_3\} \\ & \cup \left\{ x_{i,j} : i \in \{0, 2\} \text{ and } j \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\} \right\} \\ & \cup \left\{ x_{i,j} : i \in \{1, 3\} \text{ and } j \in \left\{0, \ldots, \left\lceil \frac{n}{2} \right\rceil \right\} \right\} \end{aligned}$$

and (always taking $i+1$ modulo 4) the following edge set:

$$\begin{aligned} E(H^n) = & \{\{a_i, a_{i+1}\} : i \in \mathbb{Z}_4\} \cup \{\{b_i, b_{i+1}\} : i \in \mathbb{Z}_4\} \cup \{\{a_i, b_i\} : i \in \mathbb{Z}_4\} \\ & \cup \{\{a_i, x_{i,j}\} : i \in \mathbb{Z}_4 \text{ and any } j\} \cup \{\{a_{i+1}, x_{i,j}\} : i \in \mathbb{Z}_4 \text{ and any } j\} \\ & \cup \{\{b_i, x_{i,j}\} : i \in \mathbb{Z}_4 \text{ and any } j\} \cup \{\{b_{i+1}, x_{i,j}\} : i \in \mathbb{Z}_4 \text{ and any } j\} \\ & \cup \{\{x_{i,j}, x_{i+1,j'}\} : i \in \{0, 2\} \text{ and } j' \leq j\} \\ & \cup \{\{x_{i,j}, x_{i+1,j'}\} : i \in \{1, 3\} \text{ and } j' < j\} \\ & \cup \{\{x_{i,j}, x_{i,j'}\} : i \in \{0, \ldots, 3\} \text{ and } j' \neq j\} \end{aligned}$$

Figure 1 shows three of these graphs. For the sake of clarity, some vertices (white dots) and some edges (dashed lines) have been drawn twice.

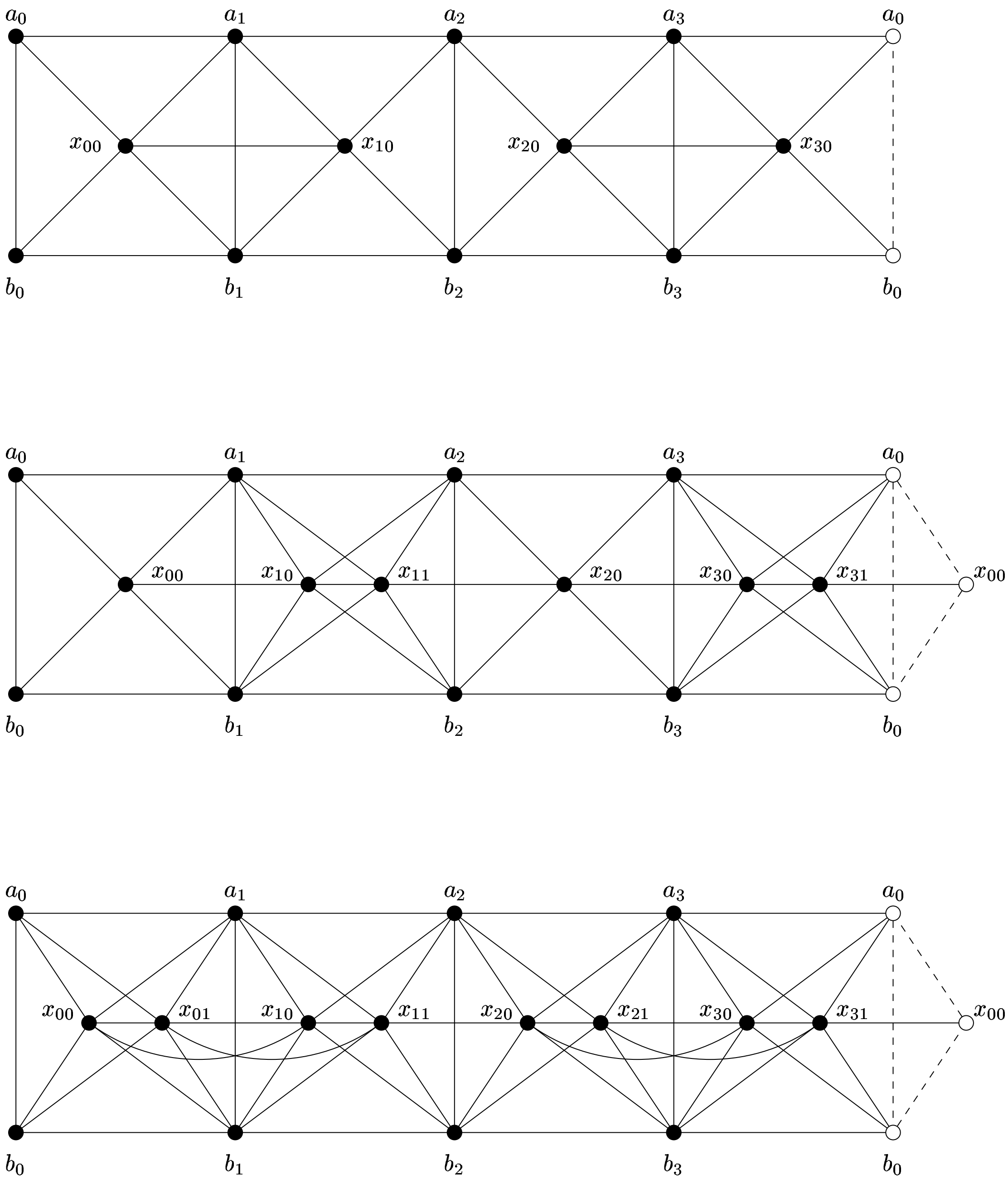

Fig. 1. From top to bottom: H^0, H^1 and H^2.

It should be clear that the function τ defined by: $\tau(a_i) = b_{i+2}$, $\tau(b_i) = a_{i+2}$ and $\tau(x_{i,j}) = x_{i+2,j}$ is an automorphism of H^n.

Now we define the $\Gamma-$graph $\mathbb{H}^n$ as $\mathbb{H}^n = (H^n, *)$, where $* : \Gamma \to Aut(H^n)$ is the injective morphism having image $\langle\tau\rangle$. The following theorem follows immediately from the theory developed in [8], but for the reader's convenience we include a proof.

Theorem 10 $k(\mathbb{H}^n) \cong \mathbb{H}^{n+1}$.

PROOF. Let's call *segments* of H^n to the sets $S_i = \{a_i, b_i, x_{i,j} : \text{ any } j\}$ for $i = 0, 1, 2, 3$. We shall say that S_i and S_{i+1} are *consecutive* segments (as always, $i+1$ taken modulo 4).

Let's calculate explicitly the cliques of H^n. It is clear, from the definition of H^n, that every segment induces a complete subgraph (but not necessarily a clique) of H^n and that every edge is contained in the union of at most two consecutive segments. Then, every clique is either one of the four segments or it intersects non-trivially two consecutive segments. Let's calculate first the cliques of the second type.

Let Q be a clique which intersects non-trivially two consecutive segments, say S_i and S_{i+1}. If $a_i \in Q$, then, as $N_{H^n}(a_i) \cap S_{i+1} = \{a_{i+1}\}$, we have $Q \cap S_{i+1} = \{a_{i+1}\}$, but then $N(a_i) \cap N(a_{i+1}) = \{x_{i,j} : \text{ any } j\}$ induces a complete subgraph. Then $Q = \{a_i, a_{i+1}\} \cup \{x_{i,j} : \text{ any } j\}$. Let's name these cliques A_i:

$$A_i = \{a_i, a_{i+1}\} \cup \{x_{i,j} : \text{ any } j\} \text{ for } i = 0, \ldots, 3.$$

Similarly, we have other four cliques when $b_i \in Q$:

$$B_i = \{b_i, b_{i+1}\} \cup \{x_{i,j} : \text{ any } j\} \text{ for } i = 0, \ldots, 3.$$

Now assume $a_i, b_i \notin Q$. As $Q \cap S_i \neq \phi$, there is at least one $x_{i,l} \in Q$. Let $x_{i,j}$ be the vertex in $Q \cap S_i$ with the smallest index j. Then as $Q \subseteq N[x_{i,j}]$, we have:

$$\begin{aligned} Q &\subseteq \{x_{i,j'} : j \leq j'\} \cup \{a_{i+1}, b_{i+1}\} \cup \{x_{i+1,j''} : j'' \leq j\} \text{ if } i \text{ is even} \\ Q &\subseteq \{x_{i,j'} : j \leq j'\} \cup \{a_{i+1}, b_{i+1}\} \cup \{x_{i+1,j''} : j'' < j\} \text{ if } i \text{ is odd} \end{aligned}$$

but in each case the right hand side set already induces a maximal complete subgraph, so we have another family of cliques:

$$\begin{aligned} X_{i,j} &= \{x_{i,j'} : j \leq j'\} \cup \{a_{i+1}, b_{i+1}\} \cup \{x_{i+1,j''} : j'' \leq j\} \\ &\qquad \text{for } i \in \{0, 2\} \text{ and } j \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\} \\ X_{i,j} &= \{x_{i,j'} : j \leq j'\} \cup \{a_{i+1}, b_{i+1}\} \cup \{x_{i+1,j''} : j'' < j\} \\ &\qquad \text{for } i \in \{1, 3\} \text{ and } j \in \left\{0, \ldots, \left\lceil \frac{n}{2} \right\rceil\right\} \end{aligned}$$

This completes the case in which the clique intersects non-trivially two consecutive segments.

It only remains to study the four segments as possible extra cliques.

Assume n to be even. Then $\left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ and therefore $S_1 \subseteq X_{0,\left\lfloor \frac{n}{2} \right\rfloor}$ and $S_3 \subseteq X_{2,\left\lfloor \frac{n}{2} \right\rfloor}$. On the other hand, S_0 is a clique because no vertex in S_3 is adjacent to $x_{0,\left\lfloor \frac{n}{2} \right\rfloor}$ and no vertex in S_1 is adjacent to both a_0 and b_0. Analogously S_2 is a clique. Let's rename these two cliques as $S_0 = X_{3,\left\lceil \frac{n+1}{2} \right\rceil}$ and $S_2 = X_{1,\left\lceil \frac{n+1}{2} \right\rceil}$

Now suppose n to be odd. Then $\left\lceil \frac{n}{2} \right\rceil > \left\lfloor \frac{n}{2} \right\rfloor$ and therefore $S_0 \subseteq X_{3,\left\lceil \frac{n}{2} \right\rceil}$ and $S_2 \subseteq X_{1,\left\lceil \frac{n}{2} \right\rceil}$. But S_1 and S_3 are cliques, because no vertex of $S_0 \cup S_2$ is adjacent to every vertex in $\left\{ x_{1,\left\lceil \frac{n}{2} \right\rceil}, a_1, b_1 \right\} \subseteq S_1$ and no vertex in $S_2 \cup S_0$ is adjacent to every vertex in $\left\{ x_{3,\left\lceil \frac{n}{2} \right\rceil}, a_3, b_3 \right\} \subseteq S_3$. Also in this case we rename these two cliques as $S_1 = X_{0,\left\lfloor \frac{n+1}{2} \right\rfloor}$ and $S_3 = X_{2,\left\lfloor \frac{n+1}{2} \right\rfloor}$.

Now, it is straight forward to verify that the intersection graph of all these cliques (i.e. the clique graph of H^n) is isomorphic to H^{n+1} via the isomorphism, ι, defined by $\iota(A_i) = a_i$, $\iota(B_i) = b_i$ and $\iota(X_{i,j}) = x_{i,j}$.

Finally, it should be clear that ι is an admissible reflexive morphism between $k(\mathbb{H}^n)$ and $\mathbb{H}^{n+1}$. $\square$

Theorem 11 *For all n, $\mathbb{H}^n$ is $k-$divergent, absolutely $(\omega, 3)-$saturated and absolutely free.*

PROOF. As $|\mathbb{H}^n| = 12 + 2n$, $k-$divergence is clear from theorem 10. It is a routine verification that $\mathbb{H}^0$ is $3-$saturated. Now we note that every vertex has exactly $n + 5$ neighbors in $\mathbb{H}^n$ and then Lemma 9 tells us that $\mathbb{H}^n$ is absolutely $3-$saturated. Absolute freedom is immediate from theorem 5 and the comment following it. $\square$

5 The Icosahedron

Consider the icosahedron I as shown in Figure 2 (please check isomorphism against your favorite drawing!). There, vertices are labeled from 1 to 12 in such a way that $\varphi = (1, 11)(2, 12)(3, 9)(4, 10)(5, 7)(6, 8)$ is the antipode automorphism. As before, white dots are duplicated vertices and the dashed line is

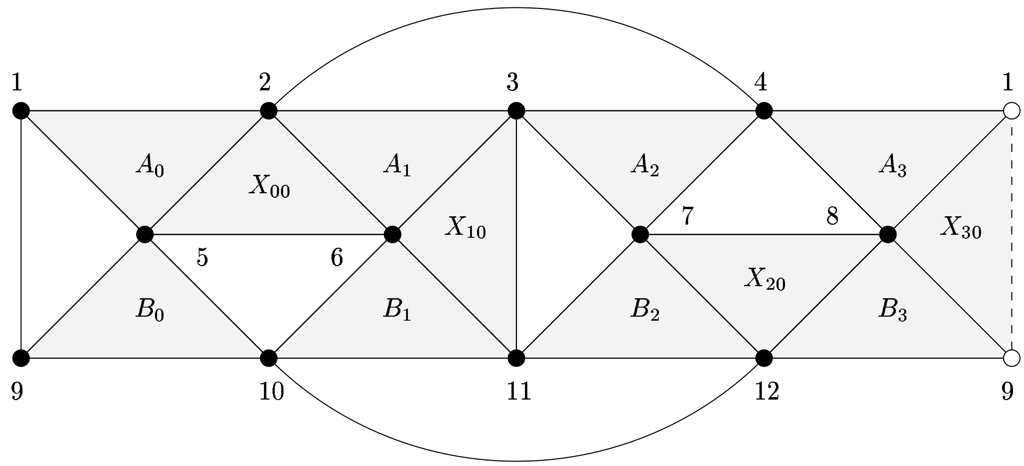

Fig. 2. The icosahedron with an embedding of H^0 in $k(I)$

a duplicated edge. The cliques of the icosahedron are precisely its faces. Let's name some cliques of I:

$$\begin{array}{lll} A_0 = \{1,2,5\} & B_0 = \{5,9,10\} & X_{00} = \{2,5,6\} \\ A_1 = \{2,3,6\} & B_1 = \{6,10,11\} & X_{10} = \{3,6,11\} \\ A_2 = \{3,4,7\} & B_2 = \{7,11,12\} & X_{20} = \{7,8,12\} \\ A_3 = \{1,4,8\} & B_3 = \{8,9,12\} & X_{30} = \{1,8,9\} \end{array}$$

Now consider the $\Gamma-$graph $\mathbb{I} = (I, \circ)$, where $\circ : \Gamma \to Aut(I)$ is the morphism having $\langle \varphi \rangle$ as image. Then $\mathbb{I}$ is a $(\omega, 3)-$coaffine $\Gamma-$graph. Now, the named cliques as shown in Figure 2 induce a subgraph of $k(\mathbb{I})$ isomorphic to $\mathbb{H}^0$ in the category $\mathcal{G}_\Gamma$. Then, theorem 8 tells us that $k(I)$ is clique divergent and so we have proved:

Theorem 12 *The icosahedron is $k-$divergent.* $\square$

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