ON SELF-CLIQUE SHOAL GRAPHS

F. LARRIÓN, M.A. PIZAÑA, AND R. VILLARROEL-FLORES

Abstract. The *clique graph* of a graph G is the intersection graph $K(G)$ of its (maximal) cliques, and G is *self-clique* if $K(G)$ is isomorphic to G. A graph G is *locally* H if the neighborhood of each vertex is isomorphic to H. Assuming that each clique of the regular and self-clique graph G is a triangle, it is known that G can only be r-regular for $r \in \{4, 5, 6\}$ and G must be, depending on r, a locally H graph for some $H \in \{P_4, P_2 \cup P_3, 3P_2\}$. The self-clique locally P_4 graphs are easy to classify, but only a family of locally H self-clique graphs was known for $H = P_2 \cup P_3$, and another one for $H = 3P_2$.

We study locally $P_2 \cup P_3$ graphs (i.e. *shoal graphs*). We show that all previously known shoal graphs were self-clique. We give a bijection from (finite) shoal graphs to 2-regular digraphs without directed 3-cycles. Under this translation, self-clique graphs correspond to self-dual digraphs, which simplifies constructions, calculations and proofs. We compute the numbers, for each $n \leq 28$, of self-clique and non-self-clique shoal graphs of order n, and also prove that these numbers grow at least exponentially with n.

1. Introduction.

Our graphs are simple and, unless they clearly are not (as, e.g. $P_2 \cup P_3$), also connected. We deal mostly with finite graphs, but some infinite graphs are also considered. We will be explicit about finiteness or infiniteness when needed. A *clique* of a graph G is a maximal complete subgraph of G, or just its set of vertices, as we identify induced subgraphs with their vertex sets. The *clique graph* of G is the intersection graph $K(G)$ of the cliques of G, and G is *self-clique* if G is connected and $K(G) \cong G$. The study of self-clique graphs began in [9] and has been pursued in [1–7, 13–16]. A graph is *locally* H if the (open) neighborhood $N(v)$ of any vertex $v \in G$ induces a subgraph isomorphic to H. We denote by P_n the path graph on n vertices and by kP_n the disjoint union of k copies of P_n.

This research was motivated by the paper [7], in which Chia and Ong propose the study of those self-clique graphs whose cliques have all the same size. For $n \geq 2$, they defined $\mathcal{G}(n)$ as the class of all, not necessarily finite, self-clique graphs having only cliques of n vertices. For $n = 2$ they proved that $\mathcal{G}(2)$ only contains the cycles C_n with $n \geq 4$, the one-way infinite path P_∞ and the two-way infinite path (or *infinite cycle*) C_∞. After this, [7] focuses into $\mathcal{G}(3)$, a much tougher proposition. For our purposes, their key results [7, Thm.2, Cor.1] are that any vertex v of a graph G in $\mathcal{G}(3)$ has, according to its degree, an open neighborhood $N(v)$ which can only be one of the graphs $P_2, P_3, P_4, P_2 \cup P_3$ or $3P_2$, and that any r-regular graph in $\mathcal{G}(3)$ must satisfy $r \in \{4, 5, 6\}$. In particular, a regular graph in $\mathcal{G}(3)$ must be locally H for some $H \in \{P_4, P_2 \cup P_3, 3P_2\}$.

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The 4-regular graphs in $\mathcal{G}(3)$, i.e. the locally P_4 graphs in $\mathcal{G}(3)$, were classified in [7]. Hall had shown in [11] that the locally P_4 graphs are just the squared cycles C_∞^2 and C_n^2 for all integers $n \geq 7$. Being self-clique, they are all in $\mathcal{G}(3)$, as implied by [7, Thm.4]. As for the remaining two cases ($r = 5, 6$) of regular graphs in $\mathcal{G}(3)$, only a family of examples was given for each type, and the question was raised in [7, §6] whether they could also be classified.

This work is devoted to the study of 5-regular graphs in $\mathcal{G}(3)$. In other words, we investigate which locally $P_2 \cup P_3$ graphs (to be renamed *shoal graphs* in the next section) are self-clique.

After a preliminary study of locally $P_2 \cup P_3$ graphs in §2, we show in §3 that their clique graphs are obtained by just flipping the diagonals of all their diamonds.

Hall [11] had proved by examples the existence of locally $P_2 \cup P_3$ graphs, and Chia an Ong's family of self-clique graphs of this type is a proper subfamily of Hall's [7, §3]. We shall prove that all graphs in Hall's family (which as far as we can tell were all the previously known locally $P_2 \cup P_3$ graphs) are indeed self-clique. In fact, Hall's graphs are "orientable", and our geometrical proof also works for the corresponding "non-orientable" analogues, see our §4.

In §5 we translate our problem into that of finding the self-dual *fishy digraphs* (i.e. balanced orientations of quartic graphs without directed 3-cycles). This greatly simplifies the analysis. The fishy digraph D associated to a locally $P_2 \cup P_3$ graph G has half the number of vertices of G, and its underlying graph is 4-regular, while G is 5-regular. Self-duality for D is simpler than self-cliqueness for G. Quartic graphs have been much more studied than locally $P_2 \cup P_3$ graphs, and there are available catalogs and computer programs to work with them.

Using fishy digraphs we give in §6 two new and easy families of locally $P_2 \cup P_3$ graphs, all of them self-clique. We now have examples of each even order greater than 12, which are all possible orders. Up to this point it could conceivably be thought that every locally $P_2 \cup P_3$ graph is self-clique. But our approach also simplified the exhaustive calculation of small examples by hand. This yielded that up to order 18 there are 16 locally $P_2 \cup P_3$ graphs, all of them self-clique, but of order 20 there are 114, and only 60 of them are self-clique. We continued these calculations using a computer and found that up to order 28 there are 3,536,172 locally $P_2 \cup P_3$ graphs and precisely 33,108 of them are self-clique (see §7).

We shall prove in §8 that the number of self-clique locally $P_2 \cup P_3$ graphs and the number of non-self-clique locally $P_2 \cup P_3$ both grow at least exponentially with the order. Also, that the numbers of self-clique and non-self-clique locally $P_2 \cup P_3$ graphs of order $\aleph_0$ is the cardinality of the continuum. It shall be quite clear that the examples constructed in this paper are just a puny fraction of the self-clique locally $P_2 \cup P_3$ graphs. In our view, these results show that self-clique locally $P_2 \cup P_3$ graphs are unclassifiable. This would solve in the negative the classification problem of 5-regular graphs in $\mathcal{G}(3)$ posed in [7, §6, Question (i)]. But of course a formal proof of unclassifiability would require a formal definition of classifiability.

A vertex v of a graph G (i.e. $v \in G$) is *universal* if v is a neighbor of every other vertex in G. A *cone* is a graph G having a universal vertex, called also an *apex* of G. Whenever we speak of a *diamond in* G we mean an *induced* one. By $X \setminus Y$ we denote difference of sets, while $G - H$ is a graph difference. A *digraph* is an oriented graph: the graph must be simple and each edge has to be oriented in *exactly one* direction, i.e. our digraphs have no loops, parallel arrows or anti-parallel arrows. The *opposite* or *dual* D^{op} of a digraph D has the same vertices as D, but all the arrows reversed: $i \to j$ in D^{op} if, and only if, $j \to i$ in D.

2. Shoals, Fishes, Heads and Tails

In a locally $P_2 \cup P_3$ graph G, the closed neighborhood $N[v]$ of each vertex $v \in G$ induces a subgraph as the graph φ in Figure 2.1, which we call the *fish* of v. The triangle $\{v, x, y\}$ is the *tail* (of the fish) of v, and the diamond $\{v, a, b, c\}$ is the *head* (of the fish) of v. By dint of using these terms we ended up saying that a locally $P_2 \cup P_3$ graph is a *shoal graph*.

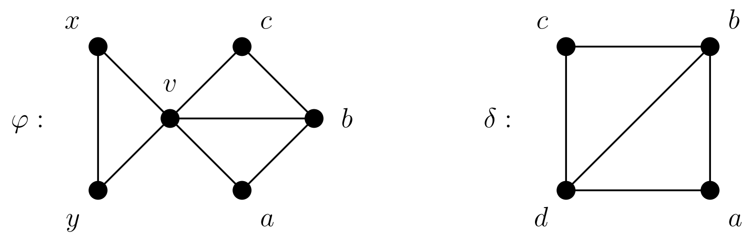


Figure 2.1. A fish φ and a diamond δ in a shoal graph G.

In the next three statements G is a shoal graph, and δ, δ' denote diamonds of G.

Lemma 2.1. *No edge of δ forms a triangle with a vertex $x \in G \setminus \delta$.*

Proof. Let the vertices of δ be labeled a, b, c and d as in Figure 2.1. If x is adjacent to both a and b, then the (open) neighborhood $N_G(b)$ contains a P_4, a contradiction. By symmetry, only the triangle $\{x, b, d\}$ remains possible, but with it $N_G(b)$ would contain a $K_{1,3}$. □

The head of a degree 3 vertex of δ (like b or d in Figure 2.1) is clearly δ. We also have:

Lemma 2.2. *The tail of a degree 2 vertex of δ is the triangle of δ incident with that vertex (like $\{a, b, d\}$ for a in Figure 2.1).*

Proof. Otherwise $\{b, d\}$ would be part of the head of a, contradicting Lemma 2.1. □

A family of (induced) diamonds of a graph is *well assembled* if any two of them that meet do so at just one vertex, and this vertex is incident to the diagonal of one of the diamonds but not to that of the other. All the diamonds of a shoal graph are well assembled:

Proposition 2.3. $|\delta \cap \delta'| \le 1$ *if* $\delta \neq \delta'$, *and* $\{d_\delta(v), d_{\delta'}(v)\} = \{2, 3\}$ *if* $\delta \cap \delta' = \{v\}$.

Proof. Let $\mathcal{I} = \delta \cap \delta'$. Assume first that $|\mathcal{I}| = 3$. By Lemma 2.1, $\mathcal{I}$ is not a triangle, so we must have (without loss of generality) $\mathcal{I} = \{a, c, d\}$ (see δ in Fig. 2.1) and $\delta' = \{a, x, c, d\}$. But then $\{a, x, c, b\}$ is a C_4 in $N_G(d)$. Assume now that $|\mathcal{I}| = 2$. Again by Lemma 2.1, $\mathcal{I}$ is not an edge. But then $\mathcal{I} = \{a, c\}$ and each of a and c would have two tails by Lemma 2.2, a contradiction. Last claim: If $\delta \cap \delta' = \{v\}$, one of δ and δ' must contain the tail of v, and the other be its head, so the set of the degrees of v in δ and δ' must be $\{2, 3\}$. □

We shall say that a shoal graph G is *diamantine* if every triangle of G is contained in some (necessarily unique) diamond. In the infinite case, our problem will be reduced in §3 to the study of diamantine shoal graphs. In the finite case, this is no restriction at all:

Proposition 2.4. *Let G be a finite shoal graph, then:*

(1) *G is diamantine, and*
(2) *the order of G is twice the number of its diamonds, so it is even.*

Proof. Let n be the order, t the number of triangles not contained in any diamond and d the number of diamonds. Note that every diamond contributes two heads (one for b and one for d in Figure 2.1) and two tails (for the vertices a and c). Likewise, any triangle not contained in a diamond contributes three tails. Since each vertex has exactly one head and one tail, we have exactly n heads and n tails, therefore $2d = n = 2d + 3t$, so $t = 0$. □

Remark. *The hypothesis that G is finite is crucial in Proposition 2.4. An infinite counterexample can be constructed as in Figure 2.2, in which the first three steps are shown. In each further step one attaches a diamond to every vertex whose neighborhood is not yet $P_2 \cup P_3$ in such a way that it achieves this neighborhood.*

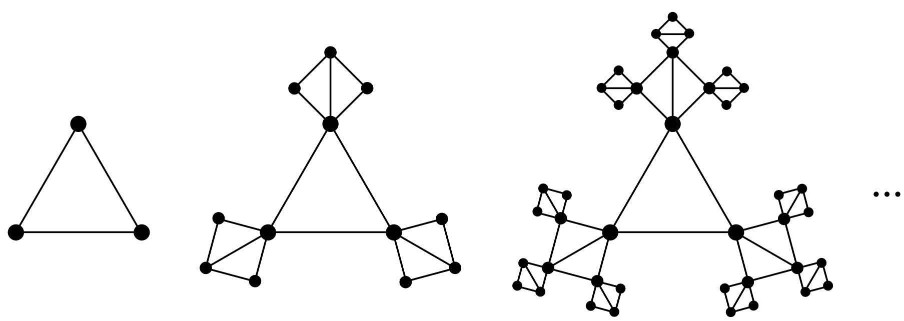


FIGURE 2.2. Finiteness is crucial in Prop. 2.4.

Theorem 2.5. *A graph G is a diamantine shoal graph if, and only if, the following four conditions are satisfied:*

(1) *Each vertex of G belongs to two diamonds.*
(2) *Each edge of G is contained in some diamond.*
(3) *Each triangle of G is contained in some diamond.*
(4) *The family of all diamonds of G is well assembled.*

Proof. If G is a diamantine shoal graph (3) holds, we know (4) by Proposition 2.3, and (1) and (2) follow from (3) for shoal graphs. Assume now that the conditions hold, and take a vertex $v \in G$. By (1), v belongs to two diamonds δ and δ', and by (4), $\delta \cap \delta' = \{v\}$ and v lies in the diagonal of (say) δ but not in that of δ'. Thus $N_G[v]$ contains a fish. These five vertices are all the neighbors of v: any other would force by (2) the existence of a third diamond δ'' meeting both δ and δ' in v, which is impossible by (4). As in Figure 2.1, let $\{v, x, y\}$ be the triangle of δ' containing v and let $\delta = \{v, a, b, c\}$. Any edge from $\{x, y\}$ to $\{a, b, c\}$ would produce a triangle, and then by (3) a third diamond δ'', meeting δ and δ' in at least two vertices, contradicting (4). Of course, a is not adjacent to c because δ is an induced diamond. Therefore G is a shoal graph, and it is diamantine by (3). □

Remark. *Some easy to see versions of Theorem 2.5 will be used: Condition* (1) *could be replaced with "Each vertex of G belongs to exactly two diamonds". Condition* (2) *could be replaced with "G is the graph union of its diamonds", but condition* (1) *would be still needed. Condition* (2) *could also be replaced, owing to condition* (4)*, with "Each edge of G is contained in a unique diamond" and, owing to condition* (3)*, with "Each edge of G is contained in a triangle". Uniqueness could also be asked for the diamond in condition* (3)*.*

3. The Clique Graph and the Flipped Graph

Clique-Helly graphs, introduced by Escalante in [9] are, from the viewpoint of clique graphs, the most widely studied and better understood interesting family of graphs. However, as we shall see, even for very restricted subfamilies of clique-Helly graphs there are difficult unsolved problems. A graph G is *clique-Helly* if the set of cliques of G satisfies the *Helly property*: every collection of pairwise intersecting cliques has a non-empty total intersection.

Using the Dragan-Szwarcfiter characterization [8,24] of clique-Hellyness (that each extended triangle is a cone) we see that any shoal graph G is clique-Helly. Indeed, let τ be a triangle of G. The extended triangle $\hat{\tau} = \{v \in G \mid |N_G(v) \cap \tau| \geq 2\}$ is just τ, if τ is not contained in a diamond, or else, by Lemma 2.1, the unique diamond δ that contains τ, which is a cone.

A vertex v in a graph G is *dominated* if $N_G(v)$ is a cone. By [9, Satz 3], a clique-Helly graph G has no dominated vertices if, and only if, $K^2(G) \cong G$ (the "if" part is invalid in the infinite case, but we shall only use the "only if" part; also remember that our main interest lies on finite graphs). As their neighborhoods are disconnected, shoal graphs do not have dominated vertices. Therefore we have:

Proposition 3.1. *If G is a shoal graph, $K^2(G) \cong G$.* □

Thus shoal graphs verge closely on self-cliqueness. In the language of [3], G is *2-self-clique* if $K^2(G) \cong G$ but $K(G) \ncong G$. We just need to distinguish between self-clique and 2-self-clique shoal graphs. The general characterizations known for a graph to be clique-Helly and self-clique (that it has a quasi-symmetric clique matrix [3], or a self-dual vertex-clique bipartite graph [15]) do not seem to help us to classify self-clique shoal graphs.

If the shoal graph G has a triangle τ not contained in any diamond, τ is a vertex of degree 6 in $K(G)$ (see Figure 2.2), so $K(G)$ is not a shoal graph. But then G cannot be self-clique. Thus, in order to study self-clique shoal graphs, we can restrict to diamantine shoal graphs. As previously observed, in the finite case this is no restriction at all by Proposition 2.4.

By *flipping* the diagonal of the diamond δ in Figure 2.1 we mean removing the edge bd and replacing it with ac. The *flipped graph* of a diamantine shoal graph G is the graph $\mathcal{F}(G)$ obtained by flipping the diagonals of all the diamonds of G. As vertex sets, the diamonds of $\mathcal{F}(G)$ are the same as those of G, and it is then clear that $\mathcal{F}(G)$ still satisfies the conditions of Theorem 2.5 and, therefore, $\mathcal{F}(G)$ is again a diamantine shoal graph.

Theorem 3.2. *Let G be a diamantine shoal graph. Then $K(G)$ is a diamantine shoal graph of the same order as G and, furthermore, $K(G) \cong \mathcal{F}(G)$.*

Proof. The cliques of G are clearly its triangles. Any vertex has a uniquely defined tail, and any triangle is the tail of some unique vertex by Theorem 2.5 and Lemma 2.2. Thus, we have a bijection $\tau\colon V(G) \to V(K(G))$ which sends each $v \in G$ to its tail $\tau(v) \in K(G)$.

That $K(G)$ is a diamantine shoal graph can be verified by observing Figure 3.1, in whose first part all the triangles meeting $\tau(v)$ and one that does not are depicted. These seven triangles are all distinct and are connected as in the figure, save for the fact that the uppermost (or the bottommost) edge could be the same that the rightmost one, so $\tau(x)$ (or $\tau(y)$) could form a diamond with $\tau(b)$. Thus the neighbors of $\tau(v)$ in $K(G)$ do induce a fish as

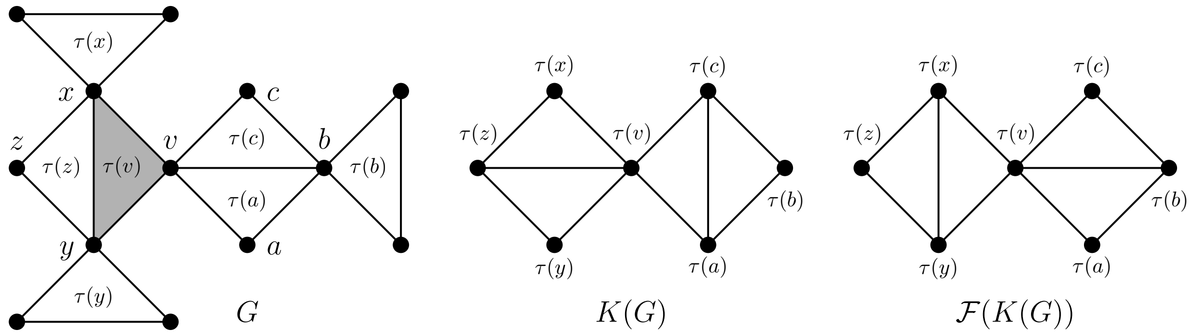


FIGURE 3.1. Surroundings of v in G and of $\tau(v)$ in $K(G)$ and $\mathcal{F}(K(G))$.

depicted in the second graph of the figure, which shows both the diamonds containing $\tau(v)$ in $K(G)$. Considering the third part of the figure we see that the vertex bijection τ is a graph isomorphism $\tau\colon G \to \mathcal{F}(K(G))$. As $\mathcal{F}^2(H) = H$ for each diamantine shoal graph H, the same bijection τ is also an isomorphism $\tau\colon \mathcal{F}(G) \to K(G)$. □

4. Hall Shoal Graphs

As far as we know, shoal graphs G were first shown to exist by Hall in [11, 4.13]. He gave an infinite graph $\tilde{H}$ in the plane (see Figure 4.1) and pointed out that it "can be rolled in many different ways onto the surface of a torus for finite G and onto the the surface of an infinitely extended cylinder for infinite G." Allowing for the also possible surfaces of the Klein bottle and the twisted cylinder (which topologically, but not geometrically, is a borderless Möbius strip), we will investigate these examples, to be called *Hall shoal graphs* in this work. We refer to [12, 22, 23] for terminology and background results.

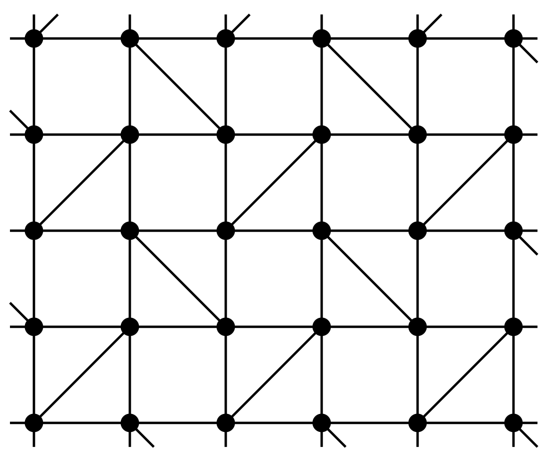


FIGURE 4.1. The infinite planar Hall shoal graph $\tilde{H}$ (which is clearly diamantine).

The infinite collection of self-clique shoal graphs given in [7, Prop.2] is properly contained in Hall's family. We shall prove in this section that every Hall shoal graph is self-clique.

Let H be a Hall shoal graph. Without prior reference to any surface, just let $p\colon \tilde{H} \to H$ be a graph morphism such that all the restrictions $p_|\colon N_{\tilde{H}}[\tilde{v}] \to N_H[p(\tilde{v})]$ are isomorphisms.

Thus, p is a *local isomorphism* (which is onto by connectedness) and this is just what we need for H to be a diamantine shoal graph which is a quotient of $\tilde{H}$. Local isomorphisms of graphs are the same as triangular covering maps (see [12]) and this permits us to see p as a topological covering map: attaching a 2-dimensional cell to each triangle of both $\tilde{H}$ and H we obtain two topological spaces (the geometric realizations of the simplicial complexes of triangles, edges and vertices) and thus $p\colon \tilde{H} \to H$, or rather its geometric realization, if you will, becomes a topological (piecewise linear, in fact) covering map.

Each vertex $\tilde{v} \in \tilde{H}$ is contained in two diamonds and two induced quadrilaterals. As no local isomorphism identifies vertices at a distance less than 4, the nine vertices encompassed by these four subgraphs induce a subgraph Q of $\tilde{H}$ such that the restriction $p_|\colon Q \to p(Q)$ is an isomorphism onto an induced subgraph of H. Thus, attaching 2-dimensional cells to the induced quadrilaterals of $\tilde{H}$ and also to their images under p in H, $\tilde{H}$ becomes the plane $\mathbb{R}^2$, H changes into some surface $\mathcal{S}$, and p extends to a topological covering map $p\colon \mathbb{R}^2 \to \mathcal{S}$.

Identifying the vertices of $\tilde{H}$ with the points of the integral lattice $\mathbb{Z} \oplus \mathbb{Z}$ we can, as in Figure 4.1, consider $\tilde{H}$ drawn in $\mathbb{R}^2$ as the integral grid graph with some extra diagonals. As $\mathbb{R}^2$ is simply connected, our topological covering map $p\colon \mathbb{R}^2 \to \mathcal{S}$ is regular [22, Chap.2]: there is a group Γ of self-homeomorphisms of $\mathbb{R}^2$ such that $\mathcal{S}$ is homeomorphic to the quotient space $\mathbb{R}^2/\,\Gamma$ and we can identify $p\colon \mathbb{R}^2 \to \mathcal{S}$ with the natural projection $p\colon \mathbb{R}^2 \to \mathbb{R}^2/\,\Gamma$. For each $\gamma \in \Gamma$ we have $\gamma(\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, and γ also sends affinely the (drawn) edges of $\tilde{H}$ into (drawn) edges of $\tilde{H}$ because $p \circ \gamma = p$ and p is piecewise linear. Therefore each $\gamma \in \Gamma$ restricts to a rigid automorphism of the drawn graph $\tilde{H} \subseteq \mathbb{R}^2$, and Γ itself can be taken to be a group of isometries of the Euclidean plane $\mathbb{R}^2$. Since the quotient space $\mathbb{R}^2/\,\Gamma$ is a surface, the group Γ must be discontinuous and fixed point free, so it contains no reflections or non-trivial rotations and can be generated by one or two of its elements [23, §2.5].

At the purely combinatorial level, the restrictions of the elements of Γ to $\mathbb{Z}\oplus\mathbb{Z}$ give us a group (denoted also by Γ) of automorphisms of the graph $\tilde{H}$ such that $H \cong \tilde{H}/\,\Gamma$ and our original graph morphism $p\colon \tilde{H} \to H$ can be identified with the natural projection $p\colon \tilde{H} \to \tilde{H}/\,\Gamma$. Since p is a triangular covering map, the group $\Gamma \leq \mathrm{Aut}(\tilde{H})$ is *admissible* in the sense that, apart from the identity, each $\gamma \in \Gamma$ satisfies $d_{\tilde{H}}(v, \gamma v) > 3$ for all $v \in \tilde{H}$. For ease of expression it will be best not to distinguish the combinatorial, topological and geometric levels anymore. We quickly describe the four non-trivial cases for the possible groups Γ:

The translations of $\tilde{H}$ are those translations of $\mathbb{Z} \oplus \mathbb{Z}$ which take the origin to a vertex in $2\mathbb{Z}\oplus 2\mathbb{Z}$; they are defined by the addition of this vertex, and can be identified with it. Toroidal Hall shoal graphs are of the form $\tilde{H}/\,\Gamma$ where the translation group $\Gamma = \langle t_1, t_2 \rangle \leq \mathrm{Aut}(\tilde{H})$ is admissible and the translations t_1, t_2 are independent. The first two parts of Figure 4.2 show fundamental domains for $\Gamma = \langle t_1, t_2 \rangle$ when the translations t_i are, in the obvious sense, $t_1 = t_{(4,0)}$ and $t_2 = t_{(0,4)}$ or $t_2 = t_{(2,4)}$. To obtain the corresponding Hall shoal graphs, we identify opposite sides of the domain so as to form a torus. (White vertices are “repeated”.)

The only glide reflections in $\mathrm{Aut}(\tilde{H})$ have a horizontal, vertical or slanted ($\pm 45°$) mirror that passes through some vertex. The associated translation must be in $(2\mathbb{Z} + 1) \oplus 0$ or in $0 \oplus (2\mathbb{Z} + 1)$ in the first two cases, or be of the form $(x, y) \mapsto (x \pm a, y + a)$ with $a \in 2\mathbb{Z}$ in the other two. A Hall shoal graph embedded in the Klein bottle can always be obtained as $\tilde{H}/\,\Gamma$, where the admissible group $\Gamma = \langle t, g \rangle \leq \mathrm{Aut}(\tilde{H})$ is generated by a translation t

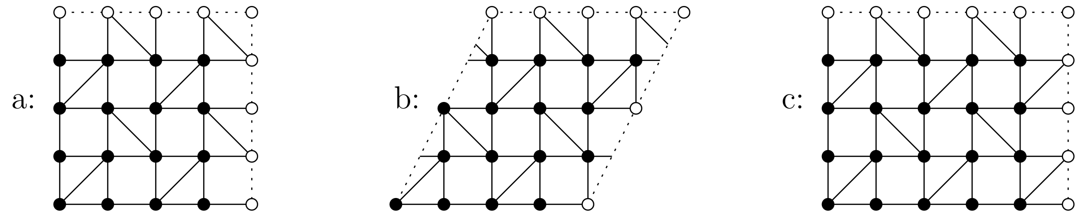


FIGURE 4.2. Fundamental domains for three admissible subgroups of $\mathrm{Aut}(\tilde{H})$.

and a glide reflection g in $\mathrm{Aut}(\tilde{H})$ in such a way that t and the mirror of g are perpendicular (and Γ is admissible). If Γ were generated by two glide reflections g_1 and g_2, using g_1g_2 and g_2 we would get our "asymmetric" presentation. The third part of Figure 4.2 shows a fundamental domain for $\Gamma = \langle t, g\rangle$ when $t = t_{(0,4)}$ and $g(x, y) = (x+5, -y)$. Here we identify the horizontal sides orientably and the vertical ones non-orientably so as to form a Klein bottle.

If the admissible group $\Gamma \leq \mathrm{Aut}(\tilde{H})$ is generated by just one translation or glide reflection, the Hall shoal graph $\tilde{H}/\Gamma$ is infinite and can be embedded in the cylinder or the twisted cylinder. We are now ready to prove:

Proposition 4.1. *Every Hall shoal graph is self-clique.*

Proof. Let $H = \tilde{H}/\Gamma$ be a Hall shoal graph, where $\Gamma \leq \mathrm{Aut}(\tilde{H})$ is admissible. By Theorem 3.2, it will be enough to show that $H \cong \mathcal{F}(H)\,(= \mathcal{F}(\tilde{H})/\Gamma)$. For the planar Hall shoal graph $\tilde{H}$, corresponding to $\Gamma = 1$, there are many isomorphisms to the flipped graph. The central inversions in the vertices $v \in \tilde{H}$ are well suited to our proof. If we put the origin at v, the central inversion in v is the isomorphism $c\colon \tilde{H} \to \mathcal{F}(\tilde{H})$ that sends each $u \in \tilde{H}$ to $c(u) = -u$. If Γ is not trivial, c still induces an isomorphism $\bar{c}\colon H \to \mathcal{F}(H)$ because c normalizes Γ. This means that $c\Gamma c^{-1} = \Gamma$, or rather (as $c^{-1} = c$) that $c\Gamma c = \Gamma$, as we will show now.

If $\Gamma = \langle t_1, t_2\rangle$ is generated by translations, we just note that if $t\colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is defined by $t(x, y) = (x + a, y + b)$, then $ctc = t^{-1}$, and therefore $c\Gamma c = \langle ct_1c, ct_2c\rangle = \left\langle t_1^{-1}, t_2^{-1}\right\rangle = \Gamma$.

If $\Gamma = \langle t, g\rangle$ is generated by the translation t and the glide reflection g, we put the origin at a vertex in the mirror of g. Then $g(x, y) = (x + a, -y)$ if the mirror is horizontal and $g(x, y) = (y + a, x + a)$ if it is slanted at 45° (the other two cases are symmetrical). Again we have that $cgc = g^{-1}$, and therefore $c\Gamma c = \langle ctc, cgc\rangle = \langle t^{-1}, g^{-1}\rangle = \Gamma$.

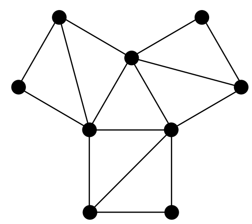
In the remaining cases of the infinitely extended cylinder or twisted cylinder the group Γ is cyclic and the above arguments also show that $c\Gamma c = \Gamma$. □

5. The Digraph of Diamonds and the Graph of Arrows.

The digraph $\mathcal{D}(G)$, which we associate to each diamantine shoal graph G, is the "tails-to-heads" (or "fish-forward") orientation of the intersection graph of the diamonds of G. In detail, the vertices of $\mathcal{D}(G)$ are the diamonds δ of G, and there is an arrow in $\mathcal{D}(G)$ from a diamond δ_1 to another δ_2 if they intersect in a vertex of G which lies in the diagonal of δ_2. Recall that in an *r-regular* digraph the in-degree and out-degree of each vertex are both r.

Proposition 5.1. *The digraph of diamonds $D = \mathcal{D}(G)$ associated to a diamantine shoal graph G is always 2-regular and never has a directed 3-cycle.*

Proof. Let δ be any vertex of D. We know that δ intersects precisely 4 other diamonds of G, one at each of its vertices. Since the diamonds of G are well assembled, two of these vertices of D send an arrow to δ, and the other two receive an arrow from it, so $d^-_D(\delta) = d^+_D(\delta) = 2$. Any directed 3-cycle $\delta_1 \to \delta_2 \to \delta_3 \to \delta_1$ in D would entail an arrangement of these three diamonds in G as in the following figure,



but then G would not be a shoal graph, and this contradiction finishes the proof. □

We say that a digraph D is *fishy* if it is 2-regular and has no directed 3-cycles. Thus, by Proposition 5.1 the digraph of diamonds of a diamantine shoal graph is always fishy.

Going the other way around, to any fishy digraph D we associate its graph of arrows $\mathcal{G}(D)$. The vertices of $\mathcal{G}(D)$ are the arrows of D, and for each vertex $x \in D$ we make adjacent in $\mathcal{G}(D)$ any two distinct arrows $f, g \in A(D)$ that go into or away from x, save for the case in which both f and g go away from x. The four arrows incident to x in D induce a diamond in $\mathcal{G}(D)$: *the diamond of* x, denoted by $\boxslash(x)$.

Theorem 5.2. *If D is a fishy digraph, $\mathcal{G}(D)$ is a diamantine shoal graph.*

Proof. We prove that $G = \mathcal{G}(D)$ satisfies the conditions (1) to (4) of Theorem 2.5.

(1): By definition, each arrow $f\colon x \to y$ of D, as a vertex of G, lies in both $\boxslash(x)$ and $\boxslash(y)$.

(2): Also by definition, $\{f, g\} \in E(G)$ implies that $\{f, g\} \subseteq \boxslash(x)$ for the vertex $x \in D$ which is incident to both f and g.

(3): Let $\tau = \{f, g, h\}$ be a triangle in G, and consider vertices $x, y, z \in D$ with $f, g \in \boxslash(x)$, $f, h \in \boxslash(y)$ and $g, h \in \boxslash(z)$. If $x \neq y$, we can assume that $f\colon x \to y$, but then we must have $g\colon z \to x$ because f and g are adjacent, and $h\colon y \to z$ because h and g are adjacent, but then f, h and g are the arrows of an directed 3-cycle in D. This contradiction implies that $x = y = z$, and then we conclude that $\tau = \{f, g, h\} \subseteq \boxslash(x)$.

(4): If the vertices $x, y \in D$ are distinct, the diamonds $\boxslash(x)$ and $\boxslash(y)$ can share at most one vertex of G: an arrow between x and y, when it exists. If $f\colon x \to y$, then f lies in the diagonal of $\boxslash(y)$ but not in that of $\boxslash(x)$. Thus to complete our proof we only need to show that the diamonds of G are all of the form $\delta = \boxslash(x)$ for some vertex $x \in D$. Let then δ be any diamond of G, and call τ_1, τ_2 its two triangles. We know by our above proof of (3) that there exist vertices $x, y \in D$ with $\tau_1 \subseteq \boxslash(x)$ and $\tau_2 \subseteq \boxslash(y)$. But $|\tau_1 \cap \tau_2| = 2$ implies that $|\boxslash(x) \cap \boxslash(y)| \geq 2$, so we conclude that $x = y$ and $\delta = \boxslash(x)$. Thus all diamonds of G are of the required form, and our proof is complete. □

Straightforward verifications show the following:

Theorem 5.3. *Let G, G' be diamantine shoal graphs, and D, D' fishy digraphs. Then:*

(1) $\mathcal{G}(\mathcal{D}(G)) \cong G$. *In particular $G \cong G'$ if, and only if, $\mathcal{D}(G) \cong \mathcal{D}(G')$.*
(2) $\mathcal{D}(\mathcal{G}(D)) \cong D$. *In particular $D \cong D'$ if, and only if, $\mathcal{G}(D) \cong \mathcal{G}(D')$.* □

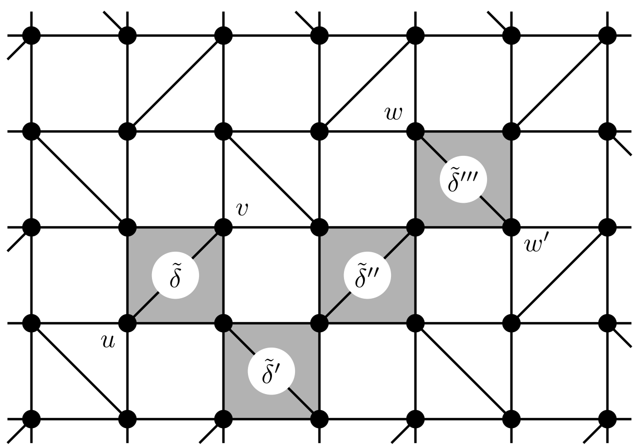
Using Theorem 3.2 and the fact that $\mathcal{D}(\mathcal{F}(G)) = \mathcal{D}(G)^{\mathrm{op}}$ for any diamantine shoal graph G, we can now describe the effect of the clique graph operator at the level of the fishy digraph of diamonds:

Theorem 5.4. *Let G be a diamantine shoal graph. Then $\mathcal{D}(K(G)) \cong \mathcal{D}(G)^{\mathrm{op}}$.* □

Theorems 5.3 and 5.4 provide an exact translation of our problem into the realm of fishy digraphs: modulo the polynomially feasible computations of the fishy digraph $\mathcal{D}$ or the graph of arrows $\mathcal{G}$, to determine the self-clique ($K(G) \cong G$) shoal graphs (which are all diamantine) is the same as determining the self-dual ($D^{\mathrm{op}} \cong D$) fishy digraphs.

Proposition 5.5. *Let H be a Hall shoal graph and $D = \mathcal{D}(H)$ its digraph of diamonds. Then the underlying graph of D is triangleless.*

Proof. Assume to the contrary that one has a triangle $\delta \to \delta' \to \delta'' \leftarrow \delta$ in D. Let $H = \tilde{H}/\Gamma$, where $\Gamma \leq \mathrm{Aut}(\tilde{H})$ is admissible. Then we must have in $\tilde{H}$, modulo some rotation or reflection, a situation as in the following figure,



where $\tilde{\delta}, \tilde{\delta}'$ and $\tilde{\delta}''$ are liftings of δ, δ' and δ'', and also $\tilde{\delta}'''$ is a lifting of δ. Thus, some $\gamma \in \Gamma$ must send $\tilde{\delta}$ into $\tilde{\delta}'''$. The diagonal $\{u, v\}$ of $\tilde{\delta}$ must go to the diagonal $\{w, w'\}$ of $\tilde{\delta}'''$, so $\gamma(v) \in \{w, w'\}$, but $d(v, w) = 3 = d(v, w')$, and the group Γ can not be admissible. □

6. Two New Families

Our families in this section enlarge the set of known shoal graphs and also illustrate the way in which fishy digraphs help to find them.

A variant of Hall's family

Figure 6.1 (left) shows the fishy digraph $\mathcal{D}(\tilde{H})$ of the Hall shoal graph $\tilde{H}$ of Figure 4.1.

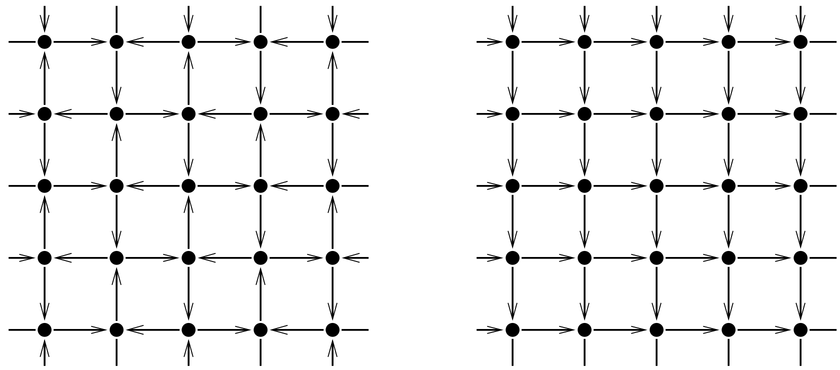


FIGURE 6.1. The infinite fishy digraphs $\mathcal{D}(\tilde{H})$ and $\mathcal{D}(\tilde{J})$.

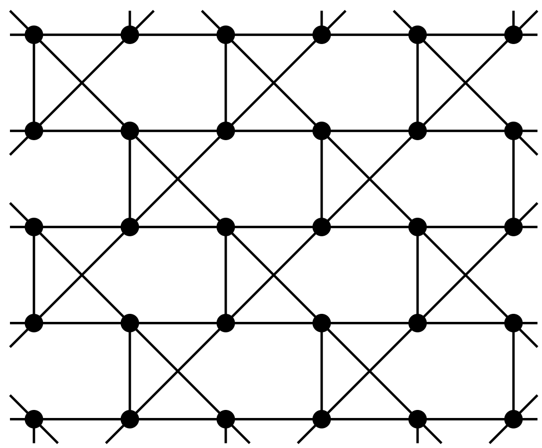


FIGURE 6.2. The infinite shoal graph $\tilde{J}$.

A glance at $\mathcal{D}(\tilde{H})$ immediately suggests the simpler and related infinite fishy digraph $\mathcal{D}(\tilde{J})$ shown in the right. Using the graph of arrows construction one easily finds the corresponding diamantine shoal graph, which is the infinite graph $\tilde{J}$ depicted in Figure 6.2.

Even if $\tilde{J}$ is not planar, it behaves in quite the same way as $\tilde{H}$. Its graph automorphisms are just the restrictions of those isometries of the Euclidean plane which apply into itself its drawing in Figure 6.2. An easy way to see this is to visualize the vertices of the fishy digraph $\mathcal{D}(\tilde{J})$ at the crossing points of pairs of edges of $\tilde{J}$: the automorphisms of $\mathcal{D}(\tilde{J})$ are much clearer.

Therefore any shoal graph J which is a quotient of $\tilde{J}$ is of the form $J \cong \tilde{J}/\Gamma$ for some admissible group $\Gamma \leq \mathrm{Aut}(\tilde{J})$ needing at most two generators and having no rotations nor reflections. Now there are two kinds of glide reflections: the mirror has to be horizontal but it can either pass through a vertex (with odd translation) or through the midpoint of a vertical edge (with even translation). When one generator of Γ is a glide reflection, the central inversion in a vertex or in the crossing point of two edges of $\tilde{J}$ will normalize Γ. We conclude that $J \cong \tilde{J}/\Gamma$ is self-clique for any admissible group $\Gamma \leq \mathrm{Aut}(\tilde{J})$.

In Figure 6.3 we show three fundamental domains for admissible subgroups $\Gamma \leq \mathrm{Aut}(\tilde{J})$. The first two are meant to be translation groups, and the third contains a glide reflection whose mirror passes trough vertices. While the graphs of order 20 in figures 4.2(c) and 6.3(c) are not isomorphic, it is interesting to notice that the graphs in 4.2(a) and 6.3(a) are isomorphic, and that also those in 4.2(b) and 6.3(b) are so. However, this is very infrequent: the graph H in Hall's family can be isomorphic to a graph J in our new family only if some vertex in $\mathcal{D}(\tilde{H})$ was identified with another at distance four while constructing H.

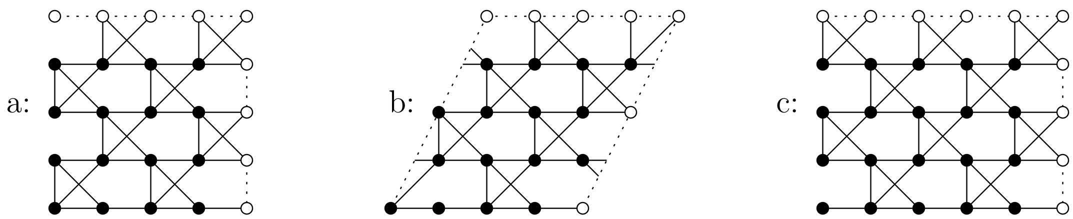


FIGURE 6.3. Fundamental domains for three admissible subgroups of $\mathrm{Aut}(\tilde{J})$.

Circulant digraphs

By Proposition 2.4 the order of any finite shoal graph is even; we shall see soon (Proposition 7.7) that this order must also be at least 14. We will introduce here a family of circulant fishy digraphs that will yield shoal graphs of every possible order, all of them self-clique. In combination with a simple gluing operation, this new family will also lend itself, in Section 8, to the construction of large quantities of shoal graphs, both self-clique and non-self-clique.

Given $n \geq 7$, consider two integers $1 \leq a < b \leq n-1$ and interpret them as modulo n residue classes in $\mathbb{Z}_n$. For the infinite case, if $n = \infty$ (or $n = \aleph_0$) let us agree that $\mathbb{Z}_n$ stands for the ring of integers $\mathbb{Z}$. Assume, further, that the following conditions hold in $\mathbb{Z}_n$:

$$\begin{aligned} 2a, 2b, 3a, 3b &\neq 0, \\ a+b, 2a+b, a+2b &\neq 0, \text{ and} \\ a, b \text{ generate } &\mathbb{Z}_n \text{ as a group.} \end{aligned}$$

For instance $a = 1$ and $b = 2$ always work, but there are other possibilities.

The *circulant digraph* $\vec{C}_n(a,b)$ has $\mathbb{Z}_n$ as vertex set, and there is an arrow $i \to j$ in $\vec{C}_n(a,b)$ if, and only if, $j - i \in \{a, b\}$. Owing to the conditions we have imposed on the jumps a and b, this is a fishy digraph (connected by the last condition). The two arrows leaving any vertex v are $v \to v+a$ and $v \to v+b$, and those arriving at v are $v-a \to v$ and $v-b \to v$. The vertex map $\varphi\colon \vec{C}_n(a,b) \to \vec{C}_n(a,b)^{\mathrm{op}}$, given by $\varphi(i) = \text{-}i$, is an isomorphism, so $\vec{C}_n(a,b)$ is self-dual. Indeed, $i \to j$ in $\vec{C}_n(a,b)$ iff $j - i \in \{a,b\}$ iff $(\text{-}i) - (\text{-}j) \in \{a,b\}$ iff $(\text{-}j) \to (\text{-}i)$ in $\vec{C}_n(a,b)$ iff $(\text{-}i) \to (\text{-}j)$ in $\vec{C}_n(a,b)^{\mathrm{op}}$. Therefore we have:

Proposition 6.1. *For each $n \geq 7$ there is a self-clique shoal graph G of order $2n$.*

Proof. Just take G to be the graph of arrows $G = \mathcal{G}(\vec{C}_n(1,2))$ as in Section 5. □

Let us remark that the examples in figures 4.2 and 6.3 have circulant fishy digraphs of diamonds: these are $\vec{C}_8(1,5)$, $\vec{C}_8(1,3)$ and $\vec{C}_{10}(1,4)$ for Figure 4.2, and $\vec{C}_8(1,5)$, $\vec{C}_8(1,3)$ and $\vec{C}_{10}(1,6)$ for Figure 6.3. On the other hand, of the 12 digraphs in Figure 7.3 only the first one will be circulant. Notice also that for the connectedness of $\vec{C}_n(a,b)$ none of a, b needs to be a generator of $\mathbb{Z}_n$: what is needed is that a and b jointly generate $\mathbb{Z}_n$ (see Figure 6.4).

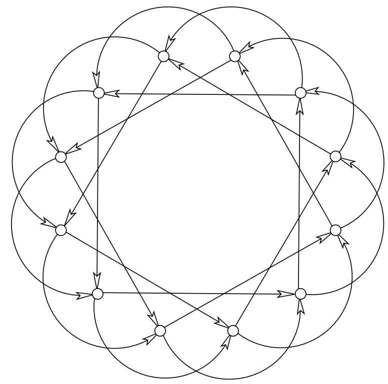


FIGURE 6.4. The circulant digraph $C_{12}(2,3)$.

7. Small Shoal Graphs

We will study the smallest shoal graphs, so here everything will be finite. By Section 5 we can study shoal graphs (diamantine by Proposition 2.4) through their fishy digraphs.

The underlying graph F of any fishy digraph is 4-regular, so we can start with such an F and look for its fishy orientations. In particular, we need balanced orientations of F. These always exist due to the classical Euler Theorem, as any Eulerian circuit in F produces one such orientation (and all of them could be obtained this way). If F is triangleless, its balanced orientations are all fishy, but certain arrangements of triangles of F can even bar the existence of fishy orientations for it.

In the next lemmas F will be the underlying graph of a fishy digraph D. A degree two vertex v of a subgraph S of F will be said to be a *pitcher* (resp. a *runner*, resp. a *catcher*) of S if, with the orientation of D, $d^+_S(v) = 2$ (resp. $d^+_S(v) = 1$, resp. $d^+_S(v) = 0$). The *ears* of a diamond of F are its vertices of degree 2; they are never runners of δ and, furthermore:

Lemma 7.1. *In any diamond δ of F one ear is a pitcher and the other a catcher.* □

Trivial as it is, Lemma 7.1 and its following immediate consequences will be rather useful. The 3-*sun* is the graph with degree sequence 2,2,2,4,4,4. The *squared path graph* P^2_n has vertices $1,\dots,n$, and its *directed* orientation has arrows $\{x \to y \mid y - x \in \{1,2\}\}$.

Lemma 7.2. *No edge is contained in three triangles of F.* □

Lemma 7.3. *There is no tetrahedron in F.* □

Lemma 7.4. *There is no induced* 3-*sun in F.* □

Lemma 7.5. *If $n \geq 5$, every P^2_n in F is directed in D.* □

Lemma 7.6. *There is no induced* 4-*wheel in F.* □

We begin by finding the minimal possible order of a shoal graph.

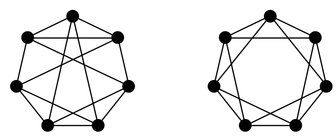
Proposition 7.7. *Every shoal graph G has at least* 14 *vertices.*

Proof. The unique quartic graph of minimal order is K_5, but this does not have fishy orientations by Lemma 7.3. The only quartic graph with six vertices is the octahedron, which does not have fishy orientations by Lemma 7.6. □

The shoal graph of order 14:

Proposition 7.8. *There is a unique (ergo self-clique) shoal graph with* 14 *vertices.*

Proof. The only two 4-regular graphs of order 7 are the complements of $C_3 \cup C_4$ and C_7:



The first one violates Lemma 7.2, and by Lemma 7.5 the second has essentially one fishy orientation: $\vec{C}_7(1,2)$. The corresponding shoal graph G is drawn in Figure 7.1, where each of the 7 diamonds is an isosceles trapezium and its diagonal is dashed for easier recognition. The up-down reflection sends G to $\mathcal{F}(G)$. Note that G is not a Hall shoal graph by Proposition 5.5, but belongs to the family of circulant shoal graphs studied in our Section 6. □

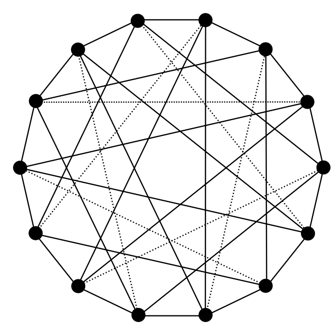
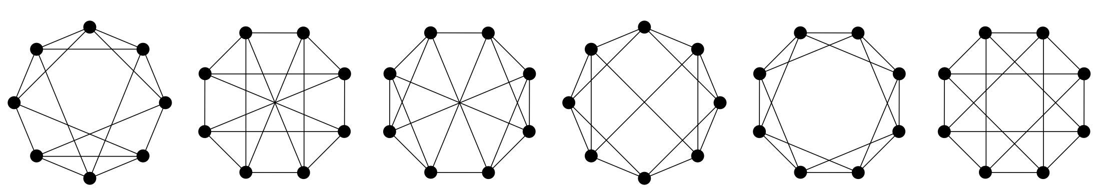


FIGURE 7.1. The smallest shoal graph.

Shoal graphs of order 16:

Proposition 7.9. *There are exactly three shoal graphs with* 16 *vertices, all self-clique.*

Proof. The following are the six quartic graphs with eight vertices. They are Q6, Q9, Q7, Q5, Q8, Q10 in [20], or #1, #2, #5, #4, #3, #6 in Meringer's list [18]:



The first violates lemmas 7.2 and 7.4, and the second violates Lemma 7.3, so they do not have fishy orientations. Using twice Lemma 7.1 one sees that the third graph does not have fishy orientations, and with two applications of Lemma 7.5 one shows the same for the fourth graph.

Up to isomorphy the fifth graph has, by Lemma 7.5, a unique (hence self-dual) fishy orientation: it is the circulant digraph $\vec{C}_8(1,2)$. The corresponding shoal graph, as well as that in Proposition 7.8, is not a Hall shoal graph by Proposition 5.5.

Our last quartic is the complete bipartite graph $F = K_{4,4}$. Denote its parts by $A = \{0,2,4,6\}$ and $B = \{1,3,5,7\}$. In order to determine a fishy orientation of F it suffices to indicate, for any vertex $i \in A$, its ex-neighborhood $B_i \subseteq B$, since the in-neighborhood is then the complement $B \setminus B_i$. Up to symmetry there are just two inequivalent possibilities:

- $B_0 = \{1,3\}$, $B_2 = \{3,5\}$, $B_4 = \{5,7\}$, $B_6 = \{7,1\}$, and
- $B_0 = B_2 = \{1,3\}$ and $B_4 = B_6 = \{5,7\}$.

These give us the circulant digraphs $\vec{C}_8(1,3)$ and $\vec{C}_8(1,5)$ which, as remarked earlier, are the fishy digraphs of the second and first examples of Hall shoal graphs in Figure 4.2. □

Shoal graphs of order 18:

We shall now consider the quartic graphs with nine vertices, which are depicted in Figure 7.2. We will denote them by $F_1, \ldots, F_{16}$. They are Q21, Q25, Q14, Q16, Q19, Q11, Q12, Q15, Q23, Q13, Q17, Q18, Q22, Q26, Q20 and Q24 in [20], or #1, …, #16 in Meringer's list [18], only that we have moved forward the sixth to put it after the ninth (thus making it the new ninth). Also our pictures are mostly Meringer's, we have only redrawn F_{10} and F_{11}.

The twelve fishy orientations afforded by these sixteen quartics of order nine will be depicted later in Figure 7.3.

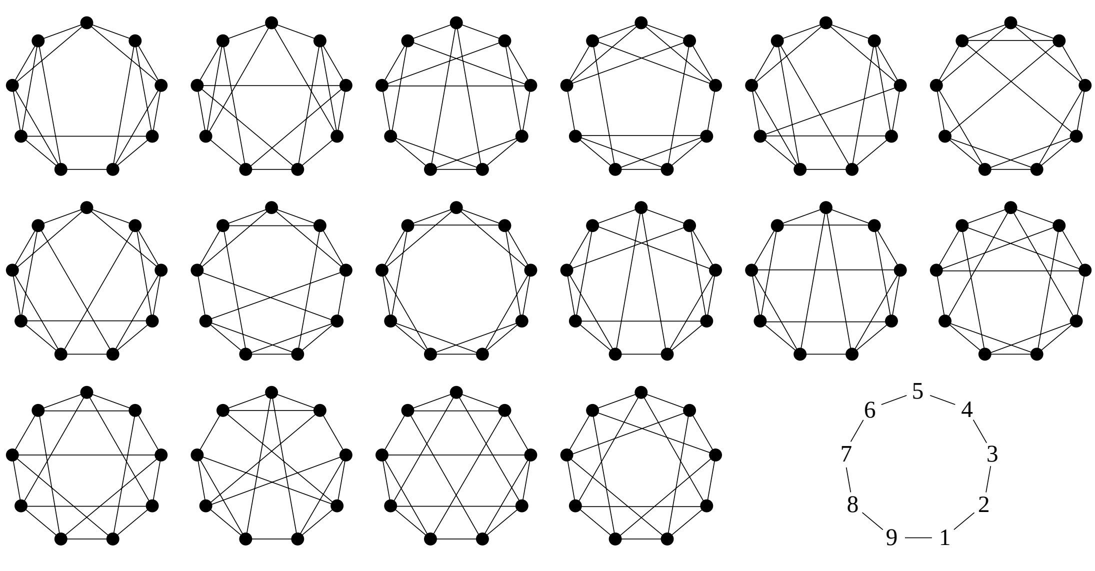


FIGURE 7.2. The sixteen quartic graphs $F_1, \ldots, F_{16}$ and a numeration of their vertices.

Our first eight quartic graphs $F_1, \ldots, F_8$ do not admit fishy orientations:

This is immediate for the first 5 quartics: F_1 violates Lemma 7.3. Both F_2 and F_3 violate Lemma 7.2. F_4 violates both Lemma 7.3 and Lemma 7.6. Finally, F_5 violates Lemma 7.4.

Easy applications of lemmas 7.1 and 7.5 show that neither of F_6, F_7 or F_8 admits any fishy orientation: If, for instance, F_8 admits one, then by Lemma 7.1 we can assume (if D is fishy, so is D^{op}) that 8 is the pitcher and 2 the catcher of the lower diamond, but this forces arrows $2 \to 3$ and $2 \to 7$, so the upper P_5^2 can not comply with Lemma 7.5.

Now we will show that each of the next four quartic graphs $F_9, \ldots, F_{12}$ admits a unique (up to isomorphy) fishy orientation, giving rise to the first four digraphs $D_1, \ldots, D_4$ in Figure 7.3. Every isomorphism of digraphs gives one between their underlying graphs, and any digraph shares the underlying graph with its dual, so we will obtain automatically that $D_1, \ldots, D_4$ are not isomorphic and also that they are self-dual. Note, however, that the first six and the last three of the twelve digraphs in Figure 7.3 are clearly self-dual, as they are sent to their duals by the left-right reflection $\mu = (1\,9)(2\,8)(3\,7)(4\,6)$.

The only fishy orientation of F_9 is clearly $D_1 = \vec{C}_9(1,2)$. Just assuming that 1 is the pitcher of the right diamond of F_{10} determines the fishy orientation D_2. Doing the same for F_{11} determines 14 arrows and the fact that the vertices of the induced quadrilateral $\{2,3,7,8\}$ are all runners in it, so it is a directed 4-cycle; it would seem that two digraphs can be obtained, but the permutation $(2\,3)(7\,8)$ sends each one to the other, so one gets only D_3. Assuming that 8 is the pitcher and 2 the catcher of the lower diamond of F_{12} we can also suppose, using the symmetry $(1\,9)(4\,6)$ if needed, that 6 is the pitcher and 4 the catcher of the upper diamond, and this determines the fishy digraph D_4.

The last four quartics have no diamonds and require a closer scrutiny.

The quartic F_{13} is made up of two induced subgraphs: the *bowtie* $\{2,4,5,6,8\}$ and the *twisted quadrilateral* $\{1,3,7,9\}$, which are connected by the *sides*, i.e. the induced quadrilaterals $\{1,2,3,4\}$ and $\{6,7,8,9\}$. The bowtie is formed by the only two triangles of F_{13}. Any symmetry of F_{13} must fix 5, and send the twisted quadrilateral to itself. Besides the left-right reflection $\mu = (1\,9)(2\,8)(3\,7)(4\,6)$, we will use $\lambda = (7\,9)$ and $\rho = (1\,3)$ to "invert" the twisted quadrilateral, and $\sigma = (2\,4)(6\,8)$ to swap the triangles.

Fix a fishy orientation D of F_{13}. In our first case, 5 is a runner of the upper triangle $\{4,5,6\}$ (hence also of the lower triangle $\{2,5,8\}$). Using μ if needed we have the arrows $4 \to 6$, $4 \to 5 \to 6$. Then we also have $7 \leftarrow 6 \to 9$, $1 \to 4 \leftarrow 3$. There are two subcases:

In the first subcase, if $2 \to 5 \to 8$ and $2 \to 8$, D must be the digraph D_5 in Figure 7.3.

In the second subcase, if $8 \to 5 \to 2$ and $8 \to 2$, besides these three, four further arrows are determined, and the twisted quadrilateral is only forced to be directed. In principle two digraphs could be obtained, but any of λ, ρ takes each one to the other, and therefore only the digraph D_6 is obtained in this subcase.

Note that $D_5 \not\cong D_6$ as they induce non-isomorphic orientations of the twisted quadrilateral.

The remaining case is that in which 5 is not a runner of either triangle. Observe that the digraphs stemming from this case won't be isomorphic to D_5 or D_6. Using σ and/or μ if needed, we have $4 \leftarrow 5 \to 6$, $6 \to 4$ and their concomitant $1 \leftarrow 4 \to 3$, $2 \to 5 \leftarrow 8$. Using λ if needed, we can also obtain $9 \to 6 \to 7$, since 6 must be a runner of the left side $\{6,7,8,9\}$. This second case has two subcases, according to whether $8 \to 2$ or $2 \to 8$.

In the first subcase, besides $8 \to 2$ we get $7 \to 8 \leftarrow 9$, and then $1 \to 9 \leftarrow 3$. Now the quadrilateral $\{1,2,3,7\}$ must be directed, and using ρ if needed we obtain D_7.

In the second subcase, besides $2 \to 8$ we only get $1 \to 2 \leftarrow 3$ and that 8 is a runner of the outer enneagon, leading to two subsubcases: $7 \to 8 \to 9$ and $9 \to 8 \to 7$.

The first subsubcase forces the twisted quadrilateral to be directed, and using ρ if needed we are led to D_8. The second subsubcase leads directly to D_9.

As they induce non-isomorphic orientations of the twisted quadrilateral, D_7, D_8 and D_9 are not isomorphic. They are also self-dual, as $\sigma\mu$, $\sigma\rho$ and $\sigma\lambda$ take them to their duals.

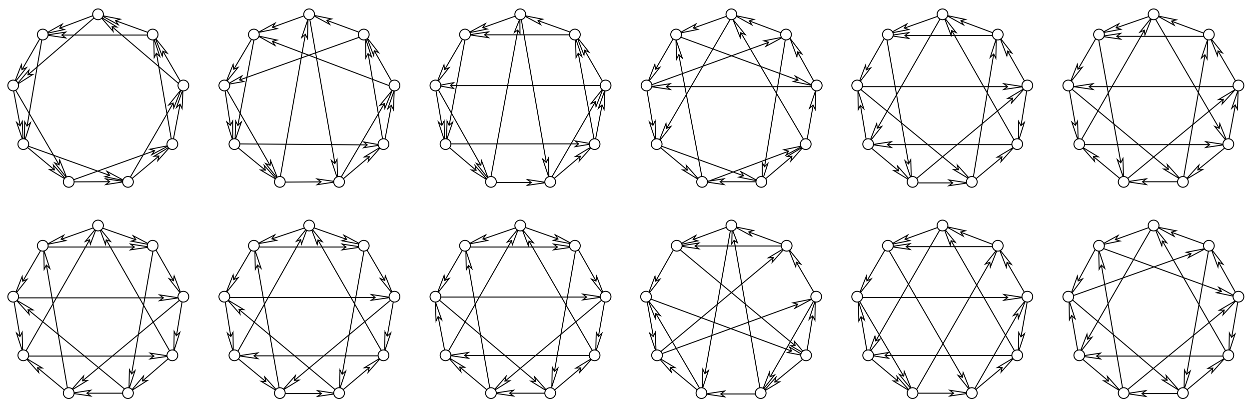


FIGURE 7.3. The twelve fishy digraphs $D_1, \dots, D_{12}$ with nine vertices.

The quartic F_{14} is the line graph of the complete bipartite graph $K_{3,3}$, so its vertices can be identified with $ax, ay, az, bz, \dots, cx$ (denoting by $\{a, b, c\}$ and $\{x, y, z\}$ the parts of $K_{3,3}$). Any symmetry of $K_{3,3}$ induces one of F_{14} in an obvious way (in fact, every symmetry of F_{14} is obtained this way by [21, 5.3], but we don't need that much). In particular, any permutation of the three upper vertices $(4, 5, 6) = (bz, bx, by)$ can be achieved by some symmetry in $\mathrm{Aut}(\{x, y, z\})$, so that we can assume (given a fishy orientation D of F_{14}) that D has arrows $4 \to 5 \to 6$ and $4 \to 6$. Then 5 is also a runner of of the triangle $\{1, 5, 9\}$ and we can also suppose that D has arrows $1 \to 5 \to 9$ and $1 \to 9$ (using if needed $(a\,c) \in \mathrm{Aut}(\{a, b, c\})$, which does not disturb the upper triangle $\{4, 5, 6\}$). But then D is forced to be D_{10}.

Given a fishy orientation D of F_{15}, we can assume, rotating or reflecting the figure if needed, that we have $2 \to 5 \to 8$ and $2 \to 8$. Using the symmetry $(1\,3)(4\,6)(7\,9)$ if needed we can also assume $4 \to 5 \to 6$ and $4 \to 6$. But then D has to be D_{11}.

Given a fishy orientation D of F_{16}, let us call a vertex a pitcher, a runner or a catcher if it is so in the outer enneagon (equivalently, if it is a catcher, a runner or a pitcher in the triangle that contains it). Then there are three vertices of each kind. Note that two pitchers (or two catchers) are never adjacent in F_{16}. We claim that no runner is isolated in the outer enneagon. Suppose, to the contrary, that 5 is an isolated runner, so we can assume that 4 is a pitcher and 6 is a catcher. Then 3 and 7 are also runners. Then 1 is a catcher, so 2 must be a pitcher (since 3, 5 and 7 are already all the runners), contradicting that 3 is a runner. It follows that the three runners are consecutive, let's say that $4, 5, 6$ are runners, 3 a pitcher and 7 a catcher. Then $D = D_{12}$. We therefore have:

Proposition 7.10. *There are exactly twelve shoal graphs with* 18 *vertices, and all of them are self-clique. Their associated fishy digraphs are those in Figure 7.3.* □

Shoal graphs of order 20:

We won't give the full details here, but certainly the case of shoal graphs with 20 vertices can still be tackled by hand using the same method.

There are 59 quartic graphs with 10 vertices, denoted by Q27 – Q85 in [20]. Twenty-seven of them do not admit fishy orientations. Indeed, one sees at first glance that eighteen of these (those with Q numbers 31–33, 49–51, 64–68, 72–75, 81–83) violate our Lemmas 7.2, 7.3, 7.4 or 7.6, while straightforward applications of Lemmas 7.1 and 7.5 take care of the remaining nine (27, 29, 34, 35, 38, 52–55).

Thirteen of these quartics admit, up to isomorphy, a unique fishy orientation, which is necessarily self-dual: Easy uses of Lemmas 7.1 and 7.5 show this for twelve of them (36, 37, 39–41, 43, 56–59, 79, 80), and Q77 also has this property even if it has no diamonds.

Lemmas 7.1 and 7.5 still help to simplify the study of most of the remaining nineteen quartics, as only nine of them do not have diamonds. In all, the order 10 quartic graphs admit 114 fishy orientations up to isomorphy, and only 60 of them are self-dual.

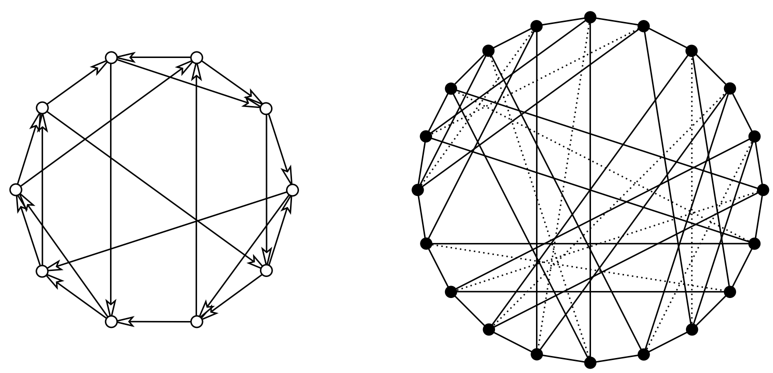


FIGURE 7.4. A non-self-dual fishy digraph D of the minimal possible order and its associated non-self-clique shoal graph G.

In Figure 7.4 we depict an example of a non-self-dual fishy digraph D and its associated shoal graph G, which is necessarily non-self-clique. The underlying graph of D is the quartic Q30, which is easily seen to have trivial symmetry group, so none of its orientations is self-dual. In fact, using Lemma 7.1 one sees easily that the depicted orientation is unique up to duality. As in the case of the unique minimal-order shoal graph in Figure 7.1, in our depiction of the shoal graph G in Figure 7.4 each of the ten diamonds is an isosceles trapezium and its diagonal is dashed for easier recognition. The quartics Q47 and Q48 are the most "productive" among those of order ten: each of them admits, up to isomorphy, fifteen fishy orientations, three self-dual and twelve non-self-dual.

We also used a computer to further explore fishy digraphs up to order 14. The corresponding quartic graphs, generated by the program GENREG, were downloaded from Meringer's site `http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html#CRG` [18]. To search for the fishy orientations of each quartic we used Gap [10], and *nauty* [17] was used for isomorphy and self-duality checking. We summarize our findings in Table 7.1.

Table 7.1. The numbers, up to isomorphy, of fishy digraphs and self-dual fishy digraphs of orders seven to fourteen:

Order:	7	8	9	10	11	12	13	14
Fishy digraphs:	1	3	12	114	1,169	15,546	219,676	3,299,651
Self-dual:	1	3	12	60	237	1,188	5,244	26,363

Quartic graphs, up to order 22, have been generated and counted up to isomorphy, using GENREG, by Markus Meringer [18] and Jason Kimberley (see [19], from where we have borrowed Table 7.2).

Table 7.2. The known numbers of quartic graphs:

Order	Quartics
5	1
6	1
7	2
8	6
9	16
10	59
11	265
12	1,544
13	10,778
14	88,168
15	805,491
16	8,037,418
17	86,221,634
18	985,870,522
19	11,946,487,647
20	152,808,063,181
21	2,056,692,014,474
22	28,566,273,166,527

8. Exponentially Many Examples

In this section we shall give, for each n with $80 \leq n \leq \aleph_0$, exponentially many examples of self-dual and non-self-dual fishy digraphs of order n, thus proving theorems 8.1 and 8.2.

As previously announced, we will apply a gluing technique to our circulant fishy digraphs of Section 6. If D, D' are digraphs, in order to *glue* D and D' *across* the arrows $f\colon x \to y$ of D and $f'\colon x' \to y'$ of D' one subdivides these arrows and identifies the new vertices, obtaining thus a digraph with $|S| + |T| + 1$ vertices. In other words, we remove f and f' and then add a new vertex v and four new arrows $x \to v \to y$ and $x' \to v \to y'$. If both D and D' are connected, 2-regular, or even fishy, then the resulting digraph is so too. We will only use circulants of the form $\vec{C}_n(1, t)$, and each of these has at any vertex x a *long arrow* $x \to x + t$ with some $t > 1$. All gluings will be made across long arrows.

Take $n \geq 80$ and put $n = 10q+r$, where $0 \leq r \leq 9$. For the infinite case, take $n = q = \aleph_0$ and $r = 0$. We rename some circulants: $A = \vec{C}_8(1, 3)$, $B = \vec{C}_8(1, 5)$, $Q = \vec{C}_q(1, 2)$, $R = \vec{C}_{r+8}(1, 2)$.

To a unique copy of Q we will glue several copies T of A, B and R. To *glue* T *to* Q *at* the vertex $x \in Q$ is to glue T and Q across the long arrows of Q at x and of T at 0. We can glue other T's at other vertices of Q, and the resulting digraph is always fishy. Figure 8.1 shows the result of gluing $R = \vec{C}_8(1,2)$ at 0 and $A = \vec{C}_8(1,3)$ at 1 to $Q = \vec{C}_{\aleph_0}(1,2)$.

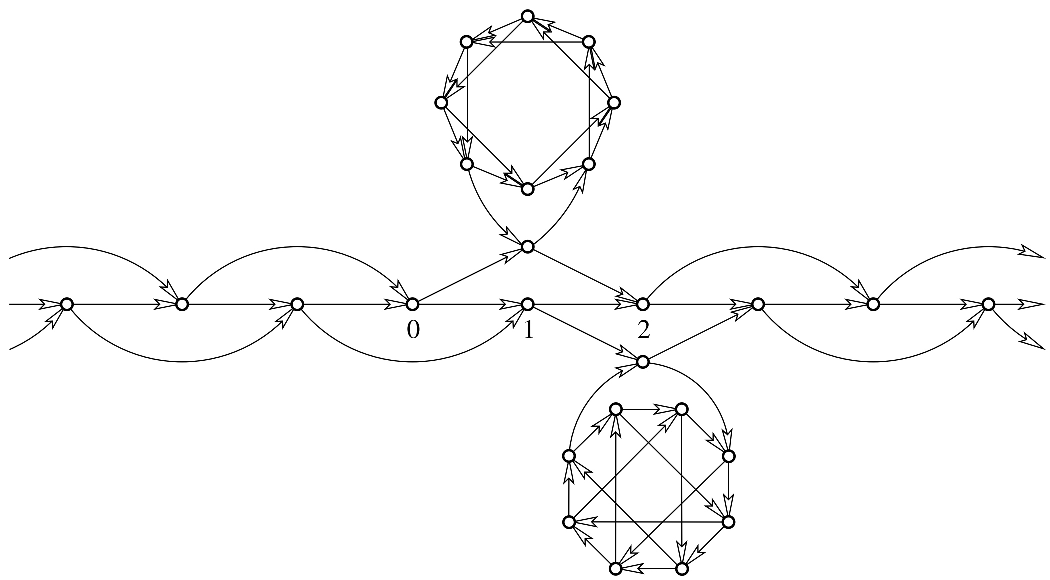


FIGURE 8.1. Result of gluing $R = \vec{C}_8(1,2)$ at 0 and A at 1 to $Q = \vec{C}_{\aleph_0}(1,2)$.

We now construct a digraph $M(X)$ for each $X \subseteq V(Q) \setminus \{0\}$. First, glue R to Q at 0. Then, for each $x \in X$, glue a copy of A at x. Finally, glue a copy of B at each remaining $x \in V(Q) \setminus (X \cup \{0\})$. The digraph $M(X)$ is fishy and has n vertices ($n = \aleph_0$ when $q = \aleph_0$). We have that $M(X) \cong M(X')$ if, and only if, $X = X'$. Besides, $M(X)$ is self-dual precisely when $X = -X$, where $-X = \{-x \mid x \in X\}$.

In the finite case, we just constructed $2^{q-1} = \Theta((2^{\frac{1}{10}})^n)$ non-isomorphic fishy digraphs of order n. Of these, $2^{\lceil \frac{q-1}{2} \rceil} = \Theta((2^{\frac{1}{20}})^n)$ are self-dual and $2^{q-1} - 2^{\lceil \frac{q-1}{2} \rceil} = \Theta((2^{\frac{1}{10}})^n)$ are non-self-dual, so we have:

Theorem 8.1. *The numbers of non-isomorphic self-clique and non-self-clique shoal graphs of finite order* $2n$ *both grow at least exponentially with* n. □

In the infinite case we constructed $\mathfrak{c} = 2^{\aleph_0}$ non-isomorphic self-dual (and also non-self-dual) fishy digraphs of order $\aleph_0$. Any connected and 2-regular infinite digraph must have this order and, as even the total number of labelled digraphs of order $\aleph_0$ is $\mathfrak{c}$, we have:

Theorem 8.2. *The number of isomorphy types of infinite connected diamantine shoal graphs (either self-clique or non-self-clique) is the cardinality of the continuum.* □

It should be clear that the examples constructed here are but an exceedingly tiny sample of shoal graphs. In our view, we have given strong evidences showing that self-clique shoal graphs can not be classified. This would answer a question in [7, §6] in the negative. However, a formal proof of unclassifiability would require a formal definition of classifiability, and this latter is lacking.

Our lower bounds $\Theta((2^{\frac{1}{10}})^n)$ and $\Theta((2^{\frac{1}{20}})^n)$ for the numbers of shoal graphs and self-clique shoal graphs of finite order $2n$ could very well be too small. On the other hand, an easy upper bound (the number of labeled graphs on $2n$ vertices and $5n$ edges) is $\binom{\binom{2n}{2}}{5n} = n^{\Theta(n)}$. Hence the following problems present themselves for the finite case:

Problem 1. *Are there better lower bounds for the number of shoal graphs and self-clique shoal graphs?*

Problem 2. *Are there better upper bounds for the number of shoal graphs and self-clique shoal graphs?*

Problem 3. *Do the number of shoal graphs and the number of self-clique shoal graphs grow exponentially or superexponentially?*

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