EQUIVARIANT COLLAPSES AND THE HOMOTOPY TYPE OF ITERATED CLIQUE GRAPHS

F. LARRIÓN, M.A. PIZAÑA, AND R. VILLARROEL-FLORES

ABSTRACT. The clique graph $K(G)$ of a simple graph G is the intersection graph of its maximal complete subgraphs, and we define iterated clique graphs by $K^0(G) = G$, $K^{n+1}(G) = K(K^n(G))$. We say that two graphs are homotopy equivalent if their simplicial complexes of complete subgraphs are so. From known results it can be easily inferred that $K^n(G)$ is homotopy equivalent to G for every n if G belongs to the class of clique-Helly graphs or to the class of dismantlable graphs. However, in both of these cases the collection of iterated clique graphs is finite up to isomorphism. In this paper we show two infinite classes of clique-divergent graphs that satisfy $G \simeq K^n(G)$ for all n, moreover $K^n(G)$ and G are simple-homotopy equivalent. We provide some results on simple-homotopy type that are of independent interest.

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1. Introduction

All of our graphs are simple and nonempty. They can be infinite, but in any case we require the order of the complete subgraphs to be uniformly bounded. Similarly, we will consider only finite-dimensional simplicial complexes.

We usually identify induced subgraphs with their vertex sets; for instance, we write $v \in G$ rather than $v \in V(G)$. We refer to complete subgraphs of G just as *completes*.

A *clique* of a graph G is a maximal complete of G. The *clique graph* $K(G)$ of G is the intersection graph of the cliques of G. More generally, we define *iterated clique graphs* by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$.

We associate to any graph G the simplicial complex $\Delta(G)$ that has as faces (i.e. simplices) the complete subgraphs of G. We denote by $|\Delta|$ the geometric realization of a simplicial complex Δ, and by $|G|$ the geometric realization of $\Delta(G)$. By means of $|G|$ one usually attaches topological concepts to G; for instance, we say that the graphs G and H are *homotopy equivalent*, and denote it by $G \simeq H$, if $|G|$ and $|H|$ are so. A simplicial complex Δ is a *Whitney complex* (also called *clique complex* and *flag complex* in the literature) if there is a graph G such that $\Delta = \Delta(G)$. In this case, G is necessarily the 1-skeleton of Δ.

A graph G is *clique-Helly* if any collection of pairwise intersecting cliques has a nonempty intersection. Let G be clique-Helly. By Prisner [11] $K(G) \simeq G$, and by Escalante [5] $K(G)$ is also clique-Helly. Hence, if G is clique-Helly we have that $K^n(G) \simeq G$ for all n.

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For a graph G and a vertex x, we denote by $N_G(x)$ the set of all neighbors of x in G. The *closed neighborhood* of x is $N_G[x] = N_G(x) \cup \{x\}$. A vertex $x \in G$ is called *dominated* if there is $x' \in G$, $x' \neq x$, such that $N_G[x] \subseteq N_G[x']$.

The class of *dismantlable* graphs can be defined recursively: the one-vertex graph is dismantlable, and a graph G with at least two vertices is dismantlable if it has a dominated vertex x with $G \setminus x$ dismantlable. Every dismantlable graph G is contractible by Prisner [11], and $K(G)$ is again dismantlable by Bandelt and Prisner [1]. Hence dismantlable graphs also satisfy that $K^n(G) \simeq G$ for all n.

The finite graph G is *clique divergent* if the number of vertices of $K^n(G)$ tends to infinity with n. Escalante also proved in [5] that each clique-Helly graph G is *eventually K-periodic*, that is, there are $n_0 \geq 0$, $p \geq 1$ such that $K^{n+p}(G) \cong K^n(G)$ for all $n \geq n_0$. Prisner proved in [11] that if G is dismantlable then G is *clique null*, that is, there is n_0 such that $K^{n_0}(G)$ is the one-vertex graph. It follows that if G is clique divergent, then no $K^n(G)$ can be clique-Helly or dismantlable. Clearly $K^n(G) \ncong K^m(G)$ for $n \neq m$ if G is clique divergent.

On the other hand, there are examples of clique divergent graphs which are not homotopy equivalent to any of its clique graphs. Indeed, let O_d be the $(2d-2)$-regular graph on $2d$ vertices; since $|O_d|$ is the suspension of $|O_{d-1}|$ and $|O_1|$ is a 0-sphere, it is clear that $|O_d|$ is a $(d-1)$-sphere. Additionally it was shown by Neumann-Lara [10, 5] that $K(O_d) \cong O_{2^{d-1}}$. Hence O_d is clique divergent for $d \geq 3$ and no two of its clique graphs are homotopy equivalent.

This paper's main purpose is to show two infinite families of clique divergent graphs G such that G and $K^n(G)$ are not only homotopy equivalent but even simple-homotopy equivalent for all n. Our examples come from the family of clockwork graphs, which was introduced in [8] (to be considered here in Section 3) and from the family of (orientable) locally C_6 graphs, which was studied in [7] and will be considered here in Section 4. In the latter case we even prove that $K^n(G)$ is collapsible to G. In order to prove homotopy equivalences between simplicial complexes, in Section 2 we shall introduce some tools of simple-homotopy theory which are interesting on their own.

2. Free faces and collapses

If Δ is a simplicial complex and $\sigma \in \Delta$, we will use the notation $[\sigma, \infty) = \{\rho \in \Delta \mid \sigma \subseteq \rho\}$. A maximal face of a simplicial complex Δ is called a *facet*. If $\sigma \in \Delta$ is not a facet, but there is only one facet which contains σ, then σ is called a *free face*. If σ is a free face, we say that the simplicial complex $\Delta' = \Delta \setminus [\sigma, \infty)$ is obtained from Δ by an *elementary collapse* or, more explicitly, by *collapsing* σ. If Δ' is obtained from Δ by a finite sequence of elementary collapses, one says that Δ *collapses* to Δ' and that Δ' *anticollapses* to Δ. We say that the complexes Δ, Δ' are *simple-homotopy equivalent* if it is possible to obtain Δ' from Δ by a finite sequence of collapses and anticollapses.

It is well known that if Δ, Δ' are simple-homotopy equivalent, then their geometric realizations are homotopy equivalent. See for instance [2].

We are mainly interested in collapses of Whitney complexes that produce another Whitney complex. The following is a result of Prisner (proof of Proposition 3.2 in [11]).

Proposition 2.1. [11] *If $x \in G$ is dominated, then $\Delta(G)$ collapses to $\Delta(G \setminus x)$.*

The following result will be useful to identify free faces. If G is a graph and $X \subseteq G$, we denote the *common (closed) neighborhood* of X in G as $N_G[X] = \cap_{x \in X} N_G[x]$.

Proposition 2.2. *Let Δ be any simplicial complex, and let G be the 1-skeleton of Δ. Let $\sigma \in \Delta$ be such that σ is a free face of the subcomplex Δ_N of Δ induced by $N = N_G[\sigma]$. Then σ is a free face in Δ, and the facet containing σ is the same in Δ and Δ_N.*

Proof: Consider the only facet $\varphi \in \Delta_N$ that contains σ, so $\sigma \neq \varphi \in \Delta$ and $\varphi \subseteq N$. If $\varphi' \in \Delta$ is any face, it follows that φ' is (induces) a complete subgraph of G. Now, supposing that $\varphi' \in \Delta$ contains σ, we must have $\varphi' \subseteq N$ and so $\varphi' \in \Delta_N$. But now the maximality of φ forces that $\varphi' \subseteq \varphi$. Hence σ is free in Δ. □

For example, in the setting of Proposition 2.2, if $\sigma \subsetneq N_G[\sigma] \in \Delta$, we get that σ is a free face of Δ. The particular case in which Δ is Whitney is useful since it gives another condition under which a collapse of Δ results in a Whitney complex. If G is a graph and e is an edge of G, we denote by $G \setminus e$ the graph with $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus \{e\}$.

Proposition 2.3. *Let $e \in E(G)$ be such that $N = N_G[e]$ is complete and such that $e \subsetneq N$. Then e is a free face of $\Delta(G)$, N is the $\Delta(G)$-facet containing e, and*

$$\Delta(G) \setminus [\, e, \infty) = \Delta(G \setminus e). \tag{2.1}$$

Proof: Since N is complete we have that $N \in \Delta(G)$, and thus $\Delta(G)_N$ is a full simplex. Since e is properly contained in N, it is a free face of $\Delta(G)_N$, and by Proposition 2.2 it is a free face of $\Delta(G)$. If $e = \{v_1, v_2\}$, equality (2.1) follows from the fact that both sides consist of the completes of G that do not contain both v_1 and v_2. □

A family $\mathcal{F}$ of free faces of a simplicial complex Δ will be called *independently free* if whenever $\sigma, \sigma' \in \mathcal{F}$ and $\tau \in \Delta$ are such that $\sigma \subseteq \tau$ and $\sigma' \subseteq \tau$, then $\sigma = \sigma'$. We will use the notation $[\,\mathcal{F}, \infty) = \cup_{\sigma \in \mathcal{F}} [\sigma, \infty)$. The following properties are immediate:

Proposition 2.4. *Let $\mathcal{F}$ be a collection of free faces in a simplicial complex Δ. Then:*

(1) *For each $\sigma \in \mathcal{F}$, let φ_σ be the only facet in Δ containing σ. Then $\mathcal{F}$ is independently free if and only if the map $\sigma \mapsto \varphi_\sigma$ is injective.*
(2) *$\mathcal{F}$ is independently free if and only if each pair $\{\sigma, \sigma'\} \subseteq \mathcal{F}$ is independently free.*
(3) *If $\mathcal{F}$ is independently free and $\mathcal{F}' \subseteq \mathcal{F}$, $\mathcal{F} \setminus \mathcal{F}'$ is independently free in $\Delta \setminus [\,\mathcal{F}', \infty)$.*
(4) *If $\mathcal{F}$ is finite and independently free, then Δ collapses to $\Delta \setminus [\,\mathcal{F}, \infty)$.* □

We shall only collapse infinite families of free faces when Δ has a simplicial group action. Let Δ be a simplicial complex on which the (possibly infinite) group Γ acts *simplicially*, i.e. Γ acts as a group of simplicial automorphisms of Δ. In this case one says that Δ, together with the action, is a Γ-*complex*. Sometimes, we shall use condition (B) of [4, p.115], that is:

> (B) If $g_0, g_1, \ldots, g_n \in \Gamma$, and $\{\, x_0, x_1, \ldots, x_n \,\}, \{\, g_0x_0, g_1x_1, \ldots, g_nx_n \,\} \in \Delta$, then there is a $g \in \Gamma$ such that $gx_i = g_ix_i$ for all i.

For this strong condition one does not assume that all x_i are distinct. Condition (B) implies the weaker condition (A$'$): If $x, gx \in \varphi$ for some $g \in \Gamma$, $\varphi \in \Delta$, then $x = gx$. This is proved in [4, p.116]: if $x, gx \in \varphi$ we have that $\{x, x\}, \{1x, gx\} \in \Delta$, so by (B) there is an $h \in \Gamma$ with $x = 1x = hx = gx$. Clearly, if a Γ-complex satisfies one of (A$'$) or (B), any Γ-subcomplex also does.

Proposition 2.5. *Let Δ be a Γ-complex which satisfies* (A′). *Let $\sigma, \varphi \in \Delta$ be such that $\sigma, g\sigma \subseteq \varphi$ for some $g \in \Gamma$. Then $\sigma = g\sigma$. In particular, whenever σ is a free face of Δ, the Γ-orbit of σ, $\Gamma\sigma = \{ g\sigma \mid g \in \Gamma \}$, is independently free.*

Proof: Let $x \in \sigma$. Then $x, gx \in \varphi$, so $x = gx$ by (A′). Hence $\sigma = g\sigma$. □

Let Δ be a Γ-complex. For a vertex x of Δ put $\Gamma x = \{ gx \mid g \in \Gamma \}$. The *orbit complex* Δ / Γ has as vertex set $\{ \Gamma x \mid x \in V(\Delta) \}$ and simplices $\Delta / \Gamma = \{ p(\sigma) \mid \sigma \in \Delta \}$ where $p : V(\Delta) \to V(\Delta / \Gamma)$ is given by $x \mapsto \Gamma x$.

Proposition 2.6. *Let σ be a free face of some Γ-complex Δ with property* (B). *Then $p(\sigma)$ is a free face of Δ / Γ, and*

$$(\Delta \setminus [\Gamma\sigma, \infty)) / \Gamma = (\Delta / \Gamma) \setminus [p(\sigma), \infty). \tag{2.2}$$

Proof: Let $p(\omega)$, where $\omega \in \Delta$, be any face of Δ / Γ with $p(\sigma) \subseteq p(\omega)$. If $x \in \sigma$, then $\Gamma x = p(x) \in p(\sigma) \subseteq p(\omega)$, so $\Gamma x = \Gamma y_x$ for some $y_x \in \omega$ and there is a $g_x \in \Gamma$ such that $x = g_x y_x$. But then $\{ y_x \mid x \in \sigma \} \in \Delta$, and since $\sigma = \{ g_x y_x \mid x \in \sigma \} \in \Delta$, from (B) we get an element $g \in \Gamma$ such that $gy_x = g_x y_x = x$ for all $x \in \sigma$. Hence $\sigma \subseteq g\omega$ and, for the unique facet τ of Δ with $\sigma \subseteq \tau$, we must have $g\omega \subseteq \tau$. It follows that $p(\omega) = p(g\omega) \subseteq p(\tau)$, and we have shown that $p(\tau)$ is the only facet of Δ / Γ which contains $p(\sigma)$. If we had $p(\sigma) = p(\tau)$ we could find $z \in \tau \setminus \sigma$ and $g \in \Gamma$ such that $gz \in \sigma \subseteq \tau$ which is absurd by property (A′). Therefore $p(\sigma)$ is a free face of Δ / Γ. The right hand side of (2.2) is $\{ p(\omega) \mid \omega \not\supseteq g\sigma \text{ for all } g \in \Gamma \}$, and this is contained in $(\Delta / \Gamma) \setminus [p(\sigma), \infty)$ by our previous argument: if $p(\omega)$ contains $p(\sigma)$, then $\sigma \subseteq g\omega$ for some $g \in \Gamma$, so $g^{-1}\sigma \subseteq \omega$. The other containment is immediate: if $\omega \supseteq g\sigma$, then $p(\omega) \supseteq p(g\sigma) = p(\sigma)$. □

If σ is a free face of the Γ-complex Δ and its Γ-orbit $\{ g\sigma \mid g \in \Gamma \}$ is independently free (for instance, by Proposition 2.5, if Δ satisfies (A′)) we will say that $\Delta \setminus [\Gamma\sigma, \infty)$ is obtained from Δ by an *elementary Γ-collapse*. Thus, Proposition 2.6 says that condition (B) allows us to apply to Δ an elementary Γ-collapse and interpret it as an elementary collapse of the orbit complex Δ / Γ. Elementary Γ-collapses were defined in [12, Def.6.1] in the context of finite groups acting on finite simplicial complexes, but note the misprint "$-\{\sigma^g\}$" in condition (c) and the absence of an explicit requirement that σ is not a facet. Moreover, we define *Γ-collapses* by means of the following

Definition 2.7. *Let $\mathcal{F}$ be a finite union of Γ-orbits of faces of the Γ-complex Δ. Then $\mathcal{F}$ is* Γ-collapsible *if $\mathcal{F} = \Gamma\sigma_1 \cup \Gamma\sigma_2 \cup \cdots \cup \Gamma\sigma_n$ in such a way that, if $\Delta_0 = \Delta$ and for $i \geq 1$ $\Delta_i = \Delta \setminus [\Gamma\sigma_1 \cup \cdots \cup \Gamma\sigma_i, \infty)$, then $\Gamma\sigma_{i+1}$ is independently free in Δ_i for all $i = 0, \ldots, n-1$.*

For instance, under condition (A′), the Γ-orbit of any free face is Γ-collapsible by Proposition 2.5. Our next result follows from Proposition 2.6:

Proposition 2.8. *If Δ is a Γ-complex with property* (B) *and $\mathcal{F} = \Gamma\sigma_1 \cup \Gamma\sigma_2 \cup \cdots \cup \Gamma\sigma_n \subseteq \Delta$ is Γ-collapsible, then Δ / Γ collapses to $(\Delta \setminus [\mathcal{F}, \infty)) / \Gamma = (\Delta / \Gamma) \setminus \cup_{i=1}^{n} [p(\sigma_i), \infty)$.* □

Proposition 2.9. *Let Δ be a Γ-complex, and $\mathcal{F}$ a finite union of Γ-orbits of faces of Δ which is independently free. Then $\mathcal{F}$ is Γ-collapsible.*

Proof: Apply Proposition 2.4.3 with any ordering of the orbits. □

Proposition 2.10. *Let Δ be a Γ-complex satisfying* (A$'$). *Let $\sigma, \tau_1, \tau_2, \dots, \tau_n$ $(n \geq 1)$ be free faces in Δ, all contained in the same facet φ. Suppose that for each $i = 1, \dots, n$ we have that $\sigma \cup \tau_i \subsetneq \varphi$ and $\sigma \setminus \tau_i = \{x_i\}$ (a singleton) and that all x_i are distinct. Then $\mathcal{F} = \Gamma\sigma \cup \Gamma\tau_1 \cup \cdots \cup \Gamma\tau_n$ is Γ-collapsible.*

Proof: We first prove that each τ_i is free in $\Delta' = \Delta \setminus [\Gamma\sigma, \infty)$. We have $\varphi \setminus \{x_i\} \in \Delta'$: if not, $\varphi \setminus \{x_i\} \supseteq g\sigma$ for some $g \in \Gamma$, but then $\sigma = g\sigma$ by Proposition 2.5 and $x_i \in \sigma = g\sigma \subseteq \varphi \setminus \{x_i\}$. Since $x_i \notin \tau_i \subseteq \varphi$, we have $\tau_i \subseteq \varphi \setminus \{x_i\}$. Let $\rho \in \Delta'$ be any face with $\tau_i \subseteq \rho$. Then $\rho \subseteq \varphi$ in Δ. Since $\sigma = (\sigma \setminus \tau_i) \cup (\sigma \cap \tau_i) \subseteq \{x_i\} \cup \tau_i$, we must have $x_i \notin \rho$ (otherwise $\sigma \subseteq \rho$, contradicting $\rho \in \Delta'$), thus $\rho \subseteq \varphi \setminus \{x_i\}$, and so $\varphi \setminus \{x_i\}$ is the unique facet of Δ' containing τ_i. We must have that $\tau_i \neq \varphi \setminus \{x_i\}$, for otherwise $\sigma \cup \tau_i = \sigma \cup (\varphi \setminus \{x_i\}) = \varphi$.

We finally prove that $\mathcal{F}' = \Gamma\tau_1 \cup \cdots \cup \Gamma\tau_n$ is independently free in the Γ-complex Δ'. Assume $g\tau_i$ and $h\tau_j$ are both contained in the same facet of Δ' for some $g, h \in \Gamma$ and $i, j \in \{1, \dots, n\}$. Since $g\tau_i \subseteq g(\varphi \setminus \{x_i\})$ and $h\tau_j \subseteq h(\varphi \setminus \{x_j\})$ we must have that $g(\varphi \setminus \{x_i\}) = h(\varphi \setminus \{x_j\})$. But then, if $\alpha = \varphi \setminus \{x_i\}$, we have that $h^{-1}g\alpha = \varphi \setminus \{x_j\} \subseteq \varphi$. As also $\alpha \subseteq \varphi$. we get, using Proposition 2.5, that $\varphi \setminus \{x_i\} = \alpha = h^{-1}g\alpha = \varphi \setminus \{x_j\}$ and hence $i = j$ because all x_i are distinct. Therefore $g\tau_i$ and $(hg^{-1})g\tau_i = h\tau_i$ are both contained in the same facet of Δ' and so $g\tau_i = h\tau_i$ by Proposition 2.5. The proof is then complete by Proposition 2.9. □

3. Clockwork graphs

Clockwork graphs were introduced in [8]. We will use the definition from [9]. A *cyclically segmented* graph G is a graph with a fixed partition $(G_0, G_1, \dots, G_{s-1})$ of its vertex set into $s \geq 3$ nonempty complete subgraphs such that every edge and every triangle of G is contained in the union of two consecutive (modulo s) segments. Each set G_i is then called a *segment* of G. If $B = (B_0, B_1, \dots, B_{s-1})$ and $C = (C_0, C_1, \dots, C_{s-1})$ are cyclically segmented graphs, their *segmented sum* is the graph G with $V(G) = V(B) \cup V(C)$ and $E(G) = E(B) \cup E(C) \cup \{\, \{b, c\} \mid b \in B_i \cup B_{i+1}, c \in C_i, 0 \leq i < s \,\}$, indices taken modulo s. This segmented sum G is cyclically segmented with segments $(B_0 \cup C_0, \dots, B_{s-1} \cup C_{s-1})$.

The cyclically segmented graph $C = (C_0, C_1, \dots, C_{s-1})$ is called a *core* if there is a linear order on each C_i in such a way that $x, y \in C_i$ and $x < y$ imply $N_C(x) \cap C_{i+1} \subseteq N_C(y) \cap C_{i+1}$ and $N_C(x) \cap C_{i-1} \supseteq N_C(y) \cap C_{i-1}$. The cyclically segmented graph $B = (B_0, B_1, \dots, B_{s-1})$ is a *crown* if each of its segments has at least two vertices, and for each i the edges of B connecting B_i and B_{i+1} constitute a perfect matching. A *clockwork graph* is a segmented sum of a core and a crown.

Theorem 3.1. (Theorem 5.4 from [8]) *If G is a clockwork graph with s segments, then its clique graph $K(G)$ is also a clockwork graph with s segments.*

The fact that $K^n(G)$ and G are simple homotopy-equivalent for all n follows from:

Theorem 3.2. *If G is a clockwork graph with s segments, then $\Delta(G)$ collapses to a cyclic graph on $2s$ vertices.*

Proof: Let G be the segmented sum of the core $C = (C_0, C_1, \dots, C_{s-1})$ and the crown $B = (B_0, B_1, \dots, B_{s-1})$. If $v \in C_i$ has a neighbor in C_{i+1}, we denote by $\lambda(v)$ the last (i.e. the greatest) of the neighbors of v in C_{i+1}.

Step 1. For a given index $0 \leq i < s$, let v_i the smallest vertex in C_i with a neighbor in C_{i+1} (if no segment of the core has a vertex with a neighbor in the next segment of the core, we can skip to Step 2). Let e_i be the edge $\{v_i, \lambda(v_i)\}$. Then

$$N_G[e_i] = \{\, v \in C_i \mid v_i \leq v \,\} \cup \{\, v \in C_{i+1} \mid v \leq \lambda(v_i) \,\} \cup B_{i+1}, \tag{3.1}$$

which is a complete of G. Therefore, by Proposition 2.3, e_i can be removed from G without altering its homotopy type, and by the definition of clockwork graphs it also follows that $G \setminus e_i$ is a clockwork graph. Repeating this step as necessary, we arrive to a clockwork graph G' with no edges between consecutive segments of its core, and such that $\Delta(G)$ collapses to $\Delta(G')$.

Step 2. Let now $v_i \in B_i$ be a vertex of the crown of G', v_{i+1} its neighbor in B_{i+1}, and let e_i be the edge $\{v_i, v_{i+1}\}$. Then $N_G[e_i] = e_i \cup C_i$, which is a clique. By Proposition 2.3, the edge e_i is free and, in fact, the collection of all such edges is independently free by Proposition 2.4.1. Hence we can remove all edges of the form e_i from G' without altering its simple-homotopy type.

Step 3. From the previous step we obtain a graph G'' which is no longer a clockwork graph but is actually a cyclically segmented graph with $2s$ segments, where the segmentation is given by $G'' = (C_0, B_0, C_1, B_1, \ldots, C_{s-1}, B_{s-1})$. Every vertex in any given segment of G'' is a neighbor of every vertex in the next and the previous segments. If there are two vertices $x \neq y$ in a given segment of G'', then y dominates x. Removing x from G'' leaves a graph with the same property we have observed in G''. Hence we can apply repeatedly Proposition 2.1 until we finally obtain the cyclic graph on $2s$ vertices. □

It is easy to construct examples of clique divergent clockwork graphs. Let G be a clockwork graph with core C and crown B as above. A segment C_i is *bad* if $\lambda(v_i)$ is the greatest element of C_{i+1} for some $v_i \in C_i$ (and, a fortiori, for the greatest $v_i \in C_i$). Otherwise, C_i is *good*. By the definition of a core, $x, y \in C_i$ and $x < y$ imply $\lambda(x) \leq \lambda(y)$ (if $\lambda(x)$ is undefined, put $\lambda(x) = \max C_i$). We say that a vertex $y \in C_i$ is (*strongly*) *covered* if there is an $x \in C_i$ with $x < y$ and $\lambda(x) = \lambda(y)$. Call $n_g(G)$ and $n_c(G)$ the numbers of good segments and covered vertices of G respectively. Since $n_g(K(G)) = n_g(G)$, $n_c(K(G)) \leq n_c(G)$ and the order of $K(G)$ is $o(K(G)) = o(G) + n_g(G) - n_c(G)$, it follows that G is clique divergent if $n_g(G) > n_c(G)$. See [8, 9] for more details.

4. Locally C_6 graphs in the torus

If G and H are graphs, we say that G is *locally* H if $N_G(x) \cong H$ for all $x \in G$. We denote by C_6 the cyclic graph on 6 vertices. In the paper [7], the authors show that any locally C_6 graph G is K-divergent: in fact, $o(K^n(G)) = (n+1) \cdot o(G)$. More generally, a graph G is *locally cyclic* if $N_G(x)$ is a cycle for each $x \in G$. It is easy to see that $|G|$ is a closed surface if and only if G is locally cyclic and $G \neq K_4$: the tetrahedron K_4 is locally cyclic but $|K_4|$ is a ball, not a surface. In other words, save for that tetrahedron, the locally cyclic graphs are precisely the 1-skeletons of the Whitney triangulations of closed surfaces. If G is locally cyclic with average degree $\bar{d}$, the Euler characteristic of $|G|$ is $\chi = \frac{n(6-\bar{d})}{6}$. Thus, if G is a regular graph (every vertex has degree $\bar{d}$), G is the 1-skeleton of a Whitney triangulation of the torus or the Klein bottle if and only if $\bar{d} = 6$, i.e. G is a locally C_6 graph. We shall prove in this section that $K^n(G) \simeq G$ for all the locally C_6 graphs that triangulate the torus. We will recall the needed material from [7], to which we refer for further details.

For graphs $\tilde{G}$ and G, a *homomorphism* $p\colon \tilde{G} \to G$ is a map of vertices that sends adjacent vertices to adjacent vertices. The homomorphism is called a *covering map* if it satisfies the *unique edge lifting property*, that is, for any edge $e = \{v_0, v_1\}$ and vertex $\tilde{v}_0 \in p^{-1}(v_0)$ there is a unique edge $\tilde{e} = \{\tilde{v}_0, \tilde{v}_1\}$ in $\tilde{G}$ such that $p(\tilde{v}_1) = v_1$. The covering map $p\colon \tilde{G} \to G$ is a *triangular cover* if it satisfies the (necessarily *unique*) *triangle lifting property*: for any triangle T in G, $v \in T$ and $\tilde{v} \in p^{-1}(v)$ there is a triangle $\tilde{T}$ in $\tilde{G}$ with $\tilde{v} \in \tilde{T}$ and $p(\tilde{T}) = T$.

Theorem 4.1. (Proposition 2.2 from [7]) *If $p\colon \tilde{G} \to G$ is a triangular covering map, then the correspondence $Q \mapsto p(Q)$ defines a triangular covering map $p_K\colon K(\tilde{G}) \to K(G)$.*

Suppose that the group Γ acts on a graph G preserving adjacencies. The *quotient graph* $G\,/\,\Gamma$ has as vertex set $\{\,\Gamma v \mid v \in G\,\}$, and Γv is declared neighbor of $\Gamma v'$ if there are $g, g' \in \Gamma$ such that gv and $g'v'$ are adjacent in G. Notice that Γ acts simplicially on $\Delta(G)$, and that the following equality is immediate:

$$\Delta(G)\,/\,\Gamma\ = \Delta(G\,/\,\Gamma\,). \tag{4.1}$$

The action of the group Γ on G is called *admissible* if $d(v, gv) \geq 4$ for all $v \in G$, $g \in \Gamma \setminus \{1\}$. The action of Γ is admissible if and only if the correspondence $v \mapsto \Gamma v$ induces a triangular cover $p_\Gamma\colon G \to G\,/\,\Gamma$ (Lemma 3.1 from [7]).

Proposition 4.2. *Let Γ act admissibly on G, and Δ be a Γ-subcomplex of $\Delta(G)$ (induced or not). Then the action of Γ on Δ satisfies condition* (B).

Proof: Assume that $\{\,x_0, x_1, \dots, x_n\,\}$ and $\{\,g_0x_0, g_1x_1, \dots, g_nx_n\,\}$ are Δ-faces. Then we have that $\{\,x_0, g_0^{-1}g_1x_1, \dots, g_0^{-1}g_nx_n\,\}$ is a complete subgraph of G, and for each $1 \leq i \leq n$ we have that $d(x_i, g_0^{-1}g_ix_i) \leq d(x_i, x_0) + d(x_0, g_0^{-1}g_ix_i) \leq 2$. By admissibility of the action, it follows that $g_0 = g_i$ for all $0 \leq i \leq n$. □

A triangular cover $p\colon \tilde{G} \to G$ is *Galois* with group $\Gamma \leq \mathrm{Aut}(\tilde{G})$ if G can be identified with $\tilde{G}\,/\,\Gamma$ in such a way that p gets identified with p_Γ. Note that by Lemma 3.1 of [7], Γ acts admissibly on $\tilde{G}$ in this case or, as we will also say, Γ *is admissible*.

Theorem 4.3. (Proposition 3.2 from [7]) *If $p\colon \tilde{G} \to G$ is Galois with group Γ, then $p_K\colon K(\tilde{G}) \to K(G)$ is Galois with group $\Gamma_K = \{\,\gamma_K \mid \gamma \in \Gamma\,\} \cong \Gamma$.*

Here $\gamma_K : K(\tilde{G}) \to K(\tilde{G})$ is defined by $\gamma_K(Q) = \{\gamma x \mid x \in Q\}$. In other words, if the action of Γ on $\tilde{G}$ is admissible, then Γ also acts admissibly on $K(\tilde{G})$, and $K(\tilde{G})\,/\,\Gamma \cong K(\tilde{G}\,/\,\Gamma\,)$.

We will define a graph $\mathcal{T}$ which was shown in [7] to cover all locally C_6 graphs. We take $V(\mathcal{T}) = \mathbb{Z} \oplus \mathbb{Z}$. Put $T = \{(0,0), (1,0), (0,1)\} \subseteq V(\mathcal{T})$ and $T - T = \{\,t - t' \mid t, t' \in T\,\}$, then the distinct vertices x, y are declared neighbors in $\mathcal{T}$ if $y - x \in T - T$ (see Figure 1).

Theorem 4.4. (Proposition 4.1 from [7] and the comment following it) *If G is a finite locally C_6 graph, then there is an admissible $\Gamma \leq \mathrm{Aut}(\mathcal{T})$ such that $G \cong \mathcal{T}\,/\,\Gamma$.*

We will describe those graphs $\mathcal{T}^n$ $(n \geq 0)$ which were shown in [7] to be isomorphic to $K^n(\mathcal{T})$. Let $p = (1,1) \in V(\mathcal{T})$ and $P = \{p\}$. Define the subsets $E_j \subseteq V(\mathcal{T})$ as follows: $E_0 = T - T$, $E_1 = T + T$, $E_2 = P + T$, $E_3 = P + P$, $E_j = \varnothing$ for $j \geq 4$ and $E_j = -E_{-j}$ for $j < 0$. Let $V(\mathcal{T}^n) = \{n\} \times \{0, 1, \dots, n\} \times V(\mathcal{T})$. We will denote $(n, i, x) \in V(\mathcal{T}^n)$ just as x_i^n. Then the distinct vertices x_i^n, y_j^n are declared adjacent in $\mathcal{T}^n$ whenever $y - x \in E_{j-i}$.

In order to describe the cliques of $\mathcal{T}^n$ we still need more notation. For $i \in \{0, 1, \dots, n\}$, denote by $\mathcal{T}_i^n$ the subgraph of $\mathcal{T}^n$ induced by $\{\,x_i^n \mid x \in V(\mathcal{T})\,\}$. Note that $\mathcal{T}_i^n \cong \mathcal{T}$.

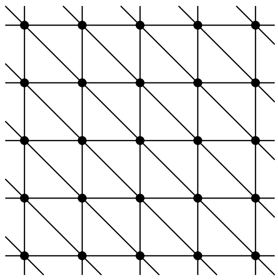

FIGURE 1. The graph $\mathcal{T}$

If $j \in \mathbb{Z} \setminus \{0, 1, \ldots, n\}$, put $\mathcal{T}_j^n = \varnothing$. For $X \subseteq V(\mathcal{T})$, let $X_j^n = \{ x_j^n \in \mathcal{T}_j^n \mid x \in X \}$. For $i \in \{0, 1, \ldots, n, n+1\}$ and $x \in V(\mathcal{T})$, put

$$Q_{i,x}^n = (x - P)_{i-2}^n \cup (x - T)_{i-1}^n \cup (x + T)_i^n \cup (x + P)_{i+1}^n. \tag{4.2}$$

The sets $Q_{i,x}^n$ are precisely the cliques of $\mathcal{T}^n$. If $0 \le i - 2 \le i + 1 \le n$, then $Q_{i,x}^n$ has eight vertices, which are marked with black dots in Figure 2.

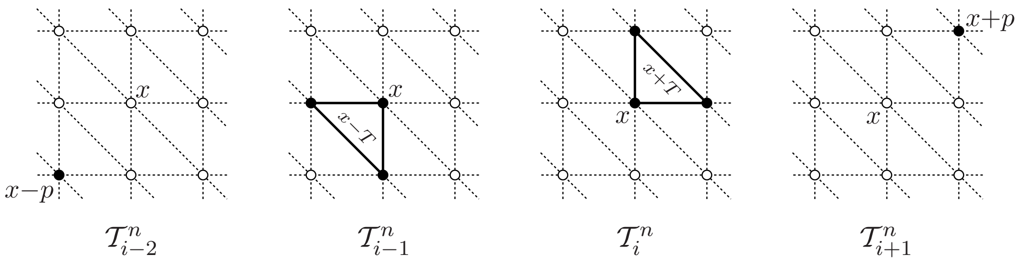

FIGURE 2. A clique of $\mathcal{T}^n$

The assignation $\phi_n(Q_{i,x}^n) = x_i^{n+1}$ defines an isomorphism of graphs $\phi_n \colon K(\mathcal{T}^n) \to \mathcal{T}^{n+1}$. Consider the obvious isomorphism $\psi_0 \colon \mathcal{T} \to \mathcal{T}^0$. One can then recursively obtain isomorphisms $\psi_{n+1} = \phi_n \circ (\psi_n)_K \colon K^{n+1}(\mathcal{T}) \to \mathcal{T}^{n+1}$ for $n \ge 0$.

We say that G is *orientable* when $|\Delta(G)|$ is an orientable surface. Note that $\Gamma \le \operatorname{Aut} \mathcal{T}$ is a group of translations when $G = \mathcal{T} / \Gamma$ triangulates the torus, and this happens if, and only if, G is orientable. In this case, as shown in [7], defining a Γ-action on $\mathcal{T}^n$ by $g(x_i^n) = (gx)_i^n$, the isomorphism $\psi_n : K^n(\mathcal{T}) \to \mathcal{T}^n$ is in fact Γ-equivariant.

Theorem 4.5. *If G is finite, locally C_6 and orientable, $K^n(G)$ collapses to $K^{n-1}(G)$ for all $n \ge 1$.*

Proof: Let $n \ge 1$ fixed. By Theorem 4.4, there is a $\Gamma \le \operatorname{Aut}(\mathcal{T})$ with Γ admissible and $G \cong \mathcal{T} / \Gamma$. By Theorem 4.3, $K^n(G) \cong K^n(\mathcal{T} / \Gamma) \cong K^n(\mathcal{T}) / \Gamma$ and, since G is orientable, $K^n(G) \cong \mathcal{T}^n / \Gamma$. Likewise, $K^{n-1}(G) \cong \mathcal{T}^{n-1} / \Gamma$. Using the mapping $x_i^{n-1} \mapsto x_i^n$ we can identify $\mathcal{T}^{n-1}$ with the subgraph of $\mathcal{T}^n$ induced by $\mathcal{T}_0^n \cup \mathcal{T}_1^n \cup \cdots \cup \mathcal{T}_{n-1}^n$, and this identification is Γ-equivariant.

We want to show that $K^n(G)$ collapses to $K^{n-1}(G)$, and by Proposition 2.8 it will be enough if, starting with $\Delta = \Delta(\mathcal{T}^n)$, we arrive at its induced Γ-subcomplex $\Delta(\mathcal{T}^{n-1})$ after successively collapsing some Γ-collapsible families of free faces. Our strategy will be to remove gradually all connections from $\mathcal{T}^n_n$ to the rest of $\mathcal{T}^n$.

Step 1. If $n \geq 3$, for each $x \in \mathcal{T}$ the edge $\varepsilon_x = \{(x-p)^n_{n-3}, (x+p)^n_n\}$ is a free face of Δ by Proposition 2.3 as $N_{\mathcal{T}^n}[\varepsilon_x] = Q^n_{n-1,x}$. Since all $Q^n_{i,x}$ are distinct, $\{\varepsilon_x\}_{x\in\mathcal{T}}$ is Γ-collapsible by Proposition 2.9 and Proposition 2.4.1. Removing those edges from Δ (and, of course, all faces containing them) we get a Whitney complex Δ_1 with no connections between vertices in $\mathcal{T}^n_n$ and those in $\mathcal{T}^n_{n-3}$.

The next steps would only involve $\mathcal{T}^n_{n-2}$, $\mathcal{T}^n_{n-1}$ and $\mathcal{T}^n_n$, and the simplices to be collapsed will always contain some vertex in $\mathcal{T}^n_n$. Therefore, in order to simplify the notation we will assume from now on that $n = 2$. We show that one obtains $\Delta(\mathcal{T}^1)$ from $\Delta_1 = \Delta(\mathcal{T}^2)$ by a finite number of Γ-collapses. As we will be working always inside $\mathcal{T}^2$, we can dispense with some upper indices: for instance, $(x+T)_2$ will mean $(x+T)^2_2$.

Step 2. Given $x \in \mathcal{T}$, the triangles $\tau_x = (x+T)_2$ and $\tau'_x = (x-T)_2$ are free in Δ_1 by Proposition 2.2, since $N_{\mathcal{T}^2}[\tau_x] = Q^2_{2,x}$ and $N_{\mathcal{T}^2}[\tau'_x] = Q^2_{3,x}$. Again, the collection $\cup_{x\in\mathcal{T}}\{\tau_x, \tau'_x\}$ is Γ-collapsible by Proposition 2.9 as all $Q^2_{i,x}$ are distinct. The Γ-collapse of this collection leads to the Γ-complex $\Delta_2 = \Delta_1 \setminus \cup_{x\in\mathcal{T}}([\tau_x, \infty) \cup [\tau'_x, \infty))$ which is no longer Whitney, but now we have that the subcomplex of Δ_2 induced by $\mathcal{T}^2_2$ is 1-dimensional.

Step 3. Let $e_1 = (1,0)$, $e_2 = (0,1)$. For every $x \in \mathcal{T}$, let $\lambda_x = \{x_2, (x+e_2)_2\} \in \Delta_2$. We have that $N_{\mathcal{T}^2}[\lambda_x] = (x-P)_0 \cup (x-T)_1 \cup \lambda_x \cup \{(x+e_1)_2, (x+e_2-e_1)_2\}$. Then the hypotheses of Proposition 2.2 are satisfied with $\Delta = \Delta_2$ and $\sigma = \lambda_x$ because the only maximal face of $(\Delta_2)_N$ containing λ_x is $(x-P)_0 \cup (x-T)_1 \cup \lambda_x$ (here we are using that none of the triangles $\tau_x = (x+T)_2$ and $\tau'_{x+e_2} = (x+e_2-T)_2$ is a face of Δ_2). Hence each λ_x is a free face of Δ_2, and similarly $\lambda'_x = \{x_2, (x+e_1)_2\}$ and $\lambda''_x = \{(x+e_1)_2, (x+e_2)_2\}$ are free in Δ_2 for each $x \in \mathcal{T}$. The collection $\cup_{x\in\mathcal{T}}\{\lambda_x, \lambda'_x, \lambda''_x\}$ is Γ-collapsible by Proposition 2.9, and their Γ-collapse leaves a Γ-complex Δ_3. Note that the subcomplex of Δ_3 induced by $\mathcal{T}^2_2$ is 0-dimensional.

Step 4. For each $x \in \mathcal{T}$, consider the face $\eta_x = \{(x-p-e_1)_0, (x-2e_1)_1, x_2\} \in \Delta_3$. We can apply Proposition 2.2 to the situation $\Delta = \Delta_3$ and $\sigma = \eta_x$, because this time we have that $N_{\mathcal{T}^2}[\eta_x] = (x-P-e_1)_0 \cup (x-e_1-T)_1 \cup (x-e_1+T)_2$, and the only $(\Delta_3)_N$-maximal face which contains η_x is $(x-P-e_1)_0 \cup (x-T-e_1)_1 \cup \{x_2\}$. Hence each η_x is free in Δ_3 and, likewise, $\eta'_x = \{(x-p-e_2)_0, (x-2e_2)_1, x_2\}$ and $\eta''_x = \{(x-p)_0, x_1, x_2\}$ are free in Δ_3 for each $x \in \mathcal{T}$. The family $\cup_{x\in\mathcal{T}}\{\eta_x, \eta'_x, \eta''_x\}$ is Γ-collapsible by Proposition 2.9. Let us call Δ_4 the Γ-complex obtained after the Γ-collapse of this family.

Step 5. Let us note first that for $x \in \mathcal{T}$, the only facet of Δ_4 that contains x_2 and either one of $(x-p)_0, (x-p-e_1)_0, (x-p-e_2)_0$ is $\Phi_x = (x-P-T)_0 \cup (x-P+T)_1 \cup \{x_2\}$. See Figure 3. Therefore, the edges $\varphi_x = \{(x-p)_0, x_2\}$, $\varphi'_x = \{(x-p-e_1)_0, x_2\}$, and $\varphi''_x = \{(x-p-e_2)_0, x_2\}$ are free in Δ_4 for each $x \in \mathcal{T}$.

Clearly, the families $\mathcal{F} = \{\varphi_x\}_{x\in\mathcal{T}}$, $\mathcal{F}' = \{\varphi'_x\}_{x\in\mathcal{T}}$ and $\mathcal{F}'' = \{\varphi''_x\}_{x\in\mathcal{T}}$ are independently free, but now $\mathcal{F} \cup \mathcal{F}' \cup \mathcal{F}''$ is not so. However, putting $\sigma = \varphi_x \cup \varphi'_x$, $\tau_1 = \varphi_x \cup \varphi''_x$, $\tau_2 = \varphi'_x \cup \varphi''_x$ and $\varphi = \Phi_x$ it follows from Proposition 2.10 that $\mathcal{F}_x = \Gamma\sigma \cup \Gamma\tau_1 \cup \Gamma\tau_2$ is Γ-collapsible. Since there are only a finite number of such $\mathcal{F}_x$ ($\Gamma x = \Gamma y$ implies $\mathcal{F}_x = \mathcal{F}_y$) we get that

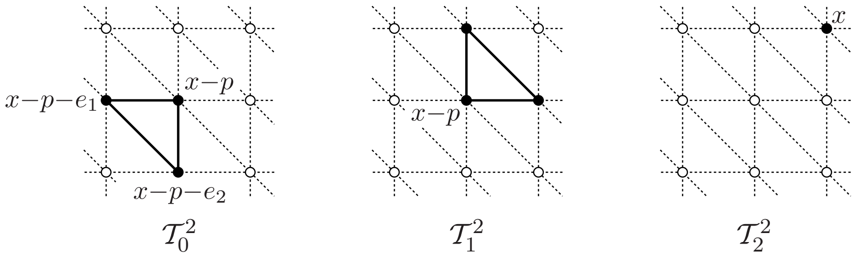

FIGURE 3. A maximal face of Δ_4

$\cup_{x\in V(\mathcal{T})}\mathcal{F}_x$ is Γ-collapsible. Now $\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}''$ is independently free in $\Delta_4\setminus[\cup_{x\in V(\mathcal{T})}\mathcal{F}_x,\infty)$. Let $\Delta_5=\Delta_4\setminus[\cup_{x\in V(\mathcal{T})}\mathcal{F}_x,\infty)\setminus[\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}'')=\Delta_4\setminus[\mathcal{F}\cup\mathcal{F}'\cup\mathcal{F}'')$.

Step 6. The complex Δ_5 is again a Whitney complex, in fact $\Delta_5=\Delta(L)$ for a graph L with $V(L)=V(\mathcal{T}^2)$. The vertex set of $\mathcal{T}_0^2\cup\mathcal{T}_1^2$ induces in L the same subgraph as in $\mathcal{T}^2$, namely the subgraph that we know can be identified with $\mathcal{T}^1$ in a Γ-equivariant way. In L, the vertices in $\mathcal{T}_2^2$ do not have neighbors in $\mathcal{T}_2^2$ nor in $\mathcal{T}_0^2$, and in fact $N_L(x_2)=(x-(T+T))_1$ for each $x_2\in\mathcal{T}_2^2$. See Figure 4. Clearly, the group Γ acts admissibly on L, since this is a Γ-invariant subgraph of $\mathcal{T}^2$.

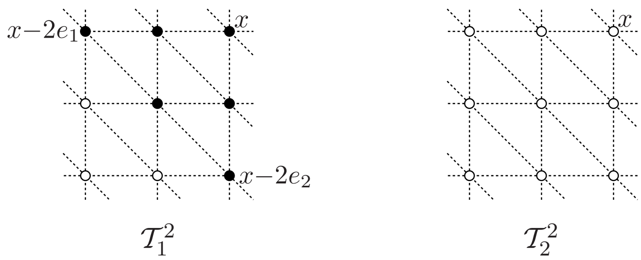

FIGURE 4. Neighbors of x_2 in L

The edges of the forms $\{x_1,x_2\}$, $\{(x-2e_1)_1,x_2\}$ and $\{(x-2e_2)_1,x_2\}$ for $x\in\mathcal{T}$ are free faces of Δ_5 by Proposition 2.3, and they form a Γ-collapsible family by Proposition 2.9. After the Γ-collapse of this family, the resulting complex is still Whitney, and in its 1-skeleton L' each vertex of the form x_2 is dominated by any of its neighbors: $(x-p)_1$, $(x-e_1)_1$, $(x-e_2)_1$. Finally we can, by Proposition 2.1, remove all vertices in $\mathcal{T}_2^2$ from L' to arrive at $\mathcal{T}^1$, as we wanted. □

5. CONCLUDING REMARKS AND OPEN PROBLEMS

We mention that there are locally C_6 graphs G triangulating the Klein bottle such that G is not even a subgraph of $K(G)$. This implies that $\Delta(K(G))$ is not collapsible to $\Delta(G)$. An example can be constructed as $G=\mathcal{T}/\langle\alpha,\beta\rangle$ where α is the translation $u\mapsto u+(4,4)$ and β is the glide reflection resulting from the translation $u\mapsto u+(-4,4)$ followed by the reflection along the mirror line $y=-x$.

Given graphs G, H we write $G \xrightarrow{\#} H$ if H is isomorphic to an induced subgraph H_0 of G such that every vertex of G is dominated by some vertex in H_0. From Proposition 2.1 it follows that $G \xrightarrow{\#} H$ implies that $\Delta(G)$ collapses to (a complex isomorphic to) $\Delta(H)$. It is proved in [6] that $G \xrightarrow{\#} H$ also implies that $K(G) \xrightarrow{\#} K(H)$. Hence, from the results in sections 3 and 4, we obtain that if $G \xrightarrow{\#} H$, where H is a clockwork graph or an orientable locally C_6 graph, then G satisfies $K^n(G) \simeq G$ for all n.

It is possible for a non-clique-Helly graph G that there exists $n_0 \geq 1$ such that $K^{n_0}(G)$ is clique-Helly, i.e., that G is *eventually clique-Helly.* We note that the smallest such n_0 was called the *Helly defect* in [3], where the authors constructed graphs with arbitrarily large Helly defect. We propose the following question:

Question 1. If G is eventually clique-Helly, do we have $K^n(G) \simeq G$ for all n?

There is even a particular case of the last question which seems rather challenging.

Question 2. If G is clique-null, is G contractible?

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F. Larrión (paco@math.unam.mx)
Instituto de Matemáticas.
Universidad Nacional Autónoma de México.
México, D.F. C.P. 04510.

M.A. Pizaña (map@xanum.uam.mx, http://xamanek.izt.uam.mx/map)
Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica.
Av. San Rafael Atlixco 186. Col. Vicentina.
México 09340 D.F.
MEXICO

R. Villarroel-Flores (rafaelv@uaeh.edu.mx)
Centro de Investigación en Matemáticas,
Universidad Autónoma del Estado de Hidalgo,
Carr. Pachuca-Tulancingo km. 4.5, Pachuca Hgo. 42184, MEXICO.