Distances and Diameters on Iterated Clique Graphs

Miguel A. Pizaña

Universidad Autónoma Metropolitana Departamento de Ingeniería Eléctrica Av. Michoacán y Purísima s/n Col. Vicentina. México D.F. 09340 MEXICO. Fax: (52) 5804-4640

Abstract

If G is a graph, its clique graph, $K(G)$, is the intersection graph of all its (maximal) cliques. Iterated clique graphs are then defined recursively by: $K^0(G) = G$ and $K^n(G) = K\left(K^{n-1}(G)\right)$. We study the relationship between distances in G and distances in $K^n(G)$. Then we apply these results to Johnson graphs to give a shorter and simpler proof of Bornstein and Szwarcfiter's theorem: For each n there exists a graph G such that $\operatorname{diam}(K^n(G)) = \operatorname{diam}(G) + n$. In the way, a new family of graphs with increasing diameters under the clique operator is shown.

Key words: iterated clique graphs, distances, diameters, Johnson graphs.

1 Introduction and Terminology

All our graphs are simple, finite and connected. A *morphism* between two graphs is a function between their vertex sets such that images of adjacent vertices are adjacent or equal. A *clique* of a graph G, is a maximal complete subgraph. We often identify induced subgraphs (hence cliques) with their vertex set. The clique graph $K(G)$ is the intersection graph of all its cliques: every clique is a vertex; two of them are adjacent if and only if they are different and have non-empty intersection. Iterated clique graphs are defined recursively by: $K^0(G) = G$ and $K^n(G) = K(K^{n-1}(G))$. Iterated clique graphs were introduced by Hedetniemi and Slater in [8] (1972). We refer to [18] and [11] for extensive bibliography on iterated clique graphs.

Email address: map@xanum.uam.mx (Miguel A. Pizaña).
URL: http://xamanek.uam.mx/map (Miguel A. Pizaña).

Preprint submitted to Elsevier Preprint *4 October 2002*

The *distance* between vertices $u, v \in V(G)$ will be written as $d_G(u,v)$ while the *distance set* between two vertex sets $A, B \subseteq V(G)$ will be denoted by $\mathcal{D}_G(A,B) = \{d_G(u,v) : u \in A, v \in B\}$. We will often omit subscripts. The following theorem was originally proved in [17], but for the reader's convenience we include a proof.

Theorem 1 *Let G be a graph and $Q, Q' \in V(K(G))$ with $Q \neq Q'$ then $d_{K(G)}(Q,Q') = \min \mathcal{D}_G(Q,Q') + 1$.*

PROOF. Call $r = \min \mathcal{D}(Q,Q')$ and $s = d(Q,Q')$. Let $(x_0, x_1, \ldots, x_r)$ be a path in G with $x_0 \in Q$ and $x_r \in Q'$. Take $Q_i \in V(K(G))$ such that $\{x_i, x_{i+1}\} \subseteq Q_i$ for $i = 0, 1, \ldots, r-1$. Then $(Q, Q_0, Q_1, \ldots, Q_{r-1}, Q')$ is a walk in $K(G)$ proving that $d(Q,Q') = s \leq r + 1$.

Now let $(Q_0, Q_1, \ldots, Q_s)$ be a minimal length path in $K(G)$ between $Q_0 = Q$ and $Q_s = Q'$. Then take $x_i \in Q_i \cap Q_{i+1}$ for $i = 0, 1, \ldots, s-1$ so $(x_0, x_1, \ldots, x_{s-1})$ is a walk in G, proving that $d(x_0, x_{s-1}) \leq s - 1$. As $x_0 \in Q$ and $x_{s-1} \in Q'$, it follows that $r + 1 = \min \mathcal{D}(Q,Q') + 1 \leq d(x_0, x_{s-1}) + 1 \leq s$. □

The previous theorem, simple as it is, has as corollaries: Theorems 1, 2 and 3 in [1]; Theorem 4.2 and Lemma 4.3 in [9]; Theorem 3.1 in [10] and Theorem 2 in [16]. Let us see some of these instances:

Observing that $\max \mathcal{D}(Q_1,Q_2) \leq \min \mathcal{D}(Q_1,Q_2) + 2$ and then taking Q_1, Q_2 satisfying $\max \mathcal{D}(Q_1,Q_2) = \mathrm{diam}(G)$, yields that for any graph G we have $\mathrm{diam}(G) - 1 \leq \mathrm{diam}(K(G))$, which is Theorem 4.2 in [9]. Likewise, it follows that $\mathrm{diam}(G) - 1 \leq \mathrm{diam}(K(G)) \leq \mathrm{diam}(G) + 1$ (Theorem 3.1 in [10]). An immediate consequence of this, is Theorem 4.1 in [10] which we shall use later: $\mathrm{diam}(G) - n \leq \mathrm{diam}(K^n(G)) \leq \mathrm{diam}(G) + n$.

Another instance: Assume that there is an injective function $f : V(G) \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq 1$ if and only if $\{x, y\} \in E(G)$. Let $a, b \in V(G)$ satisfying $f(a) \leq f(x) \leq f(b)$ for all $x \in V(G)$. Then just compute the distance between the only two cliques A and B of G satisfying $a \in A$ and $b \in B$. We conclude that for every non-trivial connected time graph G we have $\mathrm{diam}(G) > \mathrm{diam}(K(G))$. This is Lemma 4.3 in [9].

Our main result (Theorem 4) states a generalization of Theorem 1. As an example of its usability, we shall show a shorter and simpler proof of Bornstein and Szwarcfiter's theorem on diameters of iterated clique graphs:

Theorem 2 *(Bornstein and Szwarcfiter (1998)[4]). For each n there exists a graph G such that* $\mathrm{diam}(K^n(G)) = \mathrm{diam}(G) + n$.

Additionally, our proof exhibits a new family of graphs (different than that in [4]) with increasing diameters under the clique operator.

Theorem 2 was stated as a conjecture by Peyrat, Rall and Slater in [16] (1986) and the subject was also explored in [1] and [10].

2 The Distance Formula

The concept involved in the following definition was introduced (with different name and notation) by Bornstein and Szwarcfiter in [3].

Definition 3 *If G is a graph and $Q \in V(K^n(G))$, we define recursively the* basement of Q with respect to G, $B_G(Q)$, *by:*

$$B_G(Q) = \begin{cases} Q & \text{if } n = 1 \\ \bigcup_{q \in Q} B_G(q) & \text{if } n > 1 \end{cases}$$

Thus, for example, if $Q \in V(K^2(G))$, then $B_G(Q)$ is a set of vertices of G, while $B_{K(G)}(Q)$ is a set of vertices of $K(G)$ (a set of cliques of G). We will often omit the subscript so $B(Q)$ means $B_G(Q)$. Now we have our main theorem:

Theorem 4 *(The distance formula) Let Q_1 and Q_2 be vertices of $K^n(G)$ then*

$$\max \mathcal{D}(B(Q_1), B(Q_2)) - n \le d(Q_1, Q_2) \le \min \mathcal{D}(B(Q_1), B(Q_2)) + n$$

If in addition $B(Q_1) \cap B(Q_2) = \varnothing$, then

$$d(Q_1, Q_2) = \min \mathcal{D}(B(Q_1), B(Q_2)) + n$$

PROOF. Let's consider the base case $n = 1$:

Assume first that $Q_1 \cap Q_2 = \varnothing$.
Theorem 1 tells us that $d(Q_1, Q_2) = \min \mathcal{D}(B(Q_1), B(Q_2)) + 1$. It should be clear that $\max \mathcal{D}(B(Q_1), B(Q_2)) \le \min \mathcal{D}(B(Q_1), B(Q_2)) + 2$. Then:

$$\max \mathcal{D}(B(Q_1), B(Q_2)) - 1 \le \min \mathcal{D}(B(Q_1), B(Q_2)) + 1 = d(Q_1, Q_2)$$

Now, assume $Q_1 \cap Q_2 \neq \varnothing$.
Notice that $d(Q_1, Q_2) = 0$ if and only if $\mathcal{D}(B(Q_1), B(Q_2)) = \{0, 1\}$ (except,

of course, when G is the one vertex graph K_1) and also, $d(Q_1, Q_2) = 1$ if and only if $\mathcal{D}(B(Q_1), B(Q_2)) = \{0, 1, 2\}$. Thus also in these cases we have:

$$\max \mathcal{D}(B(Q_1), B(Q_2)) - 1 \leq d(Q_1, Q_2) \leq \min \mathcal{D}(B(Q_1), B(Q_2)) + 1$$

Now, assume $n \geq 2$ and apply induction:

$$\begin{aligned}
&d_{K^n(G)}(Q_1, Q_2) + (n-1) \geq \max \mathcal{D}_{K(G)}\left(B_{K(G)}(Q_1), B_{K(G)}(Q_2)\right) \\
&= \max \left\{d_{K(G)}(P_1, P_2) : P_1 \in B_{K(G)}(Q_1), P_2 \in B_{K(G)}(Q_2)\right\} \\
&\geq \max \left\{\max \mathcal{D}\left(B(P_1), B(P_2)\right) - 1 : P_1 \in B_{K(G)}(Q_1), P_2 \in B_{K(G)}(Q_2)\right\} \\
&= \max \mathcal{D}_G\left(B(Q_1), B(Q_2)\right) - 1
\end{aligned}$$

Similarly:

$$\begin{aligned}
&d_{K^n(G)}(Q_1, Q_2) - (n-1) \leq \min \mathcal{D}_{K(G)}\left(B_{K(G)}(Q_1), B_{K(G)}(Q_2)\right) \\
&= \min \left\{d_{K(G)}(P_1, P_2) : P_1 \in B_{K(G)}(Q_1), P_2 \in B_{K(G)}(Q_2))\right\} \\
&\leq \min \left\{\min \mathcal{D}\left(B(P_1), B(P_2)\right) + 1 : P_1 \in B_{K(G)}(Q_1), P_2 \in B_{K(G)}(Q_2))\right\} \\
&= \min \mathcal{D}_G\left(B(Q_1), B(Q_2)\right) + 1
\end{aligned}$$

When $B_G(Q_1) \cap B_G(Q_2) = \varnothing$, it follows that $B_{K^m(G)}(Q_1) \cap B_{K^m(G)}(Q_2) = \varnothing$ for all $m = 0, \ldots, n-1$. Then just repeat the last argument, changing every less-or-equal sign by an equal sign. □

This theorem can be used to give new proofs for Lemma 1 in [3] and Theorem 3 in [4].

Notice that if $B(Q_1) \cap B(Q_2) \neq \varnothing$ we only know that $d(Q_1, Q_2) \leq n$, but if in addition we had $B_{K(G)}(Q_1) \cap B_{K(G)}(Q_2) = \varnothing$ (for example when $B(Q_1) \cap B(Q_2)$ does not contain a clique of G) then

$$d(Q_1, Q_2) = \min \mathcal{D}\left(B_{K(G)}(Q_1), B_{K(G)}(Q_2)\right) + (n-1) = n$$

So we can determine the exact value of $d(Q_1, Q_2)$ whenever we are able to calculate basements and distances with respect to the first iterated clique graph of G where the basements do not overlap.

3 Johnson Graphs

Definition 5 *For any set M and every integer $0 \leq r \leq |M|$, the* Johnson graph *$J(M,r)$ is the graph having as vertices all the subsets of M of order r, two of them being adjacent if and only if they share exactly $r-1$ elements of M. If $0 \leq r \leq m$ we set $J(m,r) = J(\{1,\ldots,m\},r)$.*

It is well known that for every $R_1, R_2 \in V(J(m,r))$, $d(R_1,R_2) = |R_1 - R_2| = |R_2 - R_1|$. In particular, $\operatorname{diam}(J(m,r)) = \min\{r, m-r\}$. Note that $J(m,r)$ is a complete graph for $r \in \{0,1,m-1,m\}$. We refer to [2,5–7,15,19] for further results on Johnson graphs.

Now, let's prove that $K^n(J(m,r))$ often contains a subgraph isomorphic to $J(m,r+n)$:

Lemma 6 *Let $m, r \in \mathbb{N}$ with $2 \leq r \leq m-1$. Then for every $n \in \mathbb{N}$ satisfying $1 \leq n \leq \min\{r-1, m-r\}$ there is a (necessarily injective) morphism*

$$\varphi_n : J(m,r+n) \to K^n(J(m,r))$$

such that for every $S \in V(J(m,r+n))$,

$$B(\varphi_n(S)) = V(J(S,r)).$$

PROOF. Assume $n = 1$. Take $M = \{1,\ldots,m\}$ and $S \in V(J(M,r+1))$. The set $V(J(S,r))$ induces a complete subgraph of $J(M,r)$ as for any two $R_1, R_2 \in V(J(S,r))$ with $R_1 \neq R_2$, we have $|R_1 \cap R_2| = r-1$ and therefore, R_1 and R_2 are adjacent in $J(M,r)$.

In this case, $V(J(S,r))$ is already a clique of $J(M,r)$: If $R \in J(M,r) - V(J(S,r))$, then $|S - R| \geq 2$ and, as $r \geq 2$, there is always a vertex R' of $J(M,r)$ in $V(J(S,r))$ such that $|R' - R| \geq 2$.

Now we have that the function $\varphi_1 : J(M,r+1) \to K(J(M,r))$ defined by $\varphi_1(S) = V(J(S,r))$ is a morphism because if $S, S' \in J(M,r+1)$ are adjacent, then $|S \cap S'| = r$ and therefore $V(J(S,r)) \cap V(J(S',r)) \supseteq \{S \cap S'\} \neq \varnothing$. Clearly, $B(V(J(S,r))) = V(J(S,r))$.

Now assume $n \geq 2$. Inductive hypothesis gives us a morphism

$$\varphi_{n-1} : J(M,r+n-1) \to K^{n-1}(J(M,r))$$

such that $B(\varphi_{n-1}(T)) = V(J(T,r))$ for every $T \in V(J(M,r+n-1))$.

As before, for every $S \in J(M, r+n)$ the set $V(J(S, r+n-1))$ induces a complete subgraph of $J(M, r+n-1)$, so $\varphi_{n-1}(V(J(S, r+n-1)))$ induces a complete subgraph of $K^{n-1}(J(M,r))$. Define $C_S = \varphi_{n-1}(V(J(S, r+n-1)))$. For every $S \in J(M, r+n)$, let us select a fixed clique Q_S of $K^{n-1}(J(M,r))$ (a vertex in $K^n(J(M,r))$) containing C_S. Then

$$B(Q_S) = \bigcup_{q \in Q_S} B(q) \supseteq \bigcup_{q \in C_S} B(q) = \bigcup_{T \in V(J(S,r+n-1))} B(\varphi_{n-1}(T))$$
$$= \bigcup_{T \in V(J(S,r+n-1))} V(J(T,r)) = V(J(S,r)).$$

but if $R \in J(M,r) - V(J(S,r))$, then $|S - R| \geq n+1$ and therefore, as $r \geq n+1$, there is an $R' \in V(J(S,r))$ such that $d(R, R') \geq n+1$. Now, Theorem 4 forbids R to belong to $B(Q_S)$ because $\max D(B(Q_S), B(Q_S)) \leq n$. Then we have that $B(Q_S) = V(J(S,r))$.

Finally if $S, S' \in J(M, r+n)$ and $|S \cap S'| = r+n-1$, then $Q_S \cap Q_{S'} \supseteq \{\varphi_{n-1}(S \cap S')\} \neq \varnothing$.

Thus, the required morphism is given by $\varphi_n(S) = Q_S$ □

Next we give a new proof for Theorem 2:

Theorem 7 *Let $r \geq 2$, $m \geq 4r-2$ and $G = J(m,r)$, then for every n with $1 \leq n \leq r-1$: $\operatorname{diam}(K^n(G)) = \operatorname{diam}(G) + n$.*

PROOF. Let $S_1 = \{1, \ldots, r+n\}$, $S_2 = \{r+n+1, \ldots, 2r+2n\}$ and $M = \{1, \ldots, m\}$. By Lemma 6, there is a morphism $\varphi : J(M, r+n) \to K^n(J(M,r))$ such that $B(\varphi(S_1)) = V(J(S_1, r))$ and $B(\varphi(S_2)) = V(J(S_2, r))$. Since $V(J(S_1,r)) \cap V(J(S_2,r)) = \varnothing$ and $\min D_G(V(J(S_1,r)), V(J(S_2,r))) = r$, Theorem 4 tells us that $d(\varphi(S_1), \varphi(S_2)) = r+n$. Then $\operatorname{diam}(K^n(G)) \geq r+n$, but $\operatorname{diam}(G) = r$ and we already know that for every graph, $\operatorname{diam}(K^n(G)) \leq \operatorname{diam}(G) + n$. □

It can be proved using Neumann-Lara's retraction theorem [14] that $J(m,r)$ is $K-$divergent (i.e. $\lim_{n\to\infty} |K^n(J(m,r))| = \infty$) for $2 \leq r \leq m-2$ since the octahedron is a retract of any of these graphs. Thus it is still possible for some Johnson graph $J(m,r)$, that $\operatorname{diam}(K^n(J(m,r))) = \operatorname{diam}(J(m,r)) + n$ for all n, but I rather think that $\operatorname{diam}(K^n(J(m,r)))$ stabilizes at $\left\lfloor \frac{m}{2} \right\rfloor$. Unfortunately these Johnson graphs, as well as the octahedron (see [14] and [13]), have super-exponential growth rate under the clique operator. In practice, this means that computer cannot probe such conjectures because it can not calculate beyond the third iterated clique graph, in most cases not even beyond the second.

4 Concluding Remarks

Thanks to Bornstein and Szwarcfiter we know Peyrat, Rall and Slater's conjecture to be true. On the other hand Larrión and Neumann-Lara proved ([12], [11]) that there are graphs satisfying $\lim_{n\to\infty} \operatorname{diam}(K^n(G)) = \infty$. However, the diameter of these graphs grows like $\left\lfloor \frac{n}{3} \right\rfloor$. So it still remains unsolved the obvious next step:

Problem 8 *Is there a graph G such that* $\operatorname{diam}(K^n(G)) = \operatorname{diam}(G) + n$ *for all n?*

I think that Theorem 4 will play an important role in any possible solution to this problem.

Acknowledgements. I want to express my thanks to Prof. Víctor Neumann-Lara and Prof. Francisco Larrión for many enjoyable, helpful and motivating discussions on these topics. I also want to thank the referees for their valuable suggestions.

References

[1] R. Balakrishnan and P. Paulraja. *Self-clique graphs and diameters of iterated clique graphs.* Utilitas Math. **29** (1986) 263–268.

[2] A. Blokhuis and A.E. Brouwer. *Locally 4-by-4 grid graphs.* J. Graph Theory **13** (1989) 229–244.

[3] C.F. Bornstein and J.L. Szwarcfiter. *On clique convergent graphs.* Graphs Combin. **11** (1995) 213–220.

[4] C.F. Bornstein and J.L. Szwarcfiter. *Iterated clique graphs with increasing diameters.* J. Graph Theory **28** (1998) 147–154.

[5] A.E. Brouwer, A.M. Cohen and A. Neumaier. *Distance-regular graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)].* Springer-Verlag, Berlin, 1989.

[6] A.E. Brouwer and J.H. Koolen. *A new infinite series of regular uniformly geodetic code graphs.* Discrete Math. **120** (1993) 241–247.

[7] A. Dabrowski and L.S. Moss. *The Johnson graphs satisfy a distance extension property.* Combinatorica **20** (2000) 295–300.

[8] S.T. Hedetniemi and P.J. Slater. *Line graphs of triangleless graphs and iterated clique graphs.* Springer Lecture Notes in Math. **303** (1972) 139–147.

[9] B. Hedman. *Clique graphs of time graphs.* J. Combin. Theory Ser. B **37** (1984) 270–278.

[10] B. Hedman. *Diameters of iterated clique graphs.* Hadronic J. **9** (1986) 273–276.

[11] F. Larrión and V. Neumann-Lara. *Clique divergent graphs with unbounded sequence of diameters.* Discrete Math. **197/198** (1999) 491–501.

[12] F. Larrión and V. Neumann-Lara. *Locally C_6 graphs are clique divergent.* Discrete Math. **215** (2000) 159–170.

[13] V. Neumann-Lara. *A theory of expansive graphs.* Preprint.

[14] V. Neumann-Lara. *On clique-divergent graphs.* Problèmes Combinatoires et Théorie des Graphes. (Colloques internationaux du C.N.R.S., Paris) **260** (1978) 313–315.

[15] M. Numata. *A characterization of Grassmann and Johnson graphs.* J. Combin. Theory Ser. B **48** (1990) 178–190.

[16] C. Peyrat, D.F. Rall and P.J. Slater. *On iterated clique graphs with increasing diameters.* J. Graph Theory **10** (1986) 167–171.

[17] M.A. Pizaña. *The icosahedron is clique divergent.* To appear in Discrete Math.

[18] J.L. Szwarcfiter. *A survey on clique graphs.* In Recent Advances in Algorithms and Combinatorics. C. Linhares and B. Reed, eds., Springer-Verlag. To appear.

[19] P. Terwilliger. *The Johnson graph $J(d,r)$ is unique if $(d,r) \neq (2,8)$.* Discrete Math. **58** (1986) 175–189.