GRAPH HOMOTOPY AND CLIQUE GRAPHS

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Abstract. We study graph homotopy, which was introduced by Dochtermann in 2009. Among other characterizations we show that the homotopy congruence is the finest one that makes invertible the graph morphisms in certain families, and also the finest one that identifies some endomorphisms with identity maps.

On the homotopy category the clique graph operator becomes a functor. This sheds more light into the dynamical behavior of clique graphs and a new unexplored panorama emerges. We introduce a new technique, based on *unbounded morphisms*, which enables us to prove results on clique divergence that could not be afforded by previously existing methods.

1. Introduction

In *graph dynamics* [53] one considers the class of all graphs $\mathcal{G}$, some graph class $\mathcal{H} \subseteq \mathcal{G}$ and some graph operator $\Phi : \mathcal{G} \to \mathcal{G}$. The operator can be iterated on graphs, setting $\Phi^0(X) = X$ and $\Phi^{n+1}(X) = \Phi(\Phi^n(X))$, and also on classes of graphs, using $\Phi(\mathcal{H}) = \{\Phi(X) \mid X \in \mathcal{H}\}$. Natural questions in this general subject include whether we can classify the graphs in $\Phi(\mathcal{H})$, $\Phi^2(\mathcal{H})$, $\Phi^3(\mathcal{H})$,... [1, 2, 10, 11, 15, 21, 45], which graphs $X \in \mathcal{H}$ satisfy some graph equation $\Phi(X) \cong \Phi'(X)$ for some other graph operator Φ' [3–5, 7–9, 46, 59, 60], and the *dynamical behavior problem*: which graphs $X \in \mathcal{H}$ *converge* under Φ, in the sense that $\Phi^n(X) \cong \Phi^m(X)$ for some $n < m$, or *diverge* under Φ, meaning that they do not converge or, equivalently, the sequence of orders $|\Phi^n(X)|$ tends to infinity when $n \to \infty$ [12, 16, 20, 22, 41, 43, 63–67].

One of the most researched graph operators is the clique graph operator K, which sends each graph $X \in \mathcal{G}$ to its *clique graph*: the intersection graph $K(X)$ of the (maximal) cliques of X. [1, 3–5, 7–12, 14, 18, 24, 26, 28, 29, 31, 34, 40, 41, 43, 47, 48, 50, 51, 59, 62]. Applications of the clique graph operator have been made to the fixed point property for posets [24], and to loop quantum gravity [55–57]. In this work we are interested in the study of the behavior problem for the clique graph operator K from the point of view of graph homotopy.

The topological approach to graph dynamics was pioneered by Prisner in [52], where he proved that X and $K^2(X)$ have the same homotopy type when X is clique-Helly. This came by way of a general result: The pared graph $P(X)$ of any graph X satisfies $P(X) \simeq X$. Here X is seen as a topological space by means of the geometric realization of its *flag complex*, whose simplexes are the complete subgraphs of X. Also from [52]: For any $X \in \mathcal{G}$ one has that X and $K(X)$ have the same 1-dimensional modulo 2 Betti numbers, and therefore only the graphs X with $\bar{\beta}_1(X) = 0$ can be K*-null* in the sense that $K^n(X)$ has just one vertex for some $n \geq 0$. This kind of topological research of the behavior of graphs under the clique graph operator has been pursued further, among other papers, in [29, 30, 33, 35–39].

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But there is another way to speak of homotopy in the category of graphs which does not use flag complexes. Rather than topological, it is categorical: In the category $\mathcal{G}$ of graphs we have products and, even if we do not have an interval object I in $\mathcal{G}$, we do have a whole family of them, namely the paths I_n of length $n \geq 0$. The cylinder of height n over a graph X is $X \times I_n$ and we cay say that two graph morphisms $f, g : X \to Y$ are homotopic if there is a graph morphism $H : X \times I_n \to Y$ for some $n \geq 0$ such that the restriction of H to the bottom $X \times \{0\}$ of the cylinder is f and the restriction of H to the top $X \times \{n\}$ of the cylinder is g. Precisely this framework, under the name of ×-*homotopy*, was introduced and first explored by Dochtermann in [13] as the basis for his investigation of Hom complexes.

In accordance with the coloring-related interests pursued therein, the graphs of [13] are simple graphs that may have loops at some vertices. Since we are interested in clique graphs we shall work in the category $\mathcal{G}$ of simple graphs with reflexive morphisms, which is a full subcategory of the category in [13] if we decree that in our graphs every vertex has a loop. This way also the categorical product '×' in $\mathcal{G}$ will admit the same description as in [13]. As this product is the only one we shall use, we shall call '×-homotopy' just *graph homotopy*. Since our aims are different, our overlaps with [13] will be few and sometimes, mostly when occurring in straightforward material meriting no proof, they will not be mentioned.

Paving the way for homotopy in §4, we devote §2 to the graph category $\mathcal{G}$ and §3 to graph relations. Some concepts and constructions that originated in clique graph research play a central role in graph homotopy: A vertex y *dominates* x when $N[x] \subseteq N[y]$, and x and y are *twins* if they dominate each other (see §2.1). These notions motivate others, some known, some new. For instance, the minimal dominant subgraphs are just the pared graphs which, together with dominance, go back to [14]. But dominant equivalence relations and dominant subgraphs are new. Among the morphisms occurring in 4.20 and 4.22 only dominant folds, antifolds and dismantling retractions had appeared in print. The homotopy congruence is the finest one that identifies our new types of endomorphisms with identities (see 4.20) and also the finest one that makes invertible some other dominance-related maps (see 4.22).

In particular, twinhood gains new importance: Twin vertices are indistinguishable, and a transposition that just interchanges two of them is always an automorphism of the graph. If we decide to regard these transpositions of twins as if they were the identity map, all the graph homotopy relations $f \simeq g$ then follow just by the definition of a congruence (see 4.20).

Under the homotopy viewpoint (that is on the homotopy category $h\mathcal{G}$, see §4) the clique graph operator K is a functor (see §5.1) and the star morphism, long known in clique graph theory, is a natural transformation (see §5.2). Investigating the dynamical behavior of the functor K becomes a more natural and amenable theory. For instance, we will introduce the concept of an unbounded morphism in §5.4 and apply it in §5.5 to prove the K-divergence (even K-divergence in the homotopy category) of an infinite family of graphs (see 5.16). As far as we know, this result can not be established by any previously known technique.

We deal in §6 with a new and interesting problem but leave it open: Is K-divergence in the homotopy category $h\mathcal{G}$ really stronger than K-divergence in the old graph category $\mathcal{G}$? An affirmative answer is equivalent, by our 6.11, to a seemingly stricter requirement.

For terminology not explicitly defined here we refer the reader to [6] for graph theory, to [61] for algebraic topology, and to [42] for category theory.

2. The Category of Graphs

2.1. **Preliminaries.** We are only interested in finite *simple* graphs (i.e. finite, undirected, no loops, no multiple edges). However, it we will be convenient to assume that each vertex has a loop (see [25], [13]). By this *reflexivity* of our graphs we can say simply "x and y are adjacent" when we mean "x and y are either adjacent or equal." It is also helpful in view of the kind of graph morphisms that we use. We shall keep loops rather implicit, and avoid drawing or mentioning them. We say, for instance, that the *path* I_n of length n has vertices $V(I_n) = \{0, 1, \ldots, n\}$ and edges $[0,1], [1,2], \ldots, [n-1, n]$, and that the *complete graph* K_n of order n has $V(K_n) = \{1, \ldots, n\}$ and $E(K_n) = \{ij \mid i < j\}$. (Of course, $[i,j] = ij = ji = [j,i]$).

A *subgraph* X of Y ($X \leqslant Y$) satisfies $V(X) \subseteq V(Y)$ and $E(X) \subseteq E(Y)$. For a set $S \subseteq V(X)$ we denote by $\langle S \rangle$ the subgraph of X induced by S. We write $X \trianglelefteq Y$ if X is an induced subgraph of Y. We identify sets of vertices $S \subseteq V(X)$ with the subgraphs $\langle S \rangle \trianglelefteq X$ that they induce so, given a graph X, we often write $x \in X$ instead of $x \in V(X)$ and denote the order of X by $|X|$. If $x, x' \in X$ we write as usual $x \sim x'$ (or $xx' \in E(X)$) if x and x' are adjacent, but remember that this never excludes the possibility that $x = x'$. The *neighborhood* of $x \in X$ is $N[x] = N_X[x] = \{x' \in X \mid x' \sim x\}$. The *open neighborhood* of x is $N(x) = N[x] \smallsetminus \{x\} \neq N[x]$. A *complete subgraph* of X is any non-empty subset $\varnothing \neq C \subseteq X$ such that $x \sim x'$ for all $x, x' \in C$. It is important to remember that our complete subgraphs are always non-empty. We shall use the word "complete" mostly as a noun, in the sense of "complete subgraph". The term "clique", used elsewhere for this purpose, we shall reserve for "maximal complete". Note that $x \sim x'$ in X if and only if $C = \{x, x'\}$ is a complete (which we call *small*) of X.

Dominance and twinhood will be essential for us. The vertex x *is dominated* by x', denoted by $x \preccurlyeq x'$, if $N[x] \subseteq N[x']$. This was first introduced by Escalante in [14] (see also [49, 52], and §5.2 below). One also says that x' *dominates* x, and writes $x' \succcurlyeq x$. Saying that x is a *dominated vertex* (or just *dominated*, without saying by whom) means that $x \preccurlyeq x'$ for some vertex $x' \neq x$. Thus the graph X is *stiff* [25] if X has no dominated vertices, i.e. each vertex is dominated only by itself. The vertices x, x' are said to be *twins* (denoted by $x \approx x'$) if and only if $N[x] = N[x']$, i.e. $x \preccurlyeq x'$ and $x' \preccurlyeq x$. Notice that $x \approx x' \Rightarrow x \preccurlyeq x' \Rightarrow x \sim x'$.

Lemma 2.1. *If $a, b \in X \trianglelefteq A$ and $a \preccurlyeq b$ in A, then $a \preccurlyeq b$ in X too* (*that is*, $a \preccurlyeq_A b \Rightarrow a \preccurlyeq_X b$).

Proof. Since $X \trianglelefteq A$ we have $N_X[a] = N_A[a] \cap X \subseteq N_A[b] \cap X = N_X[b]$. □

A *graph morphism* $f : X \to Y$ is any adjacency-preserving vertex mapping $f : V(X) \to V(Y)$. There are several equivalent definitions given by the conditions in:

Lemma 2.2. *For a function $f : V(X) \to V(Y)$ the following are equivalent* (TFAE):

(1) $x \sim x'$ implies $f(x) \sim f(x')$ for all $x, x' \in X$ (i.e. f is a graph morphism).

(2) $f(C)$ is a complete of Y for each small complete $C = \{x, y\}$ of X.

(3) $f(C)$ is a complete of Y for each complete C of X.

(4) $f(C) \cup f(D)$ is a complete of Y if $C \cup D$ is complete in X.

(5) $f(N[x]) \subseteq N[f(x)]$ for all $x \in X$. □

We also call graph morphisms *graph maps*, often shortening to just "morphisms" or "maps". Our maps (called *reflexive* elsewhere) can send distinct adjacent vertices to a single vertex.

Our *category of graphs* $\mathcal{G}$ has all finite graphs as objects, the elements of $\mathrm{mor}(\mathcal{G})$ are the graph maps, and the composition is the usual. We denote by $\mathrm{ob}(\mathcal{G})$ the class of objects of $\mathcal{G}$ but we shorten "$X \in \mathrm{ob}(\mathcal{G})$" to just "$X \in \mathcal{G}$" and read this as "$X$ is a graph". If $X, Y \in \mathcal{G}$ we denote by $\mathcal{G}(X, Y)$ the set $\{f \mid f : X \to Y \text{ is a map}\}$. Elsewhere $\mathrm{Hom}(X, Y)$ or $\mathrm{Hom}_{\mathcal{G}}(X, Y)$ are also common, but this kind of notation is used in other ways in [25] and [13] and to these works we will often refer, explicitly or not. When $X = Y$ one also denotes $\mathcal{G}(X, X)$ as $\mathrm{End}(X)$ and calls its elements *endomorphisms* of X. Note that $\mathrm{End}(X)$ is a monoid under composition.

There is a larger category (we denote it by $\mathcal{G}^{\pm}$) of graphs with loops at some vertices, not necessarily all. Our $\mathcal{G}$ sits inside $\mathcal{G}^{\pm}$ as a full subcategory. Such works as [25] and [13] are framed within $\mathcal{G}^{\pm}$ but both focus from time to time on $\mathcal{G}$ (which is denoted by $\mathcal{G}^{\circ}$ in [13]). We are only interested in $\mathcal{G}$ but we shall occasionally refer to $\mathcal{G}^{\pm}$ for comparison purposes.

2.2. **Special types of morphisms.** By definition a graph morphism $f : X \to Y$ is a vertex mapping $f : V(X) \to V(Y)$, but it defines as well an edge mapping $f : E(X) \to E(Y)$: for any edge $e = xx' \in E(X)$ we set $f(e) = f(x)f(x') \in E(Y)$. Using those mappings one can say, e.g., that "every vertex-injective map is edge-injective" (which is obvious, by the way).

Let $f \in \mathcal{G}(X, Y)$. The *image* of f is the subgraph $\mathrm{Im}(f) = f(X) \leqslant Y$ with vertex set $V(f(X)) = f(V(X))$ and edge set $E(f(X)) = f(E(X))$. The *kernel* of f is an equivalence relation $\equiv$ in X: $x \equiv x' \Leftrightarrow f(x) = f(x')$. This equivalence relation is denoted by $\mathrm{Ker}(f)$. The equivalence class of a vertex $x \in X$ under $\mathrm{Ker}(f)$ is the fiber $f^{-1}(f(x)) \subseteq X$.

For any equivalence relation $\equiv$ in X the *quotient graph* $X/\!\equiv$ has the classes $[x]$ as vertices, and there is an edge $[x] \sim [x']$ whenever $v \sim v'$ for some representatives $v \in [x]$ and $v' \in [x']$. The *natural projection* $q : X \to X/\!\equiv$ is given by $q(x) = [x]$. Here $\mathrm{Ker}(q)$ is precisely $\equiv$.

If $S \leqslant X$, the *inclusion map* $i = i_S : S \hookrightarrow X$ has $\mathrm{Im}(i) = S$. If $f : X \to Y$ is a morphism, the *restriction* of f to S, denoted by $f_{|} : S \to Y$, is given by $f_{|} = f \circ i_S$ and has $\mathrm{Im}(f_{|}) = f(S)$. If $f(X) \leqslant T \leqslant Y$ the *co-restriction* of f to T, denoted by $f^{|} : X \to T$, has $\mathrm{Ker}(f^{|}) = \mathrm{Ker}(f)$. Notice that $f^{|}$ is the only map from X to T such that $i_T \circ f^{|} = f$.

The monomorphisms (left cancellative, *monos* or *monic*) in $\mathcal{G}$ are the vertex-injective maps, and the epimorphisms (right cancellative, *epis* or *epic*) are the vertex-surjective maps. The isomorphisms (having a two-sided inverse, or *isos*) are the vertex- and edge-bijective maps (vertex-bijective and edge-surjective is enough). A given $f \in \mathrm{End}(X)$ needs only to be monic (or epic) to be an iso (an *automorphism* of X). $\mathrm{Aut}(X) = \{f \in \mathrm{End}(X) \mid f \text{ iso}\}$ is a group.

Two monos $s : X \to A$ and $t : Y \to A$ are *equivalent* if $s = t \circ \varphi$ for some iso $\varphi : X \to Y$. Dually, two epis $r : A \to X$ and $q : A \to Y$ are *equivalent* if $q = \varphi \circ r$ for some iso $\varphi : X \to Y$. Notice that in both cases the isomorphism φ is necessarily unique since r is epic and t is monic.

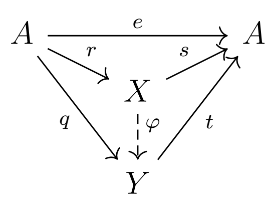
$$\begin{array}{ccc} X & \xrightarrow{s} & \\ {\scriptstyle\varphi}\downarrow & & A \\ Y & \xrightarrow{t} & \end{array} \qquad\qquad \begin{array}{ccc} & \xrightarrow{r} & X \\ A & & \downarrow{\scriptstyle\varphi} \\ & \xrightarrow{q} & Y \end{array}$$

Two monos are equivalent if and only if they have the same image, as any mono $s : X \to A$ is equivalent to the inclusion of its image. Indeed, if $Y = \mathrm{Im}(s)$, $t : Y \hookrightarrow A$ and $\varphi = s^{|} : X \to Y$, we have that $t \circ \varphi = s$ and φ is clearly vertex- and edge-bijective, so it is an isomorphism.

The *embeddings* are the "full" monomorphisms, i.e. vertex-injective maps $s : X \to A$ with $s(x) \sim s(x') \Leftrightarrow x \sim x'$. Thus, the embeddings are the monos $s : X \to A$ with $\mathrm{Im}(s) \trianglelefteq A$. A mono $s : X \to A$ is an embedding just when any equivalent mono $t : Y \to A$ is an embedding.

The *projections* are the epis $r : A \to X$ with $\mathrm{Im}(r) = X$, that is the vertex-surjective and edge-surjective maps, as for instance the natural projection $q : X \to X/\equiv$. An epi $r : A \to X$ is a projection if and only if any equivalent epi $q : A \to Y$ is a projection. The projections $r : A \to X$ are (just those) epis such that for any function $f : V(X) \to V(Y)$ the composition $c = f \circ r : V(A) \to V(Y)$ is a morphism $c : A \to Y$ if and only if $f : X \to Y$ is a morphism. Two projections are equivalent if, and only if, they have the same kernel, as any projection $q : A \to Y$ is equivalent to the natural projection $r : A \to X$ where $X = A/\mathrm{Ker}(q)$. Indeed, as $\mathrm{Ker}(r) = \mathrm{Ker}(q)$, the assignation $[a] \mapsto q(a)$ gives a well-defined and injective function $\varphi : V(X) \to V(Y)$. But φ is a map (hence a mono) because r is a projection and $\varphi \circ r = q$ is a map. And φ is edge surjective because q is so. Since φ is an iso, q is equivalent to r.

A *section* is a map $s : X \to A$ having a left inverse $r : A \to X$ and a *retraction* is a map $r : A \to X$ having a right inverse $s : X \to A$. The single equation $r \circ s = 1_X$ says that s is a section (*of* r) and r is a retraction (*of* s). We also say that (r, s) is a *retraction-section pair*, or that r and s *form* such a pair. In this case r is a projection and s an embedding. An embedding is a section just when any equivalent embedding is so, and similarly for projections: one is a retraction if and only if any equivalent projection is a retraction. The other composition $e = s \circ r : A \to A$ is always *idempotent*, i.e. $e^2 = e \circ e = e$. Indeed, $e^2 = s \circ r \circ s \circ r = s \circ 1_X \circ r = s \circ r = e$. A map $e \in \mathrm{End}(A)$ is idempotent if and only if e fixes the vertices of $e(A)$, but note that this entails $e(A) \trianglelefteq A$ and $e_{|}^{|} = 1_{e(A)} : e(A) \to e(A)$. A *splitting* of the idempotent $e : A \to A$ is a retraction-section pair (r, s) with $e = s \circ r$. Any idempotent e has a splitting: take $X = e(A)$, $r = e^{|} : A \to X$, $s = i_X : X \hookrightarrow A$. We call this the *standard* splitting. Two splittings (r, s) and (q, t) of the idempotent e are said to be *equivalent* if $q = \varphi \circ r$ and $s = t \circ \varphi$ for some isomorphism φ as in the diagram:



Proposition 2.3. *Up to equivalence any idempotent* $e : A \to A$ *has a unique splitting* (r, s).

Proof. If (q, t) is another splitting of e we use that r and q are projections to get that $s(X) = s(r(A)) = e(A) = t(q(A)) = t(Y)$. Thus s and t are equivalent and there is an iso $\varphi : X \to Y$ with $s = t \circ \varphi$. Then $t \circ \varphi \circ r = s \circ r = e = t \circ q$ and hence $\varphi \circ r = q$ as t is monic. □

A retraction r (or a section s) *splits* the idempotent e if $e = s \circ r$ for some section s of r (or for some retraction r of s). In this case we say that e is *split* by r (or by s).

Proposition 2.4. *Two retractions r and q (or two sections s and t) are equivalent if, and only if, they split the same idempotents.*

Proof. If r and q (or s and t) split the same idempotent e they are equivalent by 2.3. If r and q are equivalent, take the iso $\varphi : X \to Y$ with $q = \varphi \circ r$. Assuming that r splits the idempotent e, then $e = s \circ r$ for some section s of r. Defining $t = s \circ \varphi^{-1}$, notice that $q \circ t = \varphi \circ r \circ s \circ \varphi^{-1} = \varphi \circ 1_X \circ \varphi^{-1} = 1_Y$ and $t \circ q = s \circ \varphi^{-1} \circ \varphi \circ r = s \circ 1_X \circ r = s \circ r = e$, so t also splits e. The dual argument proves that equivalent sections split the same idempotents. □

Any morphism $r : A \to X$ with $X \leqslant A$ and $r(x) = x$ for all $x \in X$ is a retraction, which we will call *special*. If $r : A \to X$ is one such special retraction, $s : X \hookrightarrow A$ is a section of r and we shall call this section also *special*. Both the retraction and the section in the standard splitting of an idempotent endomorphism are special. Hence any retraction (or section) is equivalent to some special retraction (or special section): To find one it is enough, by 2.4, to take the standard splitting of any idempotent split by the retraction (or the section).

If $X, Y \in \mathcal{G}$, the *product* $X \times Y$ is defined on the vertex set $V(X) \times V(Y)$ by coordinatewise adjacency: $(x, y) \sim (x', y') \iff x \sim x' \;\&\; y \sim y'$. With the usual projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ given by $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$, this is the categorical product (hence *the product*) of X and Y in $\mathcal{G}$. The projections are retractions. Some sections of them embed a factor in a *slice* of the product, that is a set of the form $X \times \{y\}$ or $\{x\} \times Y$: For $y \in Y$, define $\sigma_y : X \to X \times Y$ by $\sigma_y(x) = (x, y)$ and, for $x \in X$, $\sigma_x : Y \to X \times Y$ by $\sigma_x(y) = (x, y)$. Notice that a subset $S \subseteq X \times Y$ is complete if, and only if, both projections $\pi_X(S) \subseteq X$ and $\pi_Y(S) \subseteq Y$ are complete. Only this product, *the product*, will be used here.

2.3. **The graph of morphisms.** Häggkvist, Hell, Miller and Neumann-Lara, in their paper [23], defined adjacency for parallel morphisms $f, g : X \to Y$ (and even for vertex functions $f, g : V(X) \to V(Y)$). There are several equivalent conditions for the definition:

Lemma 2.5. *Let $f, g \in \mathcal{G}(X, Y)$. Then the following conditions are equivalent:*

(1) $x \sim x'$ implies $f(x) \sim g(x')$ for all $x, x' \in V(X)$.

(2) $f(C) \cup g(C)$ is a complete of Y for each small complete C of X.

(3) $f(C) \cup g(C)$ is a complete of Y for each complete C of X.

(4) $f(C) \cup g(D)$ is a complete of Y if $C \cup D$ is complete in X.

(5) $f(N[x]) \subseteq N[g(x)]$ for all $x \in X$. □

Two morphisms $f, g \in \mathcal{G}(X, Y)$ are *adjacent* (denoted of course by $f \sim g$) if they satisfy 2.5.1 or any of the equivalent conditions in 2.5. This gives a graph structure to each set $\mathcal{G}(X, Y)$. In other words, we now have $\mathcal{G}(X, Y) \in \mathcal{G}$ whenever $X, Y \in \mathcal{G}$.

Proposition 2.6. *The* composition map $\circ : \mathcal{G}(Y, Z) \times \mathcal{G}(X, Y) \to \mathcal{G}(X, Z)$ *that sends (g, f) to $g \circ f$ is a graph morphism. In particular, $g \sim g' \Rightarrow g \circ f \sim g' \circ f$ and $f \sim f' \Rightarrow g \circ f \sim g \circ f'$.*

Proof. Let $(g, f) \sim (g', f')$ in $\mathcal{G}(Y, Z) \times \mathcal{G}(X, Y)$. Then $g \sim g' \in \mathcal{G}(Y, Z)$ & $f \sim f' \in \mathcal{G}(X, Y)$.

$$X \underset{f'}{\overset{f}{\rightrightarrows}} Y \underset{g'}{\overset{g}{\rightrightarrows}} Z.$$

If $C \subseteq X$ is complete, $f(C) \cup f'(C) \subseteq Y$ is complete by 2.5.3. Then $g(f(C)) \cup g'(f'(C)) \subseteq Z$ is complete by 2.5.4. Therefore $g \circ f \sim g' \circ f'$ by 2.5.3 again. □

2.4. **The neighborhood of the identity.** We focus here on the monoid-cum-graph $\mathrm{End}(X)$. Any function $d : V(X) \to V(X)$ such that $d(x)$ dominates x for all $x \in X$ is a *domination*. By analogy with "domination," we call $t : V(X) \to V(X)$ a *twination* if $t(x) \approx x$ for all x. Since $t(x) \approx x$ implies $t(x) \geqslant x$, twinations are particular cases of dominations.

Proposition 2.7. *Any domination $d : V(X) \to V(X)$ is a graph morphism $d : X \to X$. The neighborhood of 1_X in* $\mathrm{End}(X)$ *is the set* $N[1_X] = \{g : X \to X \mid g \text{ is a domination}\}$. *In consequence, $N[1_X]$ is a submonoid of* $\mathrm{End}(X)$ (*and so is the set of twinations*).

Proof. If $x \in X$ and $x' \in N[x]$, $x \in N[x'] \subseteq N[d(x')]$, so $d(x') \in N[x] \subseteq N[d(x)]$ and thus $d(N[x]) \subseteq N[d(x)]$, so $d : X \to X$ is a map by 2.2.5. By 2.5.5, for any $g : X \to X$ we have that $1_X \sim g$ if and only if $N[x] \subseteq N[g(x)]$ for all $x \in X$, i.e. just when g is a domination. For the last claim note that $1_X \in N[1_X]$ and dominations are closed under composition as $\geqslant$ is a transitive relation (and that the same also holds for twinations and the relation $\approx$). □

Corollary 2.8. *X is stiff (no dominated vertices) $\Leftrightarrow 1_X \in \mathrm{End}(X)$ is an isolated vertex.* □

Besides being closed under products, i.e. compositions, dominations and twinations are also closed under certain factorizations that hold for any self-map $f : V(X) \to V(X)$. These will be reviewed in Appendix A, to which we refer for more details.

Any self-map $f : V(X) \to V(X)$ is the product $f = f_a \circ f_b$ of its *acyclic part* f_a and its *bijective part* f_b, where f_b is bijective and f_a is *acyclic*: $m > 0$ and $(f_a)^m(x) = x$ imply $f_a(x) = x$. Here f is acyclic $\Leftrightarrow f = f_a \Leftrightarrow f_b = 1_X$ and f is bijective $\Leftrightarrow f = f_b \Leftrightarrow f_a = 1_X$. Both f_a and f_b (and the transpositions, idempotents and pinches below) are functions $h : V(X) \to V(X)$ with $h(x) \in \{x, f(x)\}$ for each x, so they are dominations (or twinations) as soon as f is so.

Any bijective domination $f : X \to X$ is in fact a bijective twination: Indeed, if $x \in X$ there is an $s > 0$ with $f^s(x) = x$, but then $x = f^s(x) \geqslant f^{s-1}(x) \geqslant \ldots \geqslant f(x) \geqslant x$ implies $x = f^s(x) \approx f^{s-1}(x) \approx \cdots \approx f(x) \approx x$. Any bijective twination f is a product of transpositions $f = t_1 \circ \cdots \circ t_s$ where each $t_j = (x_j, y_j)$ is a *twinning transposition*, that is $x_j \approx y_j$.

Any acyclic domination $f : X \to X$ is a product $f = e_k \circ \cdots \circ e_1$ of idempotent dominations. If $a, b \in X$, the *pinch* $p = [a, b] : X \to X$ is the map sending a to b and fixing every $x \neq a$. This pinch p is a *dominant pinch* when $b \geqslant a$, and a *twinning pinch* when $a \approx b$. Any idempotent domination $e : X \to X$ is, by 2.9 below, a product of dominant pinches.

The *twin* of a statement uses "twination"/"twinning" instead of "domination"/"dominant". Both versions are often true, and with twin proofs. We shall denote this with "(+ Twin)".

Lemma 2.9 (+ Twin). *If the idempotent dominations $e_1, \ldots, e_k \in \mathrm{End}(A)$ commute with each other, $e = \prod_{i=1}^k e_i$ is an idempotent domination. Conversely, any idempotent domination $e \in \mathrm{End}(A)$ admits such a factorization $e = \prod_{i=1}^k e_i$ where the e_i commute with each other and are dominant pinches of the form $e_i = [x_i, e(x_i)]$.*

Proof. By A.2, e is idempotent, and a domination by 2.7. Conversely, by A.2, $e = \prod_{i=1}^k e_i$ with $e_i = [x_i, e(x_i)]$ and $e_i \circ e_j = e_j \circ e_i$. As $e(x_i) \geqslant x_i$, these e_i are dominant pinches. □

2.5. **Dominant and twinning sections and retractions.** Dominations and twinations are endomorphisms: elements of $\mathrm{End}(A)$ for some $A \in \mathcal{G}$. Co-restricting such an $f \in \mathrm{End}(A)$ to its image $X = f(A) \leqslant A$ gives rise to a morphism $f^{|} : A \to X$ which, if $X \neq A$, is no longer an endomorphism of A but an epimorphism $r : A \to X$. When f is idempotent r is a special retraction which, with the special section $s : X \hookrightarrow A$, gives the standard splitting of f.

We define *dominant retractions* and *dominant sections* to be those retractions and sections that split some idempotent domination. Thus a map $r : A \to X$ is a dominant retraction just when there is a map $s : X \to A$ such that $r \circ s = 1_X$ and $e = s \circ r \in \mathrm{End}(A)$ is a domination. In the same way, a section $s : X \to A$ is dominant if for some retraction $r : A \to X$ of s the idempotent $e = s \circ r : A \to A$ is a domination. Similarly, by definition, *twinning retractions* and *twinning sections* are those retractions and sections that split idempotent twinations.

If $e \in \mathrm{End}(A)$ is an idempotent domination recall that its standard splitting has $X = e(A)$, $r = e^{|} : A \to X$ and $s : X \hookrightarrow A$. Each vertex of $A \setminus X$ is dominated by some vertex in X, so the dominant retraction r can be seen as recording in a morphism the operation of removing some dominated vertices of A (those in $A \setminus X$) and also recording, for each removed vertex, one that dominated it before the removal. Similarly, the dominant section s corresponds to adding some dominated vertices to X. These remarks (and their twins) also apply for any splitting (standard or not) of e, as all are equivalent by 2.3. For instance, any dominant retraction $r : A \to X$ "removes", up to an iso $\varphi : X \to X'$, some dominated vertices of A.

An induced subgraph $X \trianglelefteqslant A$ is a *dominant* subgraph of A if each vertex of A is dominated by some vertex in X. Of course, if each vertex of A is a twin of some vertex in X, we call $X \trianglelefteqslant A$ a *twinning subgraph* of A. An equivalence relation $\equiv$ in $V(A)$ is *dominant* if each equivalence class C contains some vertex $a_C \in C$ that dominates all C, i.e. $a_C \succcurlyeq x \ \forall x \in C$. By the twin definition $\equiv$ is *twinning* if each equivalence class C contains some vertex $a_C \in C$ which is a twin of all the vertices in C, i.e. $a_C \approx x \ \forall x \in C$, but this just means that each equivalence class C is a *twinset* in the sense that all its elements are twins: $x \approx y \ \forall x, y \in C$.

Proposition 2.10 (+ Twin)**.** *If $t : Y \to A$ is an embedding with $X = \mathrm{Im}(t)$, TFAE:*

(1) the embedding $t : Y \to A$ is a dominant section,

(2) any embedding $u : Z \to A$ with $\mathrm{Im}(u) = X$ is a dominant section,

(3) the embedding $s = i_X : X \hookrightarrow A$ is a dominant section,

(4) $X = \mathrm{Im}(e)$ for some idempotent domination $e : A \to A$, and

(5) X is a dominant subgraph of A.

Proof. The embeddings in (1), (2), and (3) have the same image X, so they are equivalent. They are sections as soon as any of them is, and by 2.4 they all split the same idempotents, be they dominations or not. Therefore (1), (2) and (3) are equivalent. To finish the proof we will prove now that (1) ⇒ (4) ⇒ (5) ⇒ (3). Using (1), t splits some idempotent domination $e : A \to A$, and so $X = \mathrm{Im}(t) = \mathrm{Im}(e)$, that is (4). By (4) $X \trianglelefteqslant A$ is dominant, hence (5). Using (5), let $e(a) = a$ if $a \in X$ and, if $a \notin X$, pick any $e(a) \in X$ such that $e(a) \succcurlyeq a$. This domination $e : A \to A$, a map by 2.7, is idempotent since it fixes its image $e(A) = X$. The standard splitting of e is $(e^{|}, s)$ and thus we have (3). □

Corollary 2.11. *If $X \trianglelefteqslant A$ is dominant, there is a dominant retraction $r = e^{|} : A \to X$.* □

Proposition 2.12 (+ Twin)**.** *If $q : A \to Y$ is a projection, let $X = A/\mathrm{Ker}(q)$. TFAE:*

(1) the projection $q : A \to Y$ is a dominant retraction,

(2) any projection $v : A \to Z$ with $Ker(v) = \mathrm{Ker}(q)$ is a dominant retraction,

(3) the natural projection $p : A \to X$ is a dominant retraction,

(4) $\mathrm{Ker}(q) = \mathrm{Ker}(e)$ for some idempotent domination $e : A \to A$, and

(5) $\mathrm{Ker}(q)$ is a dominant equivalence relation on A.

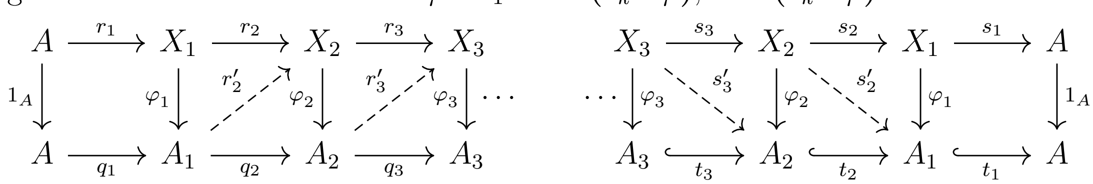
Proof. q, v, and p have the same kernel, so (1) ⇔ (2) ⇔ (3). By (1), q splits an idempotent domination $e : A \to A$. Hence (4), as $\mathrm{Ker}(q) = \mathrm{Ker}(e)$. Assuming (4) note that $e^{|} : A \to e(A)$ splits e and $\mathrm{Ker}(e^{|}) = \mathrm{Ker}(e)$. Then $\mathrm{Ker}(q) = \mathrm{Ker}(e^{|})$, so q also splits e by 2.4 and there is a section $t : Y \to A$ of q with $e = t \circ q$. If $y \in Y$, put $a_y = t(y)$. As $q(a_y) = q(t(y)) = y$, $a_y \in q^{-1}(y)$. As e is a domination, for all $x \in q^{-1}(y)$ we have $x \preccurlyeq e(x) = t(q(x)) = t(y) = a_y$, so $\mathrm{Ker}(q)$ is dominant, that is, (5) holds.

Using (5) take, for any $y \in Y$, a vertex $a_y \in q^{-1}(y)$ with $a_y \succcurlyeq x$ for all $x \in q^{-1}(y)$. Define $t : V(Y) \to V(A)$ by $t(y) = a_y$. Set $e = t \circ q : V(A) \to V(A)$. As $e(x) = a_{q(x)} \succcurlyeq x$ for all $x \in A$, e is a map by 2.7. Then t is a map since q is a projection. If $y \in Y$, $q(t(y)) = q(a_y) = y$, so $q \circ t = 1_Y$ and t is a section of q. As e is an idempotent domination split by q, (1) holds. □

2.6. **Folds and antifolds, dismantlings and assemblings.** If $x \in A$ is dominated (say $x \preccurlyeq y \neq x$) the pinch $e = [x, y] : A \to A$ is an idempotent domination. Any splitting (r, s) of e entails a dominant retraction $r : A \to X$ and a dominant section $s : X \to A$ with $e = s \circ r$ and $r \circ s = 1_X$. We call r a *dominant fold* and s a *dominant antifold*. Using the standard splitting, $r = e^{|} : A \to A \smallsetminus x$ is a special dominant fold (or just *a fold* in [25], and the special dominant antifold $s : A \smallsetminus x \hookrightarrow A$ is an *antifold* in [13]). *Twinning folds* and *twinning antifolds* split the twinning pinches $e = [x, y]$ with $x \approx y \neq x$. We will focus on dominant folds and antifolds, mostly skipping their twin versions and mentioning them only in 4.22 below. We shorten "dominant fold" to "fold." By 2.12 a projection $r : A \to X$ is a fold iff $\mathrm{Ker}(r)$ is *folding*: it has one class $\{x, y\}$ with $x \preccurlyeq y \neq x$, and all others are singletons. By 2.10 an embedding $s : X \to A$ is an antifold iff $\mathrm{Im}(s) \leqslant A$ is *antifolding*: a dominant subgraph missing just one vertex. A projection equivalent to a fold is a fold by 2.4 and likewise for sections and antifolds, so any fold (antifold) is equivalent to some special fold (antifold). A *dismantling* $r : A \to X$ is a composition of folds $r = r_k \circ \cdots \circ r_2 \circ r_1$. In this case A *dismantles* to X, or X is a *dismantling* of A [13, 17, 25, 49, 52]. An *assembling* $s : X \to A$ is a composition of antifolds $s = s_1 \circ s_2 \circ \cdots \circ s_k$. Dismantlings (assemblings) are always retractions (sections) since so are folds (antifolds), but in general they are not dominant retractions (sections).

Lemma 2.13. *A projection r is a dismantling iff some (any) equivalent projection $\varphi \circ r$ is so, and likewise for embeddings and assemblings. Any dismantling (assembling) is equivalent to a composition of special folds (special antifolds).*

Proof. If $r = r_k \circ \cdots \circ r_1$ is a dismantling, $\varphi \circ r = (\varphi \circ r_k) \circ \cdots \circ r_1$ is a dismantling too. Indeed, $\varphi \circ r_k$ is equivalent to r_k and thus it is also a fold. For an assembling $s = s_1 \circ \cdots \circ s_k$ the argument is similar: we have $s \circ \varphi = s_1 \circ \cdots \circ (s_k \circ \varphi)$, and $(s_k \circ \varphi)$ is also an antifold.



Now let $r = r_k \circ \cdots \circ r_2 \circ r_1$ be any dismantling, and take an iso $\varphi_1 : X_1 \to A_1 \trianglelefteq A$ where $q_1 = \varphi_1 \circ r_1 : A \to A_1$ is a special fold. If $k > 1$, then $r_2' = r_2 \circ \varphi_1^{-1} : A_1 \to X_2$ is a fold as it has a folding kernel just as r_2. Then there is an iso $\varphi_2 : X_2 \to A_2 \trianglelefteq A_1$ where $q_2 = \varphi_2 \circ r_2' : A_1 \to A_2$ is a special fold. Then $\varphi_2 \circ r_2 = \varphi_2 \circ r_2' \circ \varphi_1 = q_2 \circ \varphi_1$, so the first two squares commute and $\varphi_2 \circ r_2 \circ r_1 = q_2 \circ q_1$ is a composition of two special folds. Continuing this way we get an iso φ_k such that $\varphi_k \circ r = \varphi_k \circ r_k \circ \cdots \circ r_2 \circ r_1 = q_k \circ \cdots \circ q_2 \circ q_1$ is a composition of k special folds. The case of an assembling $s = s_1 \circ s_2 \circ \cdots \circ s_k$ is similar: The image of each $s_i' = \varphi_{i-1} \circ s_i$ is antifolding, hence the isomorphism φ_i and the special antifold t_i with $t_i \circ \varphi_i = s_i'$. □

Given a sequence of distinct vertices $x_1, x_2, \ldots, x_k \in A$ put $X_i = A \smallsetminus \{x_1, x_2, \ldots x_i\}$ for $0 \leq i \leq k$ and $X = X_k$, so we have a sequence of induced subgraphs $A = X_0 \trianglerighteq X_1 \trianglerighteq X_2 \trianglerighteq \ldots \trianglerighteq X_k = X$. We call $x_1, x_2, \ldots, x_k$ a *dismantling sequence* if $x_i \preccurlyeq y_i \neq x_i$ in X_{i-1} for some vertices $y_i \in X_i$ where $i \in \{1, 2, \ldots, k\}$. In this case one clearly obtains a sequence of special dominant folds $A = X_0 \xrightarrow{r_1} X_1 \xrightarrow{r_2} X_2 \xrightarrow{r_3} \cdots X_{k-1} \xrightarrow{r_k} X_k = X$ (also called a *dismantling sequence* in [25]).

Theorem 2.14. *Any dominant retraction* (*dominant section*) *is a dismantling* (*assembling*). *Any special dominant retraction* (*section*) *is a composition of special folds* (*antifolds*).

Proof. We can take (2.13) a special dominant retraction $r : A \to X$. Let $A \smallsetminus X = \{x_1, \dots, x_k\}$ and $X_j = A \smallsetminus \{x_1, x_2, \dots x_j\}$, so $X_0 = A$ and $X_k = X$. Let $y_i = r(x_i)$, so $x_i \preccurlyeq y_i \neq x_i$ in A. Then $x_i \preccurlyeq y_i$ in $X_{i-1} \trianglelefteq A$ by 2.1, so $x_1, x_2, \dots, x_k$ is a dismantling sequence and we have special folds $r_1, r_2, \dots, r_k$ as above. Then $r = r_k \circ \cdots \circ r_2 \circ r_1$ is a dismantling. For sections we take a special dominant section $s : X \hookrightarrow A$. Then $X \trianglelefteq A$ is dominant by 2.10, and adding to X the vertices in $A \smallsetminus X$ one by one yields a sequence of special antifolds $s_i : X_i \hookrightarrow X_{i-1}$ because, as $X \trianglelefteq A$ is dominant, so is each $X_i \trianglelefteq X_{i-1}$. Another proof: If $e \in \mathrm{End}(A)$ is an idempotent domination split by r or by s, put $e = \prod_{i=1}^{k} \bar{e}_i$ as in 2.9. If $\bar{e}_i = [x_i, y_i]$, take $e_i = [x_i, y_i] \in \mathrm{End}(X_{i-1})$ with standard splitting (r_i, s_i). These are the needed r_i or s_i. □

2.7. **Pared graphs.** Let $X \trianglelefteq A$ be a dominant subgraph of A. If $X \trianglelefteq Y \trianglelefteq A$, then X is also dominant in Y by 2.1. In the other direction, not all induced subgraphs $Z \trianglelefteq X$ are dominant subgraphs of A, but X must contain some minimal dominant subgraph Z of A. Dominance is a preorder $\preccurlyeq$ on $V(A)$ and twinhood $\approx$ is mutual dominance, so we know that $\approx$ is an equivalence relation in $V(A)$ and $\preccurlyeq$ induces among twinhood classes a quotient partial order $\leq$ given by $\bar{a} \leq \bar{b} \Leftrightarrow a \preccurlyeq b$, where $\bar{a}$ and $\bar{b}$ are the twinhood classes of a and b.

Proposition 2.15. *An induced subgraph is a dominant subgraph just when it contains a vertex from each of the maximal twinhood classes. In consequence, it is a minimal dominant subgraph just when its vertex set is a system of representatives of those maximal classes.*

Proof. If $X \trianglelefteq A$ is dominant and the class $\bar{a}$ is maximal, take $x \in X$ with $x \succcurlyeq a$, so $\bar{x} \geq \bar{a}$. Then $\bar{x} = \bar{a}$, so $x \in \bar{a}$ and $x \in \bar{a} \cap V(X)$. Now, if $V(X)$ contains a system S of representatives of the maximal classes in A, let $a \in A$, pick a maximal class $\bar{b}$ with $\bar{a} \leq \bar{b}$ and take $s \in S \cap \bar{b}$. We have $\bar{b} = \bar{s}$ and then $\bar{a} \leq \bar{s}$. But then $a \preccurlyeq s \in X$, and thus X is dominant. □

The *pared graph* $P(A)$, due to Escalante [14], is any minimal dominant subgraph of A. By 2.15, $P(A) = \langle S \rangle \trianglelefteq A$ for some system S of representatives of the maximal twinhood classes of A. If S' is another such system we have $\langle S \rangle \cong \langle S' \rangle$, so $P(A)$ is only well-defined up to isomorphy: "$Y = P(A)$" means "$Y = \langle S \rangle$" for some such S. By 2.11 there is a special dominant retraction $r : A \to P(A)$. We call r a *paring retraction*. It is a dismantling by 2.14. We call the dominant section $P(A) \hookrightarrow A$ a *paring section*. If $P(A)$ is not stiff, iterating the paring operator yields a sequence of graphs $A = P^0(A) \trianglerighteq P(A) \trianglerighteq P^2(A) \trianglerighteq \cdots$. The first $r \geq 0$ with $P^r(A) = P^{r+1}(A)$ is the *paring index* $\mathrm{pi}(A)$ of A, defined by Prisner [52]. Note that the *completely pared graph* $P^\infty(A) = P^{\mathrm{pi}(A)}(A)$ is stiff, and that $\mathrm{pi}(P(A)) = \max\{0, \mathrm{pi}(A) - 1\}$. Composing paring retractions we get what we still call *paring retractions* $r : A \to P^i(A)$. They admit the *paring sections* $s : P^i(A) \hookrightarrow A$ as right inverses. Then we have:

Lemma 2.16. *For* $1 \leq i \leq \mathrm{pi}(A)$, *any paring retraction* $r : A \to P^i(A)$ *is a dismantling and any paring section* $s : P^i(A) \hookrightarrow A$ *is an assembling. For* $i = 1$ *we have more:* $r : A \to P(A)$ *is a dominant retraction and* $s : P(A) \hookrightarrow A$ *is a dominant section.* □

Lemma 2.17. *For any graph A the following six conditions are equivalent:*

(1) *A is stiff.* (2) *A is its only dominant subgraph.*

(3) $A = P(A)$. (4) $A = P^\infty(A)$. (5) $\mathrm{pi}(A) = 0$.

(6) *Any dominant retraction $A \to X$ is an isomorphism.* □

2.8. **The hash arrow relation.** The mere existence of some dominant retraction $X \to Y$ is often more important than which particular one is at hand. When there is a dominant retraction $r : X \to Y$ we say that X *dismantles to* Y *in one step* and denote it by $X \xrightarrow{\#} Y$. Besides X dismantling to Y by 2.14, the removed vertices are initially dominated in X and we can either remove them sequentially (in any order) or all at once (hence "in one step"). Hash arrows were introduced by Frías-Armenta, Neumann-Lara and Pizaña in [17]. We clearly have $X \xrightarrow{\#} X$, i.e. X dismantles trivially to itself. The smallest non-trivial steps $X \xrightarrow{\#} Y$ correspond to folds $X \to Y$. Paring retractions $X \to P(X)$ are dominant by 2.16, so we always have that $X \xrightarrow{\#} P(X)$, and this is as large a step as we can take starting from X. As in [18] we shorten $X_0 \xrightarrow{\#} X_1 \xrightarrow{\#} \cdots \xrightarrow{\#} X_t$ to $X_0 \xrightarrow{\#_t} X_t$ (X_0 *dismantles to* X_t *in* t *steps*). Note that X dismantles to Y just when $X \xrightarrow{\#_t} Y$ for some t. As $Y \xrightarrow{\#} Y$, we always have:

Lemma 2.18. *If* $X \xrightarrow{\#_t} Y$ *then* $X \xrightarrow{\#_{t+1}} Y$, *so* $X \xrightarrow{\#_t} Y$ *implies* $X \xrightarrow{\#_m} Y$ *for all* $m \geq t$. □

Since $X \xrightarrow{\#_{\mathrm{pi}(X)}} P^\infty(X)$, one can reach a stiff graph from X in $\mathrm{pi}(X)$ steps, see 2.16, 2.17. At least $\mathrm{pi}(X)$ steps are needed, and the stiff graph is always $P^\infty(X)$, as seen in 2.20 below.

Proposition 2.19. *If* $X \xrightarrow{\#} Y$ *then* $Y \xrightarrow{\#} P(X)$.

Proof. Taking a special dominant retraction $r : X \to Y$ we can assume that $Y \trianglelefteq X$ is a dominant subgraph of X. Let $Z \trianglelefteq Y$ be a minimal dominant subgraph of X, so $Z = P(X)$. Hence Z is a dominant subgraph of Y by 2.1, and thus we get that $Y \xrightarrow{\#} Z$ by 2.11. □

Theorem 2.20. *If* $X \xrightarrow{\#_t} X_t$ *and* X_t *is stiff, then* $t \geq \mathrm{pi}(X)$ *and* $X_t \cong P^\infty(X)$.

Proof. ["(k)" will stand for "2.17. k"]. If $t = 1$ we have $X \xrightarrow{\#} X_1$, and $X_1 \xrightarrow{\#} P(X)$ by 2.19. As X_1 is stiff, $X_1 \cong P(X)$ by (6), so $P(X)$ is stiff, $P(X) = P^2(X)$ by (3), and $1 \geq \mathrm{pi}(X)$. As $P(X)$ is stiff, (4) yields $X_1 \cong P(X) = P^\infty(P(X)) = P^\infty(X)$. If $t > 1$, assume first that X is stiff. Then $\mathrm{pi}(X) = 0$ by (5), so certainly $t \geq \mathrm{pi}(X)$. As $X \cong X_1 \cong \cdots \cong X_t$ by (6), we have $X_t \cong X = P^\infty(X)$ by (4). If X is not stiff, $\mathrm{pi}(X) > 0$ by (5), so $\mathrm{pi}(P(X)) = \mathrm{pi}(X) - 1$.

Let us consider now the following diagram of dominant retractions, which of course is not necessarily commutative. By 2.16 we have the vertical arrows $X_i \to P(X_i)$, and by our first hypothesis we have the arrows $X_i \to X_{i+1}$ at the top row. The slanted arrows $X_i \to P(X_{i-1})$, and then also the arrows $P(X_i) \to P(X_{i+1})$ at the bottom row, come from 2.19.

$$\begin{array}{ccccccccccc}
X & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots\cdots & \longrightarrow & X_{t-1} & \longrightarrow & X_t \\
\downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \cdots\cdots & \swarrow & \downarrow & \swarrow & \downarrow \\
P(X) & \longrightarrow & P(X_1) & \longrightarrow & P(X_2) & \longrightarrow & \cdots\cdots & \longrightarrow & P(X_{t-1}) & \longrightarrow & P(X_t).
\end{array}$$

As X_t is stiff, $X_t \to P(X_{t-1})$ is an iso by (6). Thus $P(X) \xrightarrow{\#_{t-1}} P(X_{t-1})$ with $P(X_{t-1})$ stiff. By the inductive hypothesis we obtain that $t - 1 \geq \mathrm{pi}(P(X)) = \mathrm{pi}(X) - 1$, so $t \geq \mathrm{pi}(X)$. Also by induction we know that $P(X_{t-1}) \cong P^\infty(P(X)) = P^{\mathrm{pi}(P(X))}(P(X))$, but then we have that $X_t \cong P(X_{t-1}) \cong P^{\mathrm{pi}(P(X))+1}(X) = P^{\mathrm{pi}(X)}(X) = P^\infty(X)$, which ends the proof. □

In their Proposition 2.60, Hell and Nešetřil [25] proved that each graph dismantles, up to isomorphy, to a unique stiff graph. Our 2.20 yields a new proof (and statement) of that:

Corollary 2.21. *Up to isomorphy the only stiff graph to which* A *dismantles is* $P^\infty(A)$. □

3. The Relational Category

The *relational category of simple graphs* (to be denoted by $\mathcal{G}_r$) has the same objects as the category $\mathcal{G}$ of finite simple graphs, but its morphisms are the graph relations. In the context of clique graphs, graph relations first appeared in [30], but [47] used implicitly the related notion of spans (see §3.4). Graph relations are useful for the study of maps and constructions in the graph category $\mathcal{G}$, and they are so close to graph maps that from the homotopical point of view both categories $\mathcal{G}$ and $\mathcal{G}_r$ turn out to be essentially the same (see 4.12).

3.1. **The graph of relations.** A graph relation $\alpha : X \to Y$ is a particular kind of vertex relation $\alpha \subseteq V(X) \times V(Y)$, but instead of writing "$(x, y) \in \alpha$" we use "$y \in \alpha(x)$". Thus the image of $S \subseteq V(X)$ under α is $\alpha(S) = \bigcup\{\alpha(s) \mid s \in S\} \subseteq V(Y)$. All our relations are defined everywhere (i.e. $\alpha(x) \neq \varnothing\ \forall x \in X$), so they are in fact *multivalued functions*. A *graph relation* (or just a *relation*) $\alpha : X \to Y$ is any multivalued function $\alpha \subseteq V(X) \times V(Y)$ which satisfies the equivalent conditions in Lemma 3.1 below. For condition (5) in the lemma, let us define the *common neighborhood* of a set $S \subseteq X$ as $N[S] = \bigcap_{s\in S} N[s]$. Then we have:

Lemma 3.1. *For a multivalued function $\alpha \subseteq V(X) \times V(Y)$ the following are equivalent:*

(1) $x \sim x'$ in X implies $y \sim y'$ in Y for all $y \in \alpha(x)$ and $y' \in \alpha(x')$.

(2) $\alpha(C)$ is a complete of Y for each small complete C of X.

(3) $\alpha(C)$ is a complete of Y for each complete C of X.

(4) $\alpha(C) \cup \alpha(D)$ is a complete of Y if $C \cup D$ is complete in X.

(5) $\alpha(N[x]) \subseteq N[\alpha(x)]$ for all $x \in X$. □

If X and Y are graphs we shall denote by $\mathcal{G}_r(X, Y)$ the set of all graph relations $\alpha : X \to Y$. If $\alpha \in \mathcal{G}_r(X, Y)$ and $\beta \in \mathcal{G}_r(Y, Z)$ the composite $\beta \circ \alpha : X \to Z$ is given by $(\beta \circ \alpha)(x) = \beta(\alpha(x))$ and by 3.1.3 is again a graph relation. With this composition $\mathcal{G}_r$ is a category. Each graph morphism is a graph relation (cf. 2.2 and 3.1) so $\mathcal{G}(X, Y)$ is a subset of $\mathcal{G}_r(X, Y)$. Morphism composition is a particular case of composition of relations, and each identity map $1_X \in \mathcal{G}(X, X)$ is still the identity in $\mathcal{G}_r(X, X)$, so $\mathcal{G}$ is a subcategory of $\mathcal{G}_r$.

Edges among graph relations are similar to edges among graph morphisms (cf. 2.5). Two relations $\alpha, \beta \in \mathcal{G}_r(X, Y)$ are *adjacent* ($\alpha \sim \beta$) if they satisfy the equivalent conditions in 3.2:

Lemma 3.2. *Let $\alpha, \beta \in \mathcal{G}_r(X, Y)$. Then the following conditions are equivalent:*

(1) $x \sim x'$ and $y \in \alpha(x)$ and $y' \in \beta(x')$ imply $y \sim y'$ for all $x, x' \in V(X)$.

(2) $\alpha(C) \cup \beta(C)$ is a complete of Y for each small complete C of X.

(3) $\alpha(C) \cup \beta(C)$ is a complete of Y for each complete C of X.

(4) $\alpha(C) \cup \beta(D)$ is a complete of Y if $C \cup D$ is complete in X.

(5) $\alpha(N[x]) \subseteq N[\beta(x)]$ for all $x \in X$. □

Now that each set of relations $\mathcal{G}_r(X, Y)$ is a graph we have the analog of 2.6: The proof is the same as that of 2.6 but using 3.2.3 instead of 2.5.3 and 3.2.4 instead of 2.5.4:

Proposition 3.3. *The* composition map $\circ : \mathcal{G}_r(Y, Z) \times \mathcal{G}_r(X, Y) \to \mathcal{G}_r(X, Z)$ *that sends* (β, α) *to $\beta \circ \alpha$ is a graph map. In particular, $\beta \sim \beta' \Rightarrow \beta \circ \alpha \sim \beta' \circ \alpha$ and $\alpha \sim \alpha' \Rightarrow \beta \circ \alpha \sim \beta \circ \alpha'$.* □

Each $\mathcal{G}_r(X,Y)$ is partially ordered by $\subseteq$ and we have $\alpha \subseteq \beta \Leftrightarrow \alpha(x) \subseteq \beta(x)$ for all $x \in X$.

Lemma 3.4. *$\mathcal{G}_r(X,Y)$ is a lower set in the poset of multivalued functions: If $\alpha \in \mathcal{G}_r(X,Y)$ any multivalued function $\beta \subseteq V(X) \times V(Y)$ with $\beta \subseteq \alpha$ is also a graph relation $\beta \in \mathcal{G}_r(X,Y)$ and in particular any function $f : V(X) \to V(Y)$ with $f \subseteq \alpha$ is a graph morphism.*
Proof. By 3.1.3: If $C \subseteq X$ is complete so is $\alpha(C)$ and also $\beta(C)$ since $\varnothing \neq \beta(C) \subseteq \alpha(C)$. □

3.2. **Maps meet relations.** As any function is a relation we have $\mathcal{G}(X,Y) \subseteq \mathcal{G}_r(X,Y)$. Indeed, comparing 3.1 and 3.2 with 2.2 and 2.5 and using 3.4 for the last claim we have:

Lemma 3.5. *The inclusion map $\mathcal{I} = \mathcal{I}_{X,Y} : \mathcal{G}(X,Y) \hookrightarrow \mathcal{G}_r(X,Y)$ is a graph morphism. It is in fact an embedding whose image consists of the minimal elements of $\mathcal{G}_r(X,Y)$.* □

Given two morphisms $f, g : X \to Y$ their union $f \cup g \subseteq V(X) \times V(Y)$ sends each $x \in X$ to $f(x) \cup g(x) = \{f(x), g(x)\} \subseteq Y$. But we do not necessarily have that $\alpha = f \cup g \in \mathcal{G}_r(X,Y)$. Since $\alpha(C) = f(C) \cup g(C)$ for each vertex set $C \subseteq X$, comparing 2.5.3 and 3.1.3 we get:

Lemma 3.6. *If $f, g \in \mathcal{G}(X,Y)$ then $f \sim g$ in $\mathcal{G}(X,Y)$ if and only if $f \cup g \in \mathcal{G}_r(X,Y)$.* □

Comparing now 3.2.3 with 3.1.3 we generalize 3.6 to relations:

Lemma 3.7. *If $\alpha, \beta \in \mathcal{G}_r(X,Y)$ then $\alpha \sim \beta$ in $\mathcal{G}_r(X,Y)$ if and only if $\alpha \cup \beta \in \mathcal{G}_r(X,Y)$.* □

Notice that by 3.4 two relations $\alpha, \beta \in \mathcal{G}_r(X,Y)$ have a common upper bound γ in $\mathcal{G}_r(X,Y)$ (that is $\alpha \subseteq \gamma$ and $\beta \subseteq \gamma$) if and only if $\alpha \cup \beta \in \mathcal{G}_r(X,Y)$. Therefore the graph $\mathcal{G}_r(X,Y)$ is by 3.7 just the *upper bound graph*, as defined in [44], of the poset $\mathcal{G}_r(X,Y)$.

Given a relation $\alpha \in \mathcal{G}_r(X,Y)$ a *selection* of α is any morphism $f \in \mathcal{G}(X,Y)$ with $f \subseteq \alpha$.

Lemma 3.8. *Any relation has a selection. In fact for a given $\alpha \in \mathcal{G}_r(X,Y)$ any choice function $f : V(X) \to V(Y)$ of the sets $\alpha(x) \subseteq V(Y)$ is by 3.4 a selection of α.* □

The graph structures in $\mathcal{G}(X,Y)$ and $\mathcal{G}_r(X,Y)$ are nicely related via selections:

Lemma 3.9. *If $\alpha \subseteq \beta \in \mathcal{G}_r(X,Y)$ then $\alpha \sim \beta$. In particular we have $f \sim \beta$ in $\mathcal{G}_r(X,Y)$ whenever $f : X \to Y$ is a selection of $\beta \in \mathcal{G}_r(X,Y)$.*
Proof. Since $\alpha \cup \beta \subseteq \alpha$ we can apply 3.7. □

Lemma 3.10. *Let $\alpha, \beta \in \mathcal{G}_r(X,Y)$. Then the following conditions are equivalent:*

(1) $\alpha \sim \beta$ in $\mathcal{G}_r(X,Y)$.

(2) $f \sim g$ in $\mathcal{G}(X,Y)$ for all selections $f \subseteq \alpha$ and $g \subseteq \beta$.

Proof. Assume (1). Since $f \cup g \subseteq \alpha \cup \beta \in \mathcal{G}_r(X,Y)$ by 3.7, $f \cup g \in \mathcal{G}_r(X,Y)$ by 3.4 and $f \sim g$ in $\mathcal{G}(X,Y)$ by 3.6. Now assume (2). Let $x \sim x'$ in X and take $y \in \alpha(x)$ and $y' \in \beta(x')$. By 3.8 there are selections $f \subseteq \alpha$ and $g \subseteq \beta$ such that $f(x) = y$ and $g(x') = y'$. We have then by (2) that $f \sim g$ in $\mathcal{G}(X,Y)$. Hence $y \sim y'$ by 2.5.1 and finally $\alpha \sim \beta$ by 3.2.1. □

3.3. **Elementary adjacencies.** Handy for calculations, elementary edges define the same connected components by 3.11 below. Call $f, g \in \mathcal{G}(X,Y)$ *elementarily adjacent* ($f \sim_e g$) if there is some $x_0 \in X$ with $f(x_0) \sim g(x_0)$ and $f(x) = g(x)$ for $x \neq x_0$. For example, the elementary neighbors of $1_X \in \mathrm{End}(X)$ are just the dominant pinches (see 2.7 and 2.9).

As indicated by the name $f \sim_e g$ implies $f \sim g$: Indeed, let $C = \{x, x'\} \subseteq X$ be complete and let $\alpha = f \cup g$. By 2.2.2 $f(C)$ and $g(C)$ are complete. If $x_0 \notin C$ then $\alpha(C) = f(C)$ is complete, and if $x_0 \in C$ then $\alpha(C) = f(C) \cup g(C)$ is complete because $f(x_0) \sim g(x_0)$. Then $\alpha \in \mathcal{G}_r(X, Y)$ by 3.1.2, so $f \sim g$ by 3.6. The connected components of $\mathcal{G}(X, Y)$ are important for homotopy (see §4.1 below), and elementary adjacencies are enough to define those components:

Proposition 3.11. *Assume that $f \sim g$ in $\mathcal{G}(X, Y)$. Then for some finite number of graph morphisms $f_i \in \mathcal{G}(X, Y)$ we have $f = f_0 \sim_e f_1 \sim_e f_2 \sim_e \cdots \sim_e f_r = g$.*

Proof. If $f(x_0) \neq g(x_0)$, let $f_1 : V(X) \to V(Y)$ be given by $f_1(x) = f(x)$ for $x \neq x_0$ and $f_1(x_0) = g(x_0)$. Let $\alpha = f \cup g$. By 3.6 we have that $\alpha \in \mathcal{G}_r(X, Y)$. Then $f_1 : X \to Y$ is a map by 3.4 because $f_1 \subseteq \alpha$. Using that $f \cup f_1 \subseteq \alpha$ and $f_1 \cup g \subseteq \alpha$ we get from 3.6 that $f \sim f_1 \sim g$. By 2.5.1 we have that $f(x_0) \sim f_1(x_0)$ and thus $f \sim_e f_1$. We finish by induction. □

Given $f \in \mathcal{G}(X, Y)$, to construct a map g with $f \sim_e g$ in practice one picks the exceptional vertex $x_0 \in X$ and replaces $f(x_0)$ with some vertex $y_0 \in Y$ such that $f(N[x_0]) \subseteq N[y_0]$:

Lemma 3.12. *Let $f \in \mathcal{G}(X, Y)$, $x_0 \in X$, and $g : V(X) \to V(Y)$ be such that $f(x) = g(x)$ for all $x \in X \setminus x_0$. Then $f(N[x_0]) \subseteq N[g(x_0)]$ implies that $g \in \mathcal{G}(X, Y)$ and $f \sim_e g$.*

Proof. Let $x \sim x'$. If $x_0 \notin \{x, x'\}$, $g(x) = f(x) \sim f(x') = g(x')$. If $x_0 \in \{x, x'\}$ put $x_0 = x'$. As $g(N[x_0]) = g(N(x_0)) \cup \{g(x_0)\} = f(N(x_0)) \cup \{g(x_0)\} \subseteq N[g(x_0)]$ by 2.5.5 and $x \in N[x_0]$, $g(x) \in N[g(x_0)]$, i.e. $g(x) \sim g(x_0) = g(x')$. By 2.2.1, g is a map. As $f(x_0) \sim g(x_0)$, $f \sim_e g$. □

3.4. **Strong retractions and spans.** The *converse* $\alpha^\dagger : X \to A$ of a relation $\alpha : A \to X$ is given by $a \in \alpha^\dagger(x) \Leftrightarrow x \in \alpha(a)$. Note that even if α is a graph relation $\alpha^\dagger$ need not be so. We are interested here in those graph morphisms $r : A \to X$ whose converse $r^\dagger : X \to A$ is a graph relation. Notice that r must begin by being vertex-surjective for $r^\dagger(x) = r^{-1}(x)$ to be defined for all $x \in X$. In fact r needs to be also edge-surjective and even more: it must be a retraction since $r^\dagger \in \mathcal{G}_r(X, A)$ implies by 3.8 that there is a selection $s : X \to A$ of $r^\dagger$ and then s, for which $r \circ s = 1_X$ holds, is a section of r. The graph projection $r : A \to X$ will be called a *strong retraction* when its converse $r^\dagger : X \to A$ is a graph relation. Somewhat surprisingly, this coincides with the previously studied notion of a twinning retraction:

Proposition 3.13. *For a graph projection $r \in \mathcal{G}(A, X)$ the following are equivalent:*

(1) $r^\dagger : X \to A$ is a graph relation (i.e. $r : A \to X$ is a strong retraction),

(2) $r^{-1}(C) \subseteq A$ is complete for any complete $C \subseteq X$,

(3) $r^{-1}(C) \subseteq A$ is complete for any small complete $C \subseteq X$,

(4) all the fibers $r^{-1}(x)$—the equivalence classes under Ker(r)*—are twinsets, and*

(5) $r : A \to X$ is a twinning retraction □

Proof. $r^\dagger(C) = r^{-1}(C)$ if $C \subseteq V(X)$, so (2) and (3) are conditions 3.1.3 and 3.1.2 for $r^\dagger$ and (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). By the twin of 2.12 we have (4) $\Leftrightarrow$ (5). To prove (3) $\Rightarrow$ (4) take $x \in X$ and $a, a' \in r^{-1}(x)$. By symmetry $a' \preccurlyeq a$ will imply $a \approx a'$. Let $b \in N[a']$. Put $y = r(b)$. As $y \in N[r(a')] = N[x]$ by 2.2.5 $C = \{x, y\} \subseteq X$ is complete. Then $r^{-1}(C) = r^{-1}(x) \cup r^{-1}(y)$ is complete by (3), so $a \sim b$, $b \in N[a]$, and $a' \preccurlyeq a$. To prove (4) $\Rightarrow$ (3) let $C = \{x, y\} \subseteq X$ be complete. Since r is edge-surjective there exist $a \in r^{-1}(x)$ and $b \in r^{-1}(y)$ with $a \sim b$. If $a' \in r^{-1}(x)$ and $b' \in r^{-1}(y)$, $a' \approx a$ and $b \approx b'$ by (4), so $a \sim b$ implies $a' \sim b'$. As in addition the twinsets $r^{-1}(x) \neq \varnothing$ and $r^{-1}(y) \neq \varnothing$ are complete $r^{-1}(C) = r^{-1}(x) \cup r^{-1}(y)$ is complete. □

A *graph span* from X to Y in $\mathcal{G}$ is an ordered pair (q,h) of maps $h : A \to Y$ and $q : A \to X$ where the left leg $q : A \to X$ is a strong retraction. Strong retractions (via 3.13.2 and 3.13.3) and also graph spans were introduced by Neumann-Lara for the unpublished proofs of his K-divergence results in [47]. Those proofs were eventually done with relations in [30].

Figure 1. A graph span (q,h) and the graph relation $\alpha = h \circ q^\dagger$ defined by (q,h).

Proposition 3.14. *The relation $\alpha \subseteq V(X) \times V(Y)$ is a graph relation $\alpha : X \to Y$ if and only if $\alpha = h \circ q^\dagger$ for some graph span (q,h) from X to Y.*

Proof. Let (q,h) be a graph span from X to Y with $\alpha = h \circ q^\dagger$. Since q is a strong retraction $q^\dagger$ is a graph relation and therefore $\alpha = h \circ q^\dagger \in \mathcal{G}_r(X,Y)$. If $\alpha \in \mathcal{G}_r(X,Y)$ let $A = \langle \alpha \rangle = \langle \{(x,y) \mid x \in X, y \in \alpha(x)\} \rangle \trianglelefteq X \times Y$. Consider the restrictions $q = (\pi_X)_| : A \to X$ and $h = (\pi_Y)_| : A \to Y$. We will show first that (q,h) is a graph span. Since α is defined everywhere q is vertex-surjective. Let C be complete in X. By 3.1.3 $\alpha(C)$ is complete in Y. Then $q^{-1}(C)$ is complete in A as $\varnothing \neq q^{-1}(C) \subseteq C \times \alpha(C)$ and $C \times \alpha(C)$ is complete in $X \times Y$. Thus by 3.1.3 $q^\dagger : X \to A$ is a graph relation and q is a strong retraction. Hence (q,h) is a graph span from X to Y. Finally we only need to observe that for any $x \in X$ we have $(h \circ q^\dagger)(x) = h(q^\dagger(x)) = \pi_Y(\{x\} \times \alpha(x)) = \alpha(x)$ and therefore $h \circ q^\dagger = \alpha$. □

4. Homotopy

4.1. **Homotopy in graphic categories.** Graph homotopy makes sense in both categories $\mathcal{G}$ and $\mathcal{G}_r$ as we can consider homotopy of graph morphisms and also of graph relations. Both settings get unified under the notion of a graphic category, or a category enriched over $\mathcal{G}$. A category $\mathcal{A}$ is a *graphic category* if each set of morphisms $\mathcal{A}(A,B)$ is endowed with a fixed graph structure (a set of edges, or a symmetric, reflexive binary relation $\sim$) in such a way that for any $A,B,C \in \mathrm{ob}(\mathcal{A})$ the function $M : \mathcal{A}(B,C) \times \mathcal{A}(A,B) \to \mathcal{A}(A,C)$ given by $M(g,f) = g \circ f$ is a graph morphism. That is, $g \sim g'$ in $\mathcal{A}(B,C)$ and $f \sim f'$ in $\mathcal{A}(A,B)$ imply that $g \circ f \sim g' \circ f'$ in $\mathcal{A}(A,C)$. By 2.6 and 3.3 both $\mathcal{G}$ and $\mathcal{G}_r$ are graphic categories.

In a graphic category $\mathcal{A}$ two morphisms $f,g \in \mathcal{A}(A,B)$ are *homotopic* (denoted by $f \simeq g$) if f and g lie in the same connected component of the graph $\mathcal{A}(A,B)$. Thus $f \simeq g$ means that there is a walk $f = h_0 \sim h_1 \sim h_2 \sim \cdots \sim h_n = g$ in $\mathcal{A}(A,B)$ or, equivalently, some morphism $h : I_n \to \mathcal{A}(A,B)$ with $h(0) = f$ and $h(n) = g$, also called a *walk* from f to g. In this situation we say (see [13]) that f and g are n*-homotopic*: it takes at most n edges to get from f to g. Homotopy is an equivalence relation in each $\mathcal{A}(A,B)$, and it is the transitive closure of $\sim$.

If $\mathcal{A}$ and $\mathcal{B}$ are graphic categories, a functor $F : \mathcal{A} \to \mathcal{B}$ will be called a *graphic functor* if $F = F_{A,B} : \mathcal{A}(A,B) \to \mathcal{B}(F(A),F(B))$ is a graph map for all $A,B \in \mathrm{ob}(\mathcal{A})$. For instance, the inclusion functor $\mathcal{I} : \mathcal{G} \hookrightarrow \mathcal{G}_r$ is a graphic functor by 3.5. Any graphic functor $F : \mathcal{A} \to \mathcal{B}$ between graphic categories sends homotopic $\mathcal{A}$-morphisms to homotopic $\mathcal{B}$-morphisms.

A *congruence* on a category $\mathcal{C}$ is an equivalence relation $\equiv$ on each set $\mathcal{C}(A,B)$ such that whenever $g \equiv g'$ in $\mathcal{C}(B,C)$ and $f \equiv f'$ in $\mathcal{C}(A,B)$ we have that $g \circ f \equiv g' \circ f'$ in $\mathcal{C}(A,C)$. The *kernel* of any functor $G : \mathcal{C} \to \mathcal{D}$, given by $g \equiv g' \Leftrightarrow G(g) = G(g')$, is a congruence on $\mathcal{C}$.

Proposition 4.1. *The homotopy relation $\simeq$ is a congruence on any graphic category $\mathcal{A}$. In particular, it is a congruence on both categories $\mathcal{G}$ and $\mathcal{G}_r$.*

Proof. Let $g \simeq g'$ in $\mathcal{A}(B,C)$ and $f \simeq f'$ in $\mathcal{A}(A,B)$. Adding loops as needed, we have walks $g = h_0 \sim h_1 \sim h_2 \sim \cdots \sim h_n = g'$ in $\mathcal{A}(B,C)$ and $f = k_0 \sim k_1 \sim k_2 \sim \cdots \sim k_n = f'$ in $\mathcal{A}(A,B)$. Since $\mathcal{A}$ is graphic, $g \circ f = h_0 \circ k_0 \sim h_1 \circ k_1 \sim h_2 \circ k_2 \sim \cdots \sim h_n \circ k_n = g' \circ f'$ in $\mathcal{A}(A,C)$. □

The *homotopy category* of a graphic category $\mathcal{A}$ is denoted by $h\mathcal{A}$ and it is the quotient category of $\mathcal{A}$ modulo the homotopy congruence: $h\mathcal{A} = \mathcal{A}/\simeq$. This is a particular case of the quotient category $\mathcal{C}/\equiv$ of a category $\mathcal{C}$ modulo a congruence $\equiv$ on $\mathcal{C}$, see II, §8 in [42]. We recall the details, which work identically in the general case. The objects of $h\mathcal{A}$ are the same: $\mathrm{ob}(h\mathcal{A}) = \mathrm{ob}(\mathcal{A})$, but $h\mathcal{A}(A,B) = \{[f] \mid f \in \mathcal{A}(A,B)\}$ for all $A, B \in \mathrm{ob}(h\mathcal{A})$ where $[f]$ is the homotopy class of f. The composition is $[g] \circ [f] = [g \circ f]$ whenever it makes sense, and the identity in $h\mathcal{A}(A,A)$ is $[1_A]$. The *projection functor* $P = P_{\mathcal{A}} : \mathcal{A} \to h\mathcal{A}$ sends each morphism to its homotopy class: $P_{A,B} : \mathcal{A}(A,B) \to h\mathcal{A}(A,B)$ is given by $P_{A,B}(f) = P(f) = [f]$.

Lemma 4.2. *Let $\mathcal{C}, \mathcal{D}$ be any categories and let $\mathcal{A}, \mathcal{B}$ be graphic categories. Then:*

(1) If $\equiv$ is a congruence in $\mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$ is a functor such that $f \equiv g \Rightarrow G(f) = G(g)$, then there is a unique functor $G' : \mathcal{C}/\equiv \to \mathcal{D}$ making diagram (a) commute.

(2) If $F : \mathcal{A} \to \mathcal{B}$ is a graphic functor, then there is a unique functor $hF : h\mathcal{A} \to h\mathcal{B}$ which makes diagram (b) commute.

$$(a): \quad \begin{array}{ccc} \mathcal{C} & & \\ {\scriptstyle P_{\mathcal{C}}}\downarrow & \searrow^{G} & \\ \mathcal{C}/\equiv & \overset{G'}{\dashrightarrow} & \mathcal{D} \end{array} \qquad\qquad (b): \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ {\scriptstyle P_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle P_{\mathcal{B}}} \\ h\mathcal{A} & \overset{hF}{\dashrightarrow} & h\mathcal{B}. \end{array}$$

Necessarily, $G'([f]) = G(f)$ for $[f] \in \mathrm{mor}(\mathcal{C}/\equiv)$ and $hF([f]) = [F(f)]$ for $f \in \mathrm{mor}(h\mathcal{A})$.

Proof. Part (1) is standard and easy [42, II,§8]. Part (2): if $f \simeq g$ in $\mathcal{A}$, $F(f) \simeq F(g)$ in $\mathcal{B}$ and $P_{\mathcal{B}}(F(f)) = P_{\mathcal{B}}(F(g))$ in $h\mathcal{B}$. Apply now part (1) to $G = P_{\mathcal{B}} \circ F$ and take $hF = G'$. □

4.2. **The adjoint definition of homotopy.** From now on we shall confine our attention to the graph categories $\mathcal{G}$ and $\mathcal{G}_r$. In them we can recast homotopy in terms of cylinders.

Lemma 4.3. *For any three graphs $X, I, Y \in \mathcal{G}$ there is a natural graph isomorphism*

$$\lambda : \mathcal{G}(X \times I, Y) \to \mathcal{G}(I, \mathcal{G}(X,Y))$$

which is defined by $\lambda(H)(i)(x) = H(x,i)$ for all $H \in \mathcal{G}(X \times I, Y)$, $i \in I$, and $x \in X$. □

The above result is fairly standard and it is proved in [13, 7.3] for the larger category $\mathcal{G}^{\pm}$. The proof is essentially the same. We just point out that the role played by $\mathcal{G}(X,Y)$ here corresponds in [13] to the larger (and a little less natural) Cartesian exponential graph Y^X.

The *cylinder of height* n over $X \in \mathcal{G}$ is the product $X \times I_n$. Its i-th *slice* is $X \times \{i\}$. The embedding $\sigma_i = (-,i) : X \to X \times I_n$ of X into this slice is given by $\sigma_i(x) = (x,i)$ as in §2.2.

Proposition 4.4. *Two morphisms $f, g : X \to Y$ are n-homotopic if and only if there is a graph morphism $H : X \times I_n \to Y$ such that $f = H \circ \sigma_0$ and $g = H \circ \sigma_n$.*

Proof. Let $I = I_n$, $H \in \mathcal{G}(X \times I, Y)$ and $h = \lambda(H) \in \mathcal{G}(I, \mathcal{G}(X,Y))$, where λ is as in 4.3. Since $h(i)(x) = H(x,i)$ for all $i \in I$, $H \circ \sigma_0 = H(-,0) = h(0)$ and $H \circ \sigma_1 = h(1)$. Then h is a walk in $\mathcal{G}(X,Y)$ from f to g if, and only if, H satisfies the conditions in the statement. □

The map H in 4.4 is called *a homotopy from f to g*. It is a discrete version in $\mathcal{G}$ of the usual notion of a homotopy among continuous maps in elementary topology. The analog of 4.4 in $\mathcal{G}_r$ is 4.6, whose proof is the same but uses 4.5 instead of 4.3. The proof of 4.5 is quite similar to that of 4.3. We will skip the straightforward verifications for both 4.5 and 4.6.

Lemma 4.5. *Given $X, I \in \mathcal{G}$, $Y \in \mathcal{G}_r$ there is a natural graph isomorphism*

$$\psi : \mathcal{G}_r(\mathcal{I}(X \times I), Y) \to \mathcal{G}(I, \mathcal{G}_r(\mathcal{I}(X), Y))$$

which is given by $\psi(\mathcal{H})(i)(x) = \mathcal{H}(x,i) \subseteq Y$ for all $\mathcal{H} \in \mathcal{G}_r(\mathcal{I}(X \times I), Y)$, $i \in I$ and $x \in X$. □

Technicalities apart, the inclusion functor $\mathcal{I} : \mathcal{G} \hookrightarrow \mathcal{G}_r$ occurs in 4.5 to stress the fact that the product $X \times I$ is taken in $\mathcal{G}$. In $\mathcal{G}_r$ it could be nonexistent, as $I_1 \times I_1$ can be shown to be.

Proposition 4.6. *Two graph relations $\alpha, \beta : X \to Y$ are n-homotopic if and only if there is a graph relation $\mathcal{H} : X \times I_n \to Y$ ($X \times I_n$ taken in $\mathcal{G}$) such that $\alpha = \mathcal{H} \circ \sigma_0$ and $\beta = \mathcal{H} \circ \sigma_n$.* □

Prisner introduced in [52] another kind of homotopy in $\mathcal{G}$, namely topological homotopy via the flag complex [33, 36, 37]. Let us show that graph homotopy, due to Dochtermann [13] and studied here, is a finer relation. The *flag complex* $\Delta(X)$ has the same vertices as X and its simplices are the completes of X. Then $|\Delta(X)|$, the (*geometric*) *realization* of $\Delta(X)$, is an object of the category $\mathcal{T}$ of topological spaces. By 2.2.3 any $f \in \mathcal{G}(X,Y)$ is a simplicial map $\Delta(f) : \Delta(X) \to \Delta(Y)$, so it induces a continuous function $|\Delta(f)| : |\Delta(X)| \to |\Delta(Y)|$. In fact $|\Delta(\ )| : \mathcal{G} \to \mathcal{T}$ is a functor that helps us represent graphs as topological spaces. If $H : X \times I_n \to Y$ is a homotopy from f to g in $\mathcal{G}(X,Y)$ as in 4.4, consider the continuous function $|\Delta(H)| : |\Delta(X \times I_n)| \to |\Delta(Y)|$. It turns out that $|\Delta(X \times I_n)|$ is not homeomorphic to the topological product $|\Delta(X)| \times |I_n| \cong |\Delta(X)| \times I$, but it is homotopy equivalent to it and in essence $|\Delta(H)|$ is a topological homotopy $\mathcal{H} : |\Delta(X)| \times I \to |\Delta(Y)|$ from $|\Delta(f)|$ to $|\Delta(g)|$. Then the functor $|\Delta(\ )|$ preserves homotopy: $f \simeq g$ implies that f and g are also homotopic when seen as continuous maps $|\Delta(X)| \to |\Delta(Y)|$ (see §4 in [33] for details). We will resume the comparison of graph homotopy with flag complex homotopy at the end of §4.4 below.

4.3. **First characterizations.** Homotopy, both of morphisms and of relations, is described in 4.7 and 4.8 in terms of the partial order on $\mathcal{G}_r(X,Y)$ and also, via selections, in 4.9, 4.10, and 4.11. The homotopy categories $h\mathcal{G}$ and $h\mathcal{G}_r$ will be shown to be isomorphic in 4.12. The following two results, apart from using 3.4, stem directly from 3.6 and 3.7:

Proposition 4.7. *Two graph maps $f, g \in \mathcal{G}(X,Y)$ are homotopic if and only if there exist some morphisms $f_i \in \mathcal{G}(X,Y)$ and also some graph relations $\alpha_i \in \mathcal{G}_r(X,Y)$ in such a way that we have $f = f_1 \subseteq \alpha_1 \supseteq f_2 \subseteq \alpha_2 \supseteq \cdots \subseteq \alpha_{s-1} \supseteq f_s = g$.* □

Proposition 4.8. *Two graph relations $\alpha, \beta \in \mathcal{G}(X,Y)$ are homotopic if and only if there exist some relations $\alpha_i \in \mathcal{G}_r(X,Y)$ such that $\alpha = \alpha_1 \subseteq \alpha_2 \supseteq \alpha_3 \subseteq \alpha_4 \supseteq \cdots \subseteq \alpha_{s-1} \supseteq \alpha_s = \beta$.* □

Lemma 4.9. *If $\alpha \subseteq \beta \in \mathcal{G}_r(X,Y)$ then $\alpha \simeq \beta$ by 3.9. In particular $f \simeq \beta$ in $\mathcal{G}_r(X,Y)$ whenever $f : X \to Y$ is a selection of $\beta \in \mathcal{G}_r(X,Y)$.* □

Proposition 4.10. *For any relations $\alpha, \beta \in \mathcal{G}_r(X,Y)$ the following conditions are equivalent:*

(1) $\alpha \simeq \beta$ in $\mathcal{G}_r(X,Y)$.

(2) $f \simeq g$ in $\mathcal{G}(X,Y)$ for any chosen selections $f \subseteq \alpha$ and $g \subseteq \beta$.

(3) $f \simeq g$ in $\mathcal{G}(X,Y)$ for some given selections $f \subseteq \alpha$ and $g \subseteq \beta$.

Proof. Assuming (1) take a walk $\alpha = \alpha_0 \sim \alpha_1 \sim \cdots \sim \alpha_n = \beta$ in $\mathcal{G}_r(X,Y)$ and pick selections $f_i \subseteq \alpha_i$ where $f_0 = f$ and $f_n = g$ are those chosen in (2). We have a walk $f = f_0 \sim f_1 \sim \cdots \sim f_n = g$ in $\mathcal{G}(X,Y)$ by 3.10, so $f \simeq g$ in $\mathcal{G}(X,Y)$ and (2) holds. Now (2) implies (3) as any relation has a selection by 3.8. If (3) holds, any walk $f = f_0 \sim f_1 \sim \cdots \sim f_n = g$ in $\mathcal{G}(X,Y)$ can be enlarged, by 3.9, to a walk $\alpha \sim f \sim f_1 \sim \cdots \sim g \sim \beta$ in $\mathcal{G}_r(X,Y)$, so $\alpha \simeq \beta$ in $\mathcal{G}_r(X,Y)$. □

Corollary 4.11. *For any two maps $f, g \in \mathcal{G}(X,Y)$ the following conditions are equivalent:*

(1) f and g are homotopic as morphisms in $\mathcal{G}(X,Y)$.

(2) f and g are homotopic as relations in $\mathcal{G}_r(X,Y)$. □

Theorem 4.12. *The homotopy categories $h\mathcal{G}$ and $h\mathcal{G}_r$ are isomorphic.*

Proof. Denote by $[f]$ the homotopy class of a morphism $f \in \mathcal{G}(X,Y)$ and by $[\alpha]_r$ that of a relation $\alpha \in \mathcal{G}_r(X,Y)$. The inclusion functor $\mathcal{I} : \mathcal{G} \hookrightarrow \mathcal{G}_r$ is graphic by 3.5, so by 4.2 it induces a functor on the homotopy categories $h\mathcal{I} : h\mathcal{G} \to h\mathcal{G}_r$ given by $(h\mathcal{I})([f]) = [\mathcal{I}(f)]_r = [f]_r$. For any $f \in \mathcal{G}(X,Y)$ we have that $[f] = [f]_r \cap \mathcal{G}(X,Y)$ by 4.11 and therefore $h\mathcal{I}$ is faithful. Any class $[\alpha]_r \in h\mathcal{G}_r(X,Y)$ equals, by 4.9, the class $[f]_r$ of any of the selections $f \in \mathcal{G}(X,Y)$ of α that exist by 3.8 and then $h\mathcal{I}$ is also full. Since $h\mathcal{I}$ is fully faithful, it is an equivalence of categories, and being bijective (the identity) in objects it is an isomorphism of categories. Observe that the inverse of $h\mathcal{I}$ is given by $[\alpha]_r \mapsto [f]$ where f is any selection of α. □

4.4. **Homotopy equivalences.** As any congruence does, homotopy induces an equivalence relation among graphs, namely being isomorphic in the quotient category. Thus we say that the graphs X and Y are *homotopy equivalent* (or have *the same homotopy type*) and denote it by $X \simeq Y$ if there are morphisms (or relations, see 3.10 or 4.12) $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. In this case we say that f and g are *homotopy equivalences*, each of them a *homotopy inverse* of the other, or just that f and g are *homotopy inverses*.

All isomorphisms are homotopy equivalences, and the latter share with isomorphisms the *two out of three property*: If f and g are composable morphisms and two of f, g and $g \circ f$ are homotopy equivalences, so is the third. If $f \in \mathrm{End}(A)$ is a domination $f \simeq 1_A$ by 2.7, so f is by 4.1 homotopically self-inverse: $f \circ f \simeq 1_A \circ 1_A = 1_A$ and so $f : A \to A$ is a homotopy equivalence. Another example: any twinning retraction $p : A \to X$ is a homotopy equivalence. Indeed, $p^\dagger \in \mathcal{G}_r(X,A)$ by 3.13 and clearly $p \circ p^\dagger = 1_X$. As $p^\dagger \circ p \supseteq 1_A$ we get $p^\dagger \circ p \simeq 1_A$ by 4.9. Thus p and $p^\dagger$ (or p and any selection of $p^\dagger$) are homotopy inverses. This can be generalized:

Lemma 4.13. *All dismantlings and assemblings are homotopy equivalences. The following are important particular cases of dismantlings and assemblings:*

(1) folds $r : A \to X$ and antifolds $s : X \to A$,

(2) dominant retractions $r : A \to X$ and dominant sections $s : X \to A$,

(3) paring retractions $r : A \to P^i(A)$ and paring sections $s : P^i(A) \hookrightarrow A$ $(1 \leq i \leq \mathrm{pi}(A))$.

Proof. For any dominant retraction $r : A \to X$ there is a dominant section $s : X \to A$ (and viceversa) such that $r \circ s = 1_X$ and $e = s \circ r$ is a domination. By 2.7, $e \sim 1_A$, so $e \simeq 1_A$ and thus r and s are homotopy inverses. As folds (antifolds) are dominant retractions (sections), they are homotopy equivalences. Dismantlings (assemblings) are compositions of folds (antifolds), hence equivalences. The maps in (2) and (3) are particular cases by 2.14 and 2.16. □

Lemma 4.14. *Let $f \in \mathrm{End}(A)$ be an acyclic domination with stability index $s(f) = k > 0$, let $B = f(A)$ and $g = f^{|}_{|} : B \to B$. Then $g \in \mathrm{End}(B)$ is an acyclic domination with $s(g) = k-1$.*

Proof. Since g is acyclic with $s(g) = k-1$ by A.3.2, we only need to prove, given $f(x) \in B$, that $f(x) \preccurlyeq g(f(x))$ in B, that is $f(x) \preccurlyeq_B f^2(x)$. Taking $f(y) \in B$ with $f(y) \sim_B f(x)$ we will show that $f(y) \sim_B f^2(x)$. As $f^{|} : A \to B$ is a graph projection, for some vertices $x', y' \in A$ we have $f(x') = f(x)$, $f(y') = f(y)$ and $x' \sim_A y'$. Then, as $y' \sim_A x'$ and $x' \preccurlyeq_A f(x')$ we have that $y' \sim_A f(x') = f(x)$. Now 2.2.1 applied to $f^{|} : A \to B$ yields $f(y) = f(y') \sim_B f^2(x)$. □

Theorem 4.15. *Let $f \in \mathrm{End}(A)$ be a domination. Then $r = f^{|} : A \to f(A)$ is a homotopy equivalence. There is also a homotopy equivalence $q : A \to \langle f(A) \rangle$.*

Proof. In the decomposition $f = f_a \circ f_b$ both the acyclic part f_a and the bijective part f_b are dominations (see §2.4) and the isomorphism f_b is an equivalence. As $f(A) = f_a(A)$ and $f^{|} = (f_a \circ f_b)^{|} = f_a^{|} \circ f_b$ we only need to prove the result for f_a. Let then $f = f_a \neq 1_A$ be acyclic. Then $k = s(f) > 0$ by A.3.1. If $k = 1$, f is idempotent also by A.3.1. As in this case r is a dominant retraction, 4.13.2 applies. If $k > 1$, let $B = f(A)$ and $g = f^{|}_{|} : B \to B$. By 4.14 $g \in \mathrm{End}(B)$ is an acyclic domination with $s(g) = k-1$. Both $f^k \in \mathrm{End}(A)$ and $g^{k-1} \in \mathrm{End}(B)$ are (clearly acyclic) dominations by 2.7, and $s(f^k) = s(g^{k-1}) = 1$ by A.3.3. Then by the $k = 1$ case both $r_1 = (f^k)^{|} : A \to f^k(A)$ and $r_2 = (g^{k-1})^{|} : B \to g^{k-1}(B)$ are homotopy equivalences.

$$\begin{array}{ccc} A & \xrightarrow{\quad r \quad} & B \\ & {\scriptstyle r_1} \searrow \quad \swarrow {\scriptstyle r_2} & \\ & f^k(A) & \end{array}$$

Note that $g^{k-1}(B) = g^{k-1}(f(A)) = f^k(A)$, so both r_1 and r_2 have indeed the same codomain. We have $r_2 \circ r = r_1$ as these two maps send each $a \in A$ to $f^k(a)$. Therefore r is a homotopy equivalence by the two out of three property. For the second claim put $e(x) = x$ if $x \in f(a)$ and $e(x) = f(x)$ otherwise, so $e : A \to A$ is an idempotent domination with $e(A) = \langle f(A) \rangle$. Then the dominant retraction $q = e^{|} : A \to \langle f(A) \rangle$ is a homotopy equivalence by 4.13.2. □

Part "(1)⇒(4)" in the following result appeared as a consequence of Proposition 6.6 in [13]:

Theorem 4.16. *For any graphs $X, Y \in \mathcal{G}$ the following conditions are equivalent:*

(1) $X \simeq Y$.

(2) $P^\infty(X) \cong P^\infty(Y)$.

(3) X dismantles to $P^\infty(Y)$.

(4) X and Y have a common dismantling Z.

Proof. By 2.21 (2) ⇒ (3) ⇒ (4), and also (4) ⇒ (2) for, if both X and Y dismantle to Z, we must have $P^\infty(X) \cong P^\infty(Z) \cong P^\infty(Y)$. We shall prove (1) ⇔ (2): If $X \simeq Y$, by 4.13.3 $P^\infty(X) \simeq X \simeq Y \simeq P^\infty(Y)$, and thus there are homotopy equivalences $f : P^\infty(X) \to P^\infty(Y)$ and $g : P^\infty(Y) \to P^\infty(X)$ with $f \circ g \simeq 1_{P^\infty(Y)}$ and $g \circ f \simeq 1_{P^\infty(X)}$. But $P^\infty(X)$ and $P^\infty(Y)$ are stiff, so $f \circ g = 1_{P^\infty(Y)}$ and $g \circ f = 1_{P^\infty(X)}$ by 2.8, and hence $P^\infty(X) \cong P^\infty(Y)$. Conversely, if $P^\infty(X) \cong P^\infty(Y)$ then $P^\infty(X) \simeq P^\infty(Y)$, so $X \simeq P^\infty(X) \simeq P^\infty(Y) \simeq Y$ by 4.13.3 again. □

Assuming that $X \simeq Y$, take equivalences $f : X \to Y$, $g : Y \to X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Then $|\Delta(g)| \circ |\Delta(f)| = |\Delta(g \circ f)| = |\Delta(1_X)| = 1_{|\Delta(X)|}$ and likewise $|\Delta(f)| \circ |\Delta(g)| = 1_{|\Delta(X)|}$, so $|\Delta(X)| \simeq |\Delta(Y)|$ as topological spaces (see the end of §4.2 above). A stronger claim holds:

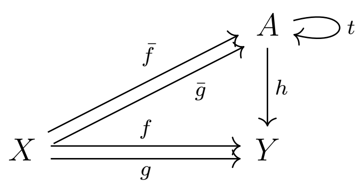
Whenever $X \simeq Y$, the flag complexes $\Delta(X)$ and $\Delta(Y)$ have the same simple homotopy type: Note first that by 4.16.4 there is a sequence (known already in [13]) of folds and antifolds

$$X \xrightarrow{r_1} X_1 \xrightarrow{r_2} \cdots \to X_{m-1} \xrightarrow{r_m} Z \xrightarrow{s_k} Y_{k-1} \to \cdots \xrightarrow{s_2} Y_1 \xrightarrow{s_1} Y.$$

Each of these maps is equivalent to a special fold $A \to A \smallsetminus a$ or to a special antifold $A \smallsetminus a \hookrightarrow A$, and so $\Delta(X)$ and $\Delta(Y)$ have the same simple homotopy type. This follows from the proof of Proposition 3.2 in [52]: in our notation, if $a \in A$ is dominated, $\Delta(A)$ collapses to $\Delta(A \smallsetminus a)$. Now, if $m > n > 3$, the cyclic flag complexes $\Delta(C_m)$ and $\Delta(C_n)$ have the same simple homotopy type, and $|\Delta(C_m)|$ and $|\Delta(C_n)|$ are even homeomorphic. However, $C_m \not\simeq C_n$ because C_m and C_n are stiff and cannot have a common dismantling as required by 4.16.4. Hence "$X \simeq Y$" is not implied by "$\Delta(X)$ and $\Delta(Y)$ have the same simple homotopy type," let alone by "$|\Delta(C_m)|$ and $|\Delta(C_n)|$ are homotopy equivalent" (or even homeomorphic).

4.5. **Further characterizations.** These will be 4.18, 4.20 and 4.22. The starting point is:

Theorem 4.17. *Let $f, g \in \mathcal{G}(X, Y)$. Then $f \sim g$ if and only if there exist $A \in \mathcal{G}$, a graph morphism $h : A \to Y$ and liftings $\bar{f}, \bar{g} : X \to A$ of f and g (that is $h \circ \bar{f} = f$ and $h \circ \bar{g} = g$) such that there is an order two bijective twination $t : A \to A$ for which $\bar{g} = t \circ \bar{f}$.*



The result also holds upon replacing "there is an order two bijective twination $t : A \to A$" in the above first version of the statement by "there is an idempotent twination $t : A \to A$."

Proof. In both versions $t \sim 1_A$ by 2.7, so $g = h \circ \bar{g} = h \circ t \circ \bar{f} \sim h \circ 1_A \circ \bar{f} = h \circ \bar{f} = f$ by 2.6. Conversely, if $f \sim g$ we have $\alpha = f \cup g \in \mathcal{G}_r(X, Y)$ by 3.6 and, by 3.14, $\alpha = h \circ q^\dagger$ for some graph span $X \xleftarrow{q} A \xrightarrow{h} Y$. Let us define $\bar{f} : X \to A$. If $x \in X$, $f(x) \in \alpha(x) = h(q^\dagger(x))$, so we can choose some $\bar{f}(x) \in q^\dagger(x)$ with $f(x) = h(\bar{f}(x))$. Then $h \circ \bar{f} = f$, and $\bar{f} \in \mathcal{G}(X, A)$ by 3.8. We have also a selection $\bar{g} : X \to A$ of $q^\dagger$ with $h \circ \bar{g} = g$. From $\{\bar{f}(x), \bar{g}(x)\} \subseteq q^\dagger(x) = q^{-1}(x)$ two facts follow: First, all the transpositions $t_x = (\bar{f}(x), \bar{g}(x))$ are disjoint (q is single-valued) so they commute with each other and their well-defined product $t = \prod_{x \in X} t_x$ is a permutation of $V(A)$ with $t^2 = 1_A$. Second, as q is a strong retraction its fibers are twinsets by 3.13, so $\bar{f}(x) \approx \bar{g}(x)$ and each $t_x \in \operatorname{End}(A)$ is a twinning transposition. Thus $t : A \to A$ is a twination of A by 2.7, and we have that $t \circ \bar{f} = \bar{g}$ just by design. For the alternative version we use the disjoint twinning pinches $p_x = [\bar{f}(x), \bar{g}(x)]$. They commute with each other by A.1, so their product $t : A \to A$, which again satisfies $t \circ \bar{f} = \bar{g}$, is an idempotent twination by 2.9. □

Theorem 4.18. *Let $f, g \in \mathcal{G}(X, Y)$. Then $f \simeq g$ if, and only if, for some $n \in \mathbb{N}$ we have:*

(1) morphisms $f_i : X \to Y$ for $i = 0, \ldots, n$ with $f = f_0$ and $f_n = g$,

(2) graphs $A_1, A_2, \ldots, A_n$ and morphisms $h_i : A_i \to Y$ for $i = 1, \ldots, n$,

(3) liftings $\bar{f}^i_{i-1}$, $\bar{f}^i_i : X \to A_i$ of f_{i-1} and f_i along h_i ($f_j = h_i \circ \bar{f}^i_j$) for $i = 1, \ldots, n$,

(4) order two bijective twinations $t_i : A_i \to A_i$ such that $t_i \circ \bar{f}^i_{i-1} = \bar{f}^i_i$ for $i = 1, \ldots, n$.

This result also holds if we replace (4) by:

(4') idempotent twinations $t_i : A_i \to A_i$ such that $t_i \circ \bar{f}^i_{i-1} = \bar{f}^i_i$ for $i = 1, \ldots, n$.

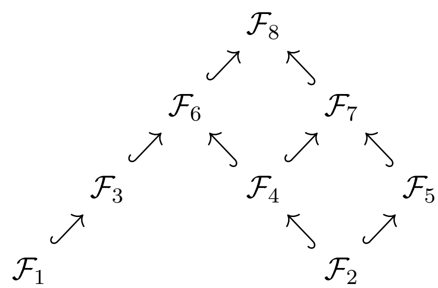
Proof. Let $f \simeq g$. For n and the f_i in (1) take a walk $f = f_0 \sim f_1 \sim \cdots \sim f_n = g$ in $\mathcal{G}(X,Y)$. Now n applications of 4.17 yield (2), (3) and (4) or (4'). For the converse, begin by fixing an $i \in \{1,\dots,n\}$. By 2.7 we have $1_{A_i} \sim t_i$ and then $\bar{f}^i_{i-1} = 1_{A_i} \circ \bar{f}^i_{i-1} \sim t_i \circ \bar{f}^i_{i-1} = \bar{f}^i_i$ by 2.6. But then $f_{i-1} = h_i \circ \bar{f}^i_{i-1} \sim h_i \circ \bar{f}^i_i = f_i$ again by 2.6. Thus $f = f_0 \sim f_1 \sim f_2 \sim \cdots \sim f_n = g$ and $f \simeq g$. □

Lemma 4.19. *Let $t \in \mathrm{End}(A)$. If t is an order order two bijective twination it is a product of twinning transpositions, and a product of twinning pinches if it is an idempotent twination.*
Proof. This (and more) follows from the factorizations in §2.4. □

A *family* $\mathcal{F}$ of endomorphisms consists of a set $\mathcal{F}_A \subseteq \mathrm{End}(A)$ for each $A \in \mathcal{G}$, but we shorten "$f \in \mathcal{F}_A$ for some $A \in \mathcal{G}$" to just "$f \in \mathcal{F}$." E.g., "$f \in \mathcal{F} \Rightarrow f \equiv 1$" means "$f \in \mathcal{F}_A \Rightarrow f \equiv 1_A$." Also, if $\mathcal{G}$ is another such family, we will use "$\mathcal{F} \subseteq \mathcal{G}$" to signify that "$\mathcal{F}_A \subseteq \mathcal{G}_A$ for all $A \in \mathcal{G}$".

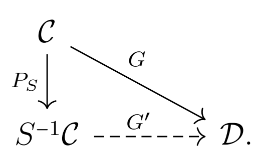
Theorem 4.20. *For any of the eight families of endomorphisms $\mathcal{F} = \mathcal{F}_i$ described below homotopy is the finest congruence on $\mathcal{G}$ which identifies the maps in $\mathcal{F}$ with identities.*

$\mathcal{F}_1 = \{f \mid f$ is a twinning transposition$\}$
$\mathcal{F}_2 = \{f \mid f$ is a twinning pinch$\}$
$\mathcal{F}_3 = \{f \mid f$ is an order two bijective twination$\}$
$\mathcal{F}_4 = \{f \mid f$ is an idempotent twination$\}$
$\mathcal{F}_5 = \{f \mid f$ is a dominant pinch$\}$
$\mathcal{F}_6 = \{f \mid f$ is a twination$\}$
$\mathcal{F}_7 = \{f \mid f$ is an idempotent domination$\}$
$\mathcal{F}_8 = \{f \mid f$ is a domination$\}$



Proof. We have that $\mathcal{F} \subseteq \mathcal{F}_8$ and $(\mathcal{F}_8)_A = N[1_A]$ by 2.7, so $f \in \mathcal{F}_A \Rightarrow f \sim 1_A \Rightarrow f \simeq 1_A$. Hence the homotopy congruence $\simeq$ does indeed identify each $f \in \mathcal{F}_A$ with the identity 1_A. Let $\equiv$ be a congruence in $\mathcal{G}$ such that $f \in \mathcal{F} \Rightarrow f \equiv 1$. As $\mathcal{F}_1 \subseteq \mathcal{F}$ or $\mathcal{F}_2 \subseteq \mathcal{F}$ we have $f \equiv 1$ for all $f \in \mathcal{F}_1$ or, alternatively, for all $f \in \mathcal{F}_2$. As $\equiv$ is a congruence 4.19 yields $f \equiv 1$ for all $f \in \mathcal{F}_3$ or, in the other case, for all $f \in \mathcal{F}_4$. Let $f \simeq g$ in $\mathcal{G}(X,Y)$. We now need to show that $f \equiv g$. We proceed by way of 4.18 using condition (4) if $\mathcal{F}_1 \subseteq \mathcal{F}$ and condition (4') if $\mathcal{F}_2 \subseteq \mathcal{F}$. Then all $t_i \in \mathcal{F}_3$ in our first case and all $t_i \in \mathcal{F}_4$ in the other. In both cases we get that all $1_{A_i} \equiv t_i$ and, as $\equiv$ is a congruence, $\bar{f}^i_{i-1} = 1_{A_i} \circ \bar{f}^i_{i-1} \equiv t_i \circ \bar{f}^i_{i-1} = \bar{f}^i_i$ and $f_{i-1} = h_i \circ \bar{f}^i_{i-1} \equiv h_i \circ \bar{f}^i_i = f_i$ for all $i = 1,\dots,n$. Thus $f = f_0 \equiv f_1 \equiv f_2 \equiv \cdots \equiv f_n = g$ and $f \equiv g$ by transitivity, as needed. □

Under some fairly general assumptions [19, 54] homotopy categories of a category $\mathcal{C}$ can be described as localizations of $\mathcal{C}$, and so can our homotopy category $h\mathcal{G}$ by 4.22 below. Recall the definitions: A functor $G : \mathcal{C} \to \mathcal{D}$ *makes invertible* a morphism $s \in \mathrm{mor}(\mathcal{C})$ if $G(s)$ is invertible (i.e. an isomorphism) in $\mathrm{mor}(\mathcal{D})$. If $S \subseteq \mathrm{mor}(\mathcal{C})$ is a class of $\mathcal{C}$-morphisms, the *localization* of $\mathcal{C}$ with respect to S (which always exists [19, I,1.1]) is a category $S^{-1}\mathcal{C}$ together with a functor $P_S : \mathcal{C} \to S^{-1}\mathcal{C}$ having the following universal property: First, P_S makes each morphism in S invertible and, second, for any functor $G : \mathcal{C} \to \mathcal{D}$ making all the morphisms in S invertible there exists a unique functor $G' : S^{-1}\mathcal{C} \to \mathcal{D}$ such that $G = G' \circ P_S$:



By the universal property $(P_S, S^{-1}\mathcal{C})$ is unique up to an isomorphism of categories.

Lemma 4.21. *Let (r, s) be a retraction-section pair in $\mathcal{C}$ with $r : A \to X$, $s : X \to A$, and composition $e = s \circ r : A \to A$. Then for any functor $G : \mathcal{C} \to \mathcal{D}$ the following are equivalent:*

(1) $F(s)$ *is invertible*

(2) $F(s)^{-1} = F(r)$

(3) $F(e) = 1_{F(A)}$

(4) $F(r)$ *is invertible*

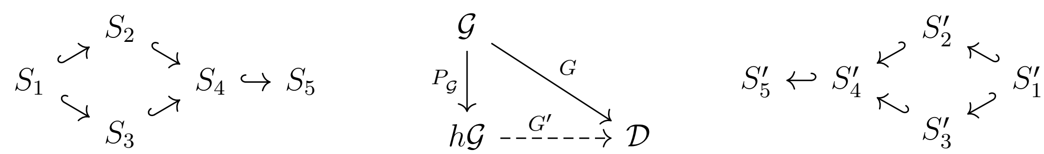
(5) $F(r)^{-1} = F(s)$

Proof. Since $r \circ s = 1_X$ we have $F(r) \circ F(s) = F(r \circ s) = F(1_X) = 1_{F(X)}$. This is already half of (2) and also half of (5). In both cases the other half is (3) which, as $e = s \circ r$, says that $F(s) \circ F(r) = F(e) = 1_{F(A)}$. Thus (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4), and these clearly imply (1) and (4). Now, to prove that (1) $\Rightarrow$ (2), let h be the inverse of $F(s)$, that is $h = F(s)^{-1} : F(A) \to F(X)$. Then $h = 1_{F(X)} \circ h = F(r) \circ F(s) \circ h = F(r) \circ 1_{F(A)} = F(r)$, and therefore $F(s)^{-1} = h = F(r)$. In an entirely similar way one can clearly prove that (4) $\Rightarrow$ (5). □

Theorem 4.22. *The homotopy category $h\mathcal{G}$, with its projection functor $P_{\mathcal{G}} : \mathcal{G} \to h\mathcal{G}$, is the localization of $\mathcal{G}$ with respect to S, where the class $S \subseteq \mathrm{mor}(\mathcal{G})$ can be any of the following:*

$S_1 = \{r \mid r \text{ is a twinning fold}\}$ | $S'_1 = \{s \mid s \text{ is a twinning antifold}\}$
$S_2 = \{r \mid r \text{ is a dominant fold}\}$ | $S'_2 = \{s \mid s \text{ is a dominant antifold}\}$
$S_3 = \{r \mid r \text{ is a twinning retraction}\}$ | $S'_3 = \{s \mid s \text{ is a twinning section}\}$
$S_4 = \{r \mid r \text{ is a dominant retraction}\}$ | $S'_4 = \{s \mid s \text{ is a dominant section}\}$
$S_5 = \{r \mid r \text{ is a dismantling}\}$ | $S'_5 = \{s \mid s \text{ is an assembling}\}$.

Proof. Fix $i \in \{1, 2, 3, 4\}$ and let f belong to S_i or to S'_i. Then f splits an idempotent domination e as $f \in S_4$ or $f \in S'_4$. Hence $e \simeq 1$ by 4.20 because $e \in \mathcal{F}_7$. Thus $P_{\mathcal{G}}(e) = [1]$, so $P_{\mathcal{G}}(f)$ is invertible by 4.21. For $i = 5$, the maps in S_5 or S'_5 are also made invertible by $\mathcal{P}_{\mathcal{G}}$ because they are compositions of maps in S_2 or S'_2.



Now let $G : \mathcal{G} \to \mathcal{D}$ be a functor that makes the maps in S invertible, so G makes either the maps in S_1 or those in S'_1 invertible. If e is a twinning pinch there is a splitting (r, s) of e with $r \in S_1$ and $s \in S'_1$. As one of $G(r)$ and $G(s)$ is invertible, $G(e) = 1$ by 4.21. Recall that the kernel of G is the congruence $\equiv$ of $\mathcal{G}$ given by $f \equiv g \Leftrightarrow G(f) = G(g)$. As $\equiv$ identifies the maps in $\mathcal{F}_2$ with identities, the homotopy congruence is finer than the kernel of G by 4.20. Then G factors uniquely trough $P_{\mathcal{G}}$ by 4.2, just as we needed to show. □

We could include in 4.22 the class T of homotopy equivalences: $P_{\mathcal{G}} : \mathcal{G} \to h\mathcal{G}$ makes the maps in T invertible (tautologically) and in fact $(P_{\mathcal{G}}, h\mathcal{G})$ is the localization of $\mathcal{G}$ with respect to T. This class T is what is called in [19] the *saturation* of any of the classes S in 4.22.

5. Clique Graphs

Recall that the complete subgraphs of the graph X (or just *the completes* of X) are always non-empty. A *clique* of X is a maximal complete of X. The *clique graph* of X is the intersection graph $K(X)$ of the cliques of X, so the vertices of $K(X)$ are the cliques of X and two cliques q_1, q_2 of X are adjacent in $K(X)$ if and only if $q_1 \cap q_2 \neq \varnothing$. On the homotopy category the operator K is a functor (§5.1), an old transformation becomes natural (§5.2), and new concepts, techniques and results for clique behavior emerge (§5.3, §5.4, §5.5).

5.1. **The clique graph functor.** Given a graph relation $\alpha : X \to Y$ and a clique $q \in K(X)$, $\alpha(q)$ is a complete of Y by 3.1.3 but it is not always a clique: $\alpha(q)$ is not necessarily maximal. We define, as in [30], a graph relation $K(\alpha) : K(X) \to K(Y)$ by sending each $q \in K(X)$ to all the cliques of Y containing $\alpha(q)$, that is $K(\alpha)(q) = \{\hat{q} \in K(Y) \mid \alpha(q) \subseteq \hat{q}\}$. Then $K(\alpha)$ is indeed a graph relation, but $K(\alpha)$ is not necessarily a graph morphism even if α is a graph morphism $\alpha = f : X \to Y$, as for some $q \in K(X)$ there can be several $\hat{q} \in K(Y)$ with $\alpha(q) \subseteq \hat{q}$. Note that K reverses inclusions: $\alpha \subseteq \beta : X \to Y$ implies $K(\alpha) \supseteq K(\beta) : K(X) \to K(Y)$.

The *clique graph operator* $K : \mathrm{ob}\,\mathcal{G}_r \to \mathrm{ob}\,\mathcal{G}_r$ given by $X \mapsto K(X)$ can be supplemented with $\alpha \mapsto K(\alpha)$ and thus K is an assignation of objects and morphisms $K : \mathcal{G}_r \to \mathcal{G}_r$. But it is not a functor, as Figure 2 shows: Let h be the composite $h = g_1 \circ f_1 = g_2 \circ f_2 : X \to Z$. Then one would need that $K(g_1) \circ K(f_1) = K(h) = K(g_2) \circ K(f_2)$, but this does not hold:

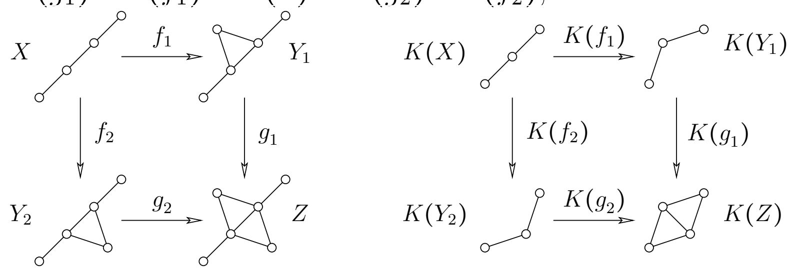


FIGURE 2. The operator $K : \mathcal{G}_r \to \mathcal{G}_r$ is not a functor. Notice that in this example we have $h = g_1 \circ f_1 = g_2 \circ f_2$ but, on the other hand, $K(g_1) \circ K(f_1) \neq K(g_2) \circ K(f_2)$.

There is an older approach [47] to the action of K on maps: If $f \in \mathcal{G}(X,Y)$ choose, for each $q \in K(X)$, some $f_K(q) \in K(Y)$ with $f(q) \subseteq f_K(q)$. This map $f_K : K(X) \to K(Y)$ is not uniquely determined but it is always a selection of the relation $K(f)$. Figure 2 shows that this "single-valued version" $K : \mathcal{G} \to \mathcal{G}$ also fails to be a functor, as f_1, f_2, g_1 and g_2 are maps and their images in $\mathcal{G}_r$ under K are maps again, so they are their only possible selections. We can also write $K(f)$ instead of f_K even when f_K is not unique: the context usually helps one to decide whether $K(f) \in \mathcal{G}(K(X), K(Y))$ or $K(f) \in \mathcal{G}_r(K(X), K(Y))$ is meant. For more clarity, however, we can use expressions such as "choose a selection of $K(f)$" or "fix a map $K(f)$" when we refer to the older, single-valued version of $K(f)$. By §3.2 and §4.3 there is little difference between both versions $K : \mathcal{G}_r \to \mathcal{G}_r$ & $K : \mathcal{G} \to \mathcal{G}$, even less when passing to the homotopy categories $h\mathcal{G}$ and $h\mathcal{G}_r$ which by 4.12 are isomorphic. The isomorphism $h\mathcal{G}(X,Y) \to h\mathcal{G}_r(X,Y)$ was given by $[f] \mapsto [f]_r$ and the inverse by $[\alpha]_r \mapsto [f]$, where $f \subseteq \alpha$ is any selection of α. In what follows we shall simplify $[\ \ ]_r$ to just $[\ \ ]$.

Theorem 5.1. *The clique graph operator $K : \mathcal{G}_r \to \mathcal{G}_r$ is a functor up to homotopy. In other words $K : h\mathcal{G}_r \to h\mathcal{G}_r$ is a functor or, which in essence is the same, $K : h\mathcal{G} \to h\mathcal{G}$ is a functor.*

Proof. If $\alpha \simeq \beta$ in $\mathcal{G}_r(X,Y)$ we have $\alpha = \alpha_1 \subseteq \alpha_2 \supseteq \alpha_3 \subseteq \alpha_4 \supseteq \cdots \subseteq \alpha_{s-1} \supseteq \alpha_s = \beta$ by 4.8. Then we get $K(\alpha) \subseteq K(\alpha_1) \supseteq K(\alpha_2) \subseteq K(\alpha_3) \supseteq K(\alpha_4) \subseteq \cdots \supseteq K(\alpha_{s-1}) \subseteq K(\alpha_s) \supseteq K(\beta)$, so $K(\alpha) \simeq K(\beta)$ again by 4.8. Thus $K : \mathcal{G}_r \to \mathcal{G}_r$ preserves homotopy and $K : h\mathcal{G}_r \to h\mathcal{G}_r$ is well-defined: the image of any $[\alpha] \in h\mathcal{G}_r(X,Y)$, i.e. $K([\alpha]) = [K(\alpha)] \in h\mathcal{G}_r(K(X), K(Y))$, does not depend on the chosen representative α of the homotopy class $[\alpha]$.

If $\alpha \in \mathcal{G}_r(X,Y)$, $\beta \in \mathcal{G}_r(Y,Z)$ and $q \in K(X)$, let $\hat{\hat{q}} \in (K(\beta) \circ K(\alpha))(q) = K(\beta)(K(\alpha)(q))$. Since $K(\beta)(K(\alpha)(q)) = \bigcup_{\hat{q} \in K(\alpha)(q)} K(\beta)(\hat{q})$ there must be some $\hat{q} \in K(\alpha)(q)$ with $\beta(\hat{q}) \subseteq \hat{\hat{q}}$. As $\hat{q} \in K(\alpha)(q)$ we have $\alpha(q) \subseteq \hat{q}$ and then $(\beta \circ \alpha)(q) = \beta(\alpha(q)) \subseteq \beta(\hat{q}) \subseteq \hat{\hat{q}}$. Since $\hat{\hat{q}} \in K(Z)$, we have $\hat{\hat{q}} \in K(\beta \circ \alpha)(q)$. Thus $K(\beta) \circ K(\alpha) \subseteq K(\beta \circ \alpha)$, and $K(\beta) \circ K(\alpha) \simeq K(\beta \circ \alpha)$ by 4.9.

Hence $K : \mathcal{G}_r \to \mathcal{G}_r$ preserves compositions up to homotopy, and for $K : h\mathcal{G}_r \to h\mathcal{G}_r$ we have $K([\beta]\circ[\alpha]) = K([\beta\circ\alpha]) = [K(\beta\circ\alpha)] = [K(\beta)\circ K(\alpha)] = [K(\beta)]\circ[K(\alpha)] = K([\beta])\circ K([\alpha])$. If $q \in K(X)$, $1_X(q) = q$ is already a clique of X, so $K(1_X)(q) = q$ and $K(1_X) = 1_{K(X)}$. But then $K([1_X]) = [1_{K(X)}]$, so $K : h\mathcal{G}_r \to h\mathcal{G}_r$ preserves identities and hence it is a functor. □

The clique graph functor could be denoted by $hK : h\mathcal{G} \to h\mathcal{G}$, but then again we prefer to use K instead of hK for simplicity's sake. (See, however, "hK-behavior" in §5.3 below.)

Corollary 5.2. *If $X \simeq Y$ then $K(X) \simeq K(Y)$.* □

5.2. **The star transformation.** The *star* of a vertex $x \in X$ is $x^* = \{q \in K(X) \mid x \in q\}$. Stars of vertices have always been important for clique graphs. For instance, in [58, Thm.2], $L_i = \{h_j \mid g_i \in K_j\}$ is just g_i^*. Also $S_G(x)$, in the notation of [14], is just x^*. By the way, Escalante [14] originally defined the dominance relation $u \preccurlyeq v$ by the condition $u^* \subseteq v^*$, and the equivalence of this with $N[u] \subseteq N[v]$ has belonged to the folklore ever since [52].

Each x^* is a set of cliques of X, but $x \mapsto x^*$ is not a graph relation $X \to K(X)$: even for the path I_3 of length three 3.1.1 fails. Nevertheless, the cliques in x^* intersect each other, so x^* is a complete of $K(X)$ and we can then define the *star relation* $* : X \to K^2(X)$ by $*(x) = \{Q \in K^2(X) \mid Q \supseteq x^*\}$. This is a graph relation: given a complete $C \subseteq X$, $*(C) = \bigcup_{c\in C}(*(c))$ so, if $Q, Q' \in *(C)$ we have $c, c' \in C$ with $Q \in *(c)$ and $Q' \in *(c')$. Thus $Q \supseteq c^*$ and $Q' \supseteq c'^*$, but $c \sim c'$ implies $c, c' \in q$ for some $q \in K(X)$, and then $q \in c^* \cap c'^* \subseteq Q \cap Q'$ and $Q \cap Q' \neq \varnothing$. Then $*(C)$ is a complete of $K^2(X)$ and thus $* \in \mathcal{G}_r(X, K^2(X))$ by 3.1.2.

We call any fixed selection of the star relation *the star morphism* and also denote it by $*$. This causes no confusion and it is harmless: all star maps are homotopic by 4.10.

Theorem 5.3. *The star relation (and the star morphism) $* : X \to K^2(X)$ is a natural transformation from the identity functor to K^2 on the homotopy category $h\mathcal{G}_r \cong h\mathcal{G}$.*

Proof. Let $f : X \to Y$ be a graph map and let us fix maps $K(f) : K(X) \to K(Y)$ and $K^2(f) : K^2(X) \to K^2(Y)$ (recall §5.1) and star maps $*_X : X \to K^2(X)$, $*_Y : Y \to K^2(Y)$. We show that the next diagram commutes up to homotopy, so its image commutes in $h\mathcal{G}$:

$$\begin{array}{ccc} X & \xrightarrow{*_X} & K^2(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle K^2(f)} \\ Y & \xrightarrow{*_Y} & K^2(Y). \end{array}$$

Let $x \sim x'$ in X, $x, x' \in q \in K(X)$, and $\hat{q} = K(f)(q) \in K(Y)$. As $x \in q$, $q \in x^* \subseteq *_X(x)$, so $\hat{q} = K(f)(q) \in K(f)(*_X(x)) \subseteq K^2(f)(*_X(x))$. As $x' \in q$ and $f(q) \subseteq \hat{q}$, $f(x') \in \hat{q}$, so $\hat{q} \in *_Y(f(x'))$. Thus $K^2(f)(*_X(x)) \cap *_Y(f(x')) \neq \varnothing$, so $K^2(f)(*_X(x)) \sim *_Y(f(x'))$ in $K^2(Y)$. By 2.5.1 $(K^2(f)\circ *_X) \sim (*_Y \circ f)$ in $\mathcal{G}(X, K^2(Y))$, hence $K^2(f)\circ *_X \simeq *_Y \circ f$. □

A graph is *clique-Helly* (see [14], [52]) if every family of pairwise intersecting cliques has a non-empty intersection. The *cliques of cliques* of these graphs are always stars of vertices: If Q is a clique of $K(X)$ and $x \in \bigcap Q$, then $Q = x^*$. But not all stars are cliques: one could have $x^* \subsetneq y^*$. In fact, $x^* \subseteq y^* \Leftrightarrow x \preccurlyeq y$, so $x^* = y^* \Leftrightarrow x \approx y$, and $x^* \in K^2(X)$ if and only if $\bar{x}$ is a maximal twinhood class, so the restriction $\psi_X = (*_X)_{|} : P(X) \to K^2(X)$ of the star relation to the pared graph (§2.7) of X is single-valued and vertex-bijective. But $x^* \sim y^*$ in $K^2(X)$ if and only if $x, y \in q$ for some $q \in K(X)$, if and only if $x \sim y$ in X, so ψ_X is an isomorphism. We have just proved Satz 2 of [14]: If X is clique-Helly, then $K^2(X) \cong P(X)$.

Proposition 5.4. *Let* CH *be the full subcategory of $h\mathcal{G}$ induced by the clique-Helly graphs. Then the functor K^2 is naturally isomorphic, over* CH, *to the identity functor* 1_{CH}.

Proof. Let $X \in \mathrm{CH}$. By [14, Satz 1] $K(X) \in \mathrm{CH}$, so $K^2 : \mathrm{CH} \to \mathrm{CH}$ is a functor. The paring section $s_X : P(X) \hookrightarrow X$ is a homotopy equivalence (4.13.3) and $*_X \circ s_X = \psi_X : P(X) \to K^2(X)$ is an isomorphism, so $*_X$ is a homotopy equivalence by the two out of three property. Then, using 5.3, the restriction of $*$ to CH is a natural isomorphism from 1_{CH} to K^2 over CH. □

5.3. **Clique behavior.** Let $K^0(X) = X$, $K^1(X) = K(X)$, $K^2(X) = K(K^1(X))$, etc. The *K-orbit* of X is $\mathcal{O}_X = \{K^n(X) \mid n \geq 0\}$. If $\mathcal{O}_X$ contains only a finite number of graphs up to isomorphy then $K^m(X) \cong K^n(X)$ for some $m > n \geq 0$ and, if n and $p = m-n$ are minimal, one says that X *converges* (in n steps) to the *p-periodic circuit* $\{K^n(X), K^{n+1}(X), \ldots, K^{m-1}(X)\}$ or just that X is *K-convergent*. In the remaining case X is said to be *K-divergent*.

As $X \cong Y$ implies $|X| = |Y|$, $X \in \mathcal{G}$ is K-convergent if and only if the sequence $(\,|K^n(X)|\,)$ is bounded or, equivalently, eventually periodic: indeed, if $M \in \mathbb{N}$, $|K^n(X)|$ can only visit $\{1, 2, \ldots, M\}$ so many times without $K^n(X)$ getting trapped in a circuit. Note that then X is K-divergent just when $(\,|K^n(X)|\,)$ is unbounded or, equivalently, $|K^n(X)| \to \infty$ as $n \to \infty$.

The *K-behavior* of X can be either K-convergence or K-divergence, so we say that two graphs have the *same K-behavior* whenever both are K-divergent or both are K-convergent. This is a major topic in the theory of clique graphs, e.g. [12, 14, 18, 28, 40, 41, 43, 47, 48]. By 2.13 our dismantlings equal those in [17], a paper whose results will be very useful here.

Theorem 5.5. [17, Thm.5] *If X dismantles to Z, X and Z have the same K-behavior.* □

Theorem 5.6. *If $X \simeq Y$, then X and Y have the same clique behavior.* □

Proof. By 4.16, X and Y have a common dismantling Z, so 5.5 can be applied. □

The notion of clique behavior carries over to $h\mathcal{G}$. The K-orbit of X in $h\mathcal{G}$ is the same class $\mathcal{O}_X \subseteq \mathrm{ob}(\mathcal{G}) = \mathrm{ob}(h\mathcal{G})$, but one says that X is *K-convergent in $h\mathcal{G}$* if $\mathcal{O}_X$ contains only a finite number of graphs up to isomorphy in $h\mathcal{G}$. Isomorphy in $h\mathcal{G}$ is homotopy equivalence in $\mathcal{G}$, so X is K-convergent in $h\mathcal{G}$ if and only if $\mathcal{O}_X$ only contains representatives from some finite set of homotopy types of graphs. Otherwise, of course, X is *K-divergent in $h\mathcal{G}$*. When we refer to K-behavior without specifying in which category, our usual $\mathcal{G}$ is meant. As for $h\mathcal{G}$, we can also speak of hK-behavior, hK-convergence, etc. with the obvious meanings.

The order of an object $X \in \mathcal{G}$ is $|X| = |\mathcal{G}(K_1, X)|$. The analog in $\mathcal{G}_r$ is $|\mathcal{G}_r(K_1, X)|$, which equals the number of connected components of X and hardly serves as "the order" of $X \in \mathcal{G}_r$. If $[X] \subseteq \mathrm{ob}\,\mathcal{G}$ is the homotopy class of X, then $P^\infty(X) \in [X]$ and $P^\infty(Y) \cong P^\infty(X)$ for each $Y \in [X]$ so, as $|P^\infty(Z)| \leq |Z|$ always, $|P^\infty(X)| \leq |Y|$ for all $Y \in [X]$ (see 2.21, 4.13.3, 4.16). Then $|P^\infty(X)|$, a construct in $\mathcal{G}$, is the minimum order of the graphs in $[X]$ and it can act as "the order" of the object X of $h\mathcal{G}$. We say that $X \in \mathcal{G}$ is *K-convergent up to homotopy* if the sequence $(\,|P^\infty K^n(X)|\,)$ is bounded, which is a condition on X in $\mathcal{G}$. It follows that X is K-convergent in $h\mathcal{G}$ if and only if X is K-convergent up to homotopy, and X is K-divergent in $h\mathcal{G}$ if and only if X is *K-divergent up to homotopy*, i.e. $|P^\infty K^n(X)| \to \infty$ as $n \to \infty$.

Note that hK-divergence implies K-divergence, as $|P^\infty K^n(X)| \leq |K^n(X)|$. Whether there is any K-divergent graph X which is hK-convergent is an open problem: we only have partial results for it (see §6 below). But by known results this same problem has an affirmative answer in the setting of the topological (even simple) homotopy type of the flag complex:

Let X be any of the graphs G^0 of [27] reviewed, together with $G^n \cong K^n(X)$, in §5.5 below. By [27], X is K-divergent. But, by [36], the topological homotopy type of $|\Delta(K^n(X))|$ is the same as that of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ for all n. Indeed, all the flag complexes $\Delta(K^n(X))$ have the same simple homotopy type as $\Delta(X)$. Hence, using the topology of the flag complexes, X is K-divergent and "homotopy-K-convergent." However, this example X does not settle our original problem: $K^n(X)$ is stiff for all $n \geq 0$ (see 5.13 below), so $P^\infty(K^n(X)) = K^n(X)$ for all $n \geq 0$ by 2.17. Under this last condition K-divergence equals hK-divergence, and thus the graph X is both K-divergent and hK-divergent, not K-divergent and hK-convergent.

5.4. **Unbounded morphisms.** The *norm* of $f \in \mathcal{G}(X,Y)$ is $\|f\| = \min_{\hat{f} \simeq f} |\mathrm{Im}(\hat{f})|$. This is a homotopy invariant: $f \simeq g$ implies $\|f\| = \|g\|$. Notice that $\|K(f)\|$ is well-defined since all possible choices of the map $K(f)$, being selections of the relation $K(f)$, are homotopic by 4.10. The map f will be said to be an *unbounded morphism* if the sequence of norms $(\|K^n(f)\|)$ is unbounded. In this subsection we will shorten "$r \leq s$ and $r \leq t$" to just "$r \leq s, t$."

Theorem 5.7. *Given maps f, g, h with $f \simeq g \circ h$, we have that $\|f\| \leq \|g\|, \|h\|$.*

Proof. Note first that $|\mathrm{Im}(f)| \leq |\mathrm{Im}(g)|, |\mathrm{Im}(h)|$ whenever $f = g \circ h$. Now assume $f \simeq g \circ h$. Take $\hat{g} \simeq g$ and $\hat{h} \simeq h$ with $|\mathrm{Im}(\hat{g})| = \|g\|$ and $|\mathrm{Im}(\hat{h})| = \|h\|$. By 4.1 we have $f \simeq g \circ h \simeq \hat{g} \circ \hat{h}$, and thus $\|f\| = \|\hat{g} \circ \hat{h}\| \leq |\mathrm{Im}(\hat{g} \circ \hat{h})| \leq |\mathrm{Im}(\hat{g})|, |\mathrm{Im}(\hat{h})|$. Therefore $\|f\| \leq \|g\|, \|h\|$. □

Theorem 5.8. *If f is unbounded and $f \simeq g \circ h$, then h and g are unbounded.*

Proof. By 5.1 K, and then each K^n, are functors on $h\mathcal{G}$. Hence $K^n(f) \simeq K^n(g) \circ K^n(h)$. But then $\|K^n(f)\| \leq \|K^n(g)\|, \|K^n(h)\|$ by 5.7, and therefore g and h are unbounded. □

Theorem 5.9. *If $f \simeq g \circ h \circ k$ and g and k are homotopy equivalences, then $\|f\| = \|h\|$.*

Proof. By 5.7, $\|f\| \leq \|g\|, \|h \circ k\|$ and also $\|h \circ k\| \leq \|h\|, \|k\|$, so we get that $\|f\| \leq \|h\|$. Take homotopy inverses $\bar{g}$ of g and $\bar{k}$ of k. By 4.1 we have $h = 1 \circ h \circ 1 \simeq \bar{g} \circ g \circ h \circ k \circ \bar{k} \simeq \bar{g} \circ f \circ \bar{k}$, so using again 5.7 as above we get now that $\|h\| \leq \|f\|$, and therefore $\|f\| = \|h\|$. □

Theorem 5.10. *If $f : X \to Y$ is unbounded then X and Y are K-divergent in $h\mathcal{G}$.*

Proof. For each $n \geq 0$ choose a graph map $K^n(f) : K^n(X) \to K^n(X)$, a paring retraction $r_n : K^n(Y) \to P^\infty K^n(Y)$, and a paring section $s_n : P^\infty K^n(X) \hookrightarrow K^n(X)$ (see §5.1 and 2.16).

$$\begin{array}{ccc} K^n(X) & \xrightarrow{K^n(f)} & K^n(Y) \\ {\scriptstyle s_n}\uparrow & & \downarrow{\scriptstyle r_n} \\ P^\infty K^n(X) & \xrightarrow{g_n} & P^\infty K^n(Y) \end{array}$$

Let $g_n = r_n \circ K^n(f) \circ s_n$. As r_n and s_n are homotopy equivalences by 4.13.3, we get by 5.9 that $\|g_n\| = \|K^n(f)\|$, and therefore the sequence $(\|g_n\|)$ is unbounded. Now we note that $\|g_n\| \leq |\mathrm{Im}(g_n)| \leq |P^\infty K^n(X)|, |P^\infty K^n(Y)|$ and from this we obtain that $|P^\infty K^n(X)| \to \infty$ and $|P^\infty K^n(Y)| \to \infty$ when $n \to \infty$, but these mean that X and Y are hK-divergent. □

Theorem 5.11. $\|1_X\| = |P^\infty(X)|$. *In particular,* $|\mathrm{Im}(f)| \geq |P^\infty(X)|$ *for any f with $f \simeq 1_X$.*

Proof. $P^\infty(X)$ is stiff by 2.17, so by 2.8 we have that $\|1_{P^\infty(X)}\| = |\mathrm{Im}(1_{P^\infty(X)})| = |P^\infty(X)|$. Now take (see 4.13.3) paring equivalences $r : X \to P^\infty(X)$ and $s : P^\infty(X) \hookrightarrow X$ such that $1_{P^\infty(X)} = r \circ s = r \circ 1_X \circ s$. Finally, by 5.9 we obtain $\|1_X\| = \|1_{P^\infty(X)}\| = |P^\infty(X)|$. □

5.5. **An application.** Using the graphs G^n $(n \geq 0)$ of [27], we shall show that certain maps $j_p^{n,m} : G^n \to G^m$ are unbounded. Then 5.10 will yield new K-divergence results in 5.16 below. We recall first the needed content from [27]. We start with a "rectangular" triangulation $\mathcal{T}$ of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ with $r \times s$ vertices, where $r \geq s \geq 4$. This is our graph $\mathcal{T} = G^0$, and the other graphs G^n will be isomorphic to its iterated clique graphs. The conditions on r and s ensure that the only triangles of the graph $\mathcal{T}$ (whose order is $r \cdot s$) are those seen in Figure 3 below, and in fact the open neighborhood of every vertex of $G^0 = \mathcal{T}$ induces an hexagon C_6.

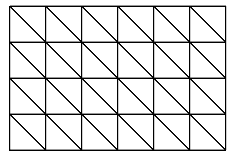
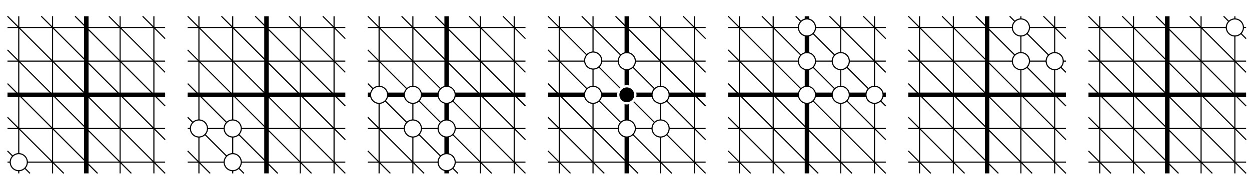


FIGURE 3. The graph $\mathcal{T}$ when $r = 6$ and $s = 4$. Identify opposite borders in order to form a torus. In this example $\mathcal{T}$ has 24 vertices, 72 edges and 48 triangles.

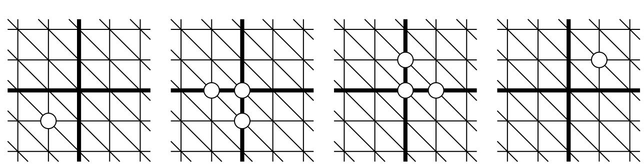
For $A, B \subseteq \mathbb{Z}_r \oplus \mathbb{Z}_s$ put $A + B = \{a + b \mid a \in A, b \in B\}$, $-A = \{-a \mid a \in A\}$ and, for $x \in \mathbb{Z}_r \oplus \mathbb{Z}_s$, we also define $x + A = \{x + a \mid a \in A\}$. Let us specify the following subsets of $\mathbb{Z}_r \oplus \mathbb{Z}_s$: $T = \{(0,0),(1,0),(0,1)\}$, $P = \{(1,1)\}$, $E_0 = T - T$, $E_1 = T + T$, $E_2 = P + T$, $E_3 = P + P$, $E_k = \varnothing$ for $k > 3$ and $E_k = -E_{-k}$ for all $k < 0$. Finally, for $n \geq 0$ and $r \geq s \geq 4$, the graph G^n has vertex set $\{x_i^n \mid x \in \mathbb{Z}_r \oplus \mathbb{Z}_s, 0 \leq i \leq n\}$ and an adjacency $x_i^n \sim y_j^n$ if and only if $y - x \in E_{j-i}$. If $A \subseteq \mathbb{Z}_r \oplus \mathbb{Z}_s$ and $i \in \mathbb{Z}$ put $A_i^n = \{a_i^n \mid a \in A\} \cap V(G^n)$. Note that $A_i^n = \varnothing$ when $i < 0$ or $i > n$. The *layers* of G^n are its subgraphs $\mathcal{T}_0^n, \mathcal{T}_1^n, \ldots, \mathcal{T}_n^n \trianglelefteq G^n$, where $\mathcal{T}_i^n = \langle(\mathbb{Z}_r \oplus \mathbb{Z}_s)_i^n\rangle$. Each layer is isomorphic to the above triangulation $\mathcal{T}$ of the torus, and G^n is the disjoint union of the $n+1$ layers plus some additional edges between layers described by the sets E_k. Call *homogeneous* any vertex set (or subgraph) contained in a layer. The closed neighborhood of each vertex $x_i^n \in G^n$ is the union of the non-empty homogeneous sets of the form $(x+E_k)_{i+k}^n$ with $-3 \leq k \leq 3$. Of these, at least $(x+E_0)_i^n$ is always present for any n, i and x. See Figure 4.



$(x + E_{-3})_{i-3}^n$ $(x + E_{-2})_{i-2}^n$ $(x + E_{-1})_{i-1}^n$ $(x + E_0)_i^n$ $(x + E_1)_{i+1}^n$ $(x + E_2)_{i+2}^n$ $(x + E_3)_{i+3}^n$

FIGURE 4. A vertex $x_i^n \in G^n$ (black dot) and its neighbors. Only the relevant parts of the layers are depicted. $N[x_i^n] \subseteq \mathcal{T}_{i-3}^n \cup \mathcal{T}_{i-2}^n \cup \cdots \cup \mathcal{T}_{i+3}^n$, but $\mathcal{T}_j^n = \varnothing$ if $j \notin \{0, 1, \ldots n\}$.

The cliques of G^n are $Q_{i,x}^n = (x-P)_{i-2}^n \cup (x-T)_{i-1}^n \cup (x+T)_i^n \cup (x+P)_{i+1}^n$ for $x \in \mathbb{Z}_r \oplus \mathbb{Z}_s$ and $0 \leq i \leq n+1$, see Figure 5. Each clique q of G^n contains one or two non-empty homogeneous triangles, and q is determined by any such triangle: If $q \supseteq (x+T)_i^n \neq \varnothing$ then $0 \leq i \leq n$ and $q = Q_{i,x}^n$, and if $q \supseteq (x-T)_{i-1}^n \neq \varnothing$ then $1 \leq i \leq n+1$ and again $q = Q_{i,x}^n$ (see 3.1-3.2 in [27]).



$(x - P)_{i-2}^n$ $(x - T)_{i-1}^n$ $(x + T)_i^n$ $(x + P)_{i+1}^n$

FIGURE 5. The clique $Q_{i,x}^n$ of G^n is contained in $\mathcal{T}_{i-2} \cup \mathcal{T}_{i-1} \cup \mathcal{T}_i \cup \mathcal{T}_{i+1}$.

It was shown in [27] that $\varphi_n : K(G^n) \to G^{n+1}$, given by $\varphi_n(Q_{i,x}^n) = x_i^{n+1}$, is an isomorphism.

Lemma 5.12. *If $N[v] \supseteq (x+E_0)_i^n$ for some $v \in G^n$, then $v = x_i^n$.*

Proof. Any $z_j^n \in G^n$ has, including itself, seven neighbors in its layer $\mathcal{T}_j^n$ and at most six in the others, so $v \in \mathcal{T}_i^n$ and in fact $v \in N[x_i^n] \cap \mathcal{T}_i^n = (x+E_0)_i^n$. If $v \neq x_i^n$, then $v \in N(x_i^n) \cap \mathcal{T}_i^n$. But then $\langle N(x_i^n) \cap \mathcal{T}_i^n \rangle$ is not an hexagon, but a cone with apex v: a contradiction. □

Corollary 5.13. *The graph G^n is stiff for any $n \geq 0$.* □

For $n \leq m$ and $0 \leq p \leq m-n$ define $j = j_p^{n,m} : G^n \to G^m$ by $j_p^{n,m}(x_i^n) = x_{i+p}^m$. Then $j : G^n \to G^m$ is an embedding. It identifies each layer $\mathcal{T}_i^n$ of G^n with the layer $\mathcal{T}_{i+p}^m$ of G^m, and its image is $j(G^n) = \left\langle \mathcal{T}_p^m \cup \mathcal{T}_{1+p}^m \cup \cdots \cup \mathcal{T}_{n+p}^m \right\rangle \trianglelefteq G^m$. When it is safe we will write just j instead of $j_p^{n,m}$.

We do not need it here, but let us just mention that the particular case $j_1^{n,n+2} : G^n \to G^{n+2}$ is in essence the star map $\ast : G^n \to K^2(G^n)$ of §5.2. Indeed, using the iso $\psi : K^2(G^n) \to G^{n+2}$ given by $\psi = \varphi_{n+1} \circ K(\varphi_n)$ one can show that $\ast \circ \psi = j_1^{n,n+2}$.

If $A = (x \pm T)_j^n$ is a homogeneous triangle of G^n, then $\hat{A} = j(T)$ is the homogeneous triangle $\hat{A} = j(T) = (x \pm T)_{j+p}^m$ of G^m. If $A \subseteq q \in K(G^n)$ and $\hat{q} = K(j)(q) \in K(G^m)$, then $\hat{A} \subseteq \hat{q}$. It follows that if $q = Q_{i,x}^n$ then $\hat{q} = Q_{i+p,x}^m$, so the left side diagram below commutes in $\mathcal{G}$, hence in $h\mathcal{G}$. Then the right side diagram below also commutes in $h\mathcal{G}$ for all $t \geq 1$ because $K^{t-1} : h\mathcal{G} \to h\mathcal{G}$ is a functor by 5.1.

$$
\begin{array}{ccc}
K(G^n) & \xrightarrow{K(j_p^{n,m})} & K(G^m) \\
\varphi_n \downarrow & & \uparrow \varphi_m^{-1} \\
G^{n+1} & \xrightarrow{j_p^{n+1,m+1}} & G^{m+1}
\end{array}
\qquad
\begin{array}{ccc}
K^t(G^n) & \xrightarrow{K^t(j_p^{n,m})} & K^t(G^m) \\
K^{t-1}(\varphi_n) \downarrow & & \uparrow K^{t-1}(\varphi_m^{-1}) \\
K^{t-1}(G^{n+1}) & \xrightarrow{K^{t-1}(j_p^{n+1,m+1})} & K^{t-1}(G^{m+1}).
\end{array}
$$

The isomorphisms φ_n and φ_m^{-1} are equivalences, so $K^{t-1}(\varphi_n)$ and $K^{t-1}(\varphi_m^{-1})$ are equivalences for all $t \geq 1$. Then the horizontal morphisms in each diagram have the same norm by 5.9. By induction on t, we get that $\|K^t(j_p^{n,m})\| = \|j_p^{n+t,m+t}\|$. Indeed, for $t=1$, see the diagram on the left. For $t > 1$, using the diagram on the right and the induction hypothesis we obtain that $\|K^t(j_p^{n,m})\| = \|K^{t-1}(j_p^{n+1,m+1})\| = \|j_p^{n+1+t-1,m+1+t-1}\| = \|j_p^{n+t,m+t}\|$.

Lemma 5.14. *For any $n \leq m$ and $0 \leq p \leq m-n$, the norm of $j = j_p^{n,m}$ is $\|j\| = (n+1) \cdot r \cdot s$.*

Proof. Since $j = j_p^{n,m} : G^n \to G^m$ is an embedding, $|\mathrm{Im}(j)| = |G^n| = (n+1) \cdot r \cdot s$. Hence it is enough to show that j is homotopic only to itself or, by 3.11, that $j \sim_e f \Rightarrow f = j$. We have $f(x) = j(x)$ away from some $x_i^n \in G^n$, and $f(x_i^n) \sim j(x_i^n)$, so $j(x_i^n) \in N[f(x_i^n)]$. By 2.2.5, $N[f(x_i^n)] \supseteq f(N[x_i^n]) = j(N(x_i^n)) \cup \{f(x_i^n)\}$, so $N[f(x_i^n)] \supseteq j(N(x_i^n))$ and we have that $N[f(x_i^n)] \supseteq j(N(x_i^n)) \cup \{j(x_i^n)\} = j(N[x_i^n])$. As $N[f(x_i^n)] \supseteq j((x+E_0)_i^n) = (x+E_0)_{i+p}^m$, we get that $f(x_i^n) = x_{i+p}^m = j(x_i^n)$ by 5.12, so $f = j$ and j is indeed homotopic only to j. □

Theorem 5.15. *If $n \leq m$ and $0 \leq p \leq m-n$, the embedding $j_p^{n,m} : G^n \hookrightarrow G^m$ is unbounded.*

Proof. Applying 5.14 for $n+t$ and $m+t$ instead of n and m, from what we already know we get that $\|K^t(j_p^{n,m})\| = \|j_p^{n+t,m+t}\| = (n+t) \cdot r \cdot s$, which is unbounded as t grows. □

Theorem 5.16. *Let $X \leqslant G^n$ contain some layer $\mathcal{T}_p^n \leqslant G^n$. Then X is K-divergent in $h\mathcal{G}$.*

Proof. Note that $j = j_p^{0,n} : G^0 \to G^n$ is unbounded by 5.15 and its image is $j(G^0) = \mathcal{T}_p^n \leqslant G^n$. Since j decomposes as $G^0 \to \mathcal{T}_p^n \hookrightarrow X \hookrightarrow G^n$ where the first map is $j^{|} : G^0 \to \mathcal{T}_p^n$, the inclusion map $X \hookrightarrow G^n$ is unbounded by 5.8. Therefore X is K-divergent in $h\mathcal{G}$ by 5.10. □

6. K-Divergence vs hK-Divergence

6.1. **Hash arrows and the operators K and KC.** We start with two known results:

Theorem 6.1. [47, Prop.3]. *If (r, s) is a retraction-section pair in $\mathcal{G}$, then any chosen selections of $K(r)$ and $K(s)$ form also a retraction-section pair $(K(r), K(s))$.* □

Theorem 6.2. [17, Thm.3]. *If $X \xrightarrow{\#} Y$, then $K(X) \xrightarrow{\#} K(Y)$.* □

The *graph of completes* of a graph X is the intersection graph $C(X)$ of all its complete subgraphs: $V(C(X)) = \{u \subseteq V(X) \mid u \text{ is complete in } X\}$ and $E(C(X)) = \{uv \mid u \cap v \neq \varnothing\}$. Note that $K(X) \trianglelefteq C(X)$. If $u, v \in C(X)$ and $u \supseteq v$, then $u \geqslant v$. As any complete is contained in some clique, $K(X)$ is a dominant subgraph of $C(X)$ and $C(X) \xrightarrow{\#} K(X)$ by 2.11. The *complete graph functor* $C : \mathcal{G} \to \mathcal{G}$ is given by $C(f)(u) = f(u)$ for $f \in \mathcal{G}(X, Y)$, $u \in C(X)$. Note that $C(f) : C(X) \to C(Y)$ is a map because $u \cap u' \neq \varnothing \Rightarrow f(u) \cap f(u') \neq \varnothing$. Also, $C(g \circ f)(u) = g(f(u)) = (C(g) \circ C(f))(u)$, so $C(g \circ f) = C(g) \circ C(f)$. Finally, since clearly $C(1_X) = 1_{C(X)}$, we get that $C : \mathcal{G} \to \mathcal{G}$ is indeed a functor. We will combine this functor C with the clique operator K to obtain the operator $KC = K \circ C$ of the graph category $\mathcal{G}$.

Lemma 6.3. *If $a, b \in A$, $q \in K(A)$, $a \geqslant b$, and $b \in q$, then $a \in q$. $(a \geqslant b \in q \in K(A) \Rightarrow a \in q)$.*

Proof. $b \in q$ implies $q \subseteq N[b] \subseteq N[a]$, so $q \cup \{a\}$ is complete and $a \in q$ as q is maximal. □

In general $X \xrightarrow{\#} Y$ does not imply $C(X) \xrightarrow{\#} C(Y)$, so 6.2 does not hold for C. Nevertheless:

Theorem 6.4. *If $X \xrightarrow{\#} Y$, then $KC(X) \xrightarrow{\#} KC(Y)$.*

Proof. We have a dominant retraction $r : X \to Y$, so for some map $s : Y \to X$ we have that $r \circ s = 1_Y$ and the idempotent $s \circ r : X \to X$ is a domination. We shall show the existence of $\hat{r} : KC(X) \to KC(Y)$ and $\hat{s} : KC(Y) \to KC(X)$ with the same corresponding properties. Since C is a functor and (r, s) is a retraction-section pair, so is $(C(r), C(s))$. Now we fix maps $\hat{r} = KC(r)$ and $\hat{s} = KC(s)$. By 6.1 we know that $(\hat{r}, \hat{s})$ is also a retraction-section pair.

We will prove that the idempotent $\hat{s} \circ \hat{r} : KC(X) \to KC(Y)$ is a domination, and therefore $\hat{r} : KC(X) \to KC(Y)$ will be a dominant retraction, thus ending the proof. Let $Q \in KC(X)$. We need $\hat{s}(\hat{r}(Q)) \geqslant Q$ so, if $Q' \sim Q$, we need that $Q' \sim \hat{s}(\hat{r}(Q))$. Take a complete $u \in C(X)$ with $u \in Q \cap Q'$, and a clique $q \in K(X)$ with $q \supseteq u$, so $q \geqslant u$ in $C(X)$.

Four uses of 6.3 will yield $q \in Q' \cap \hat{s}(\hat{r}(Q))$, so $Q' \sim \hat{s}(\hat{r}(Q))$. First of all, $q \geqslant u \in Q' \in KC(X)$ implies $q \in Q'$, so now we only need that $q \in \hat{s}(\hat{r}(Q))$. For any $x \in q$, $s(r(x)) \geqslant x \in q \in K(X)$ implies $s(r(x)) \in q$, so $s(r(q)) \subseteq q$ and then we have $q \geqslant s(r(q))$ in $C(X)$; on the other hand $q \geqslant u \in Q \in KC(X)$ implies $q \in Q$, and thus $s(r(q)) \in s(r(Q)) \subseteq s(\hat{r}(Q)) \subseteq \hat{s}(\hat{r}(Q))$. Then $q \geqslant s(r(q)) \in \hat{s}(\hat{r}(Q)) \in KC(X)$ implies $q \in \hat{s}(\hat{r}(Q))$, as we needed to show. □

Theorem 6.5. [17, Thm.4]. *If $X \xrightarrow{\#} Y$ then $KC(Y) \xrightarrow{\#} K^2(X)$.* □

Theorem 6.6. *If $X \xrightarrow{\#_t} Y$, then $(KC)^t(Y) \xrightarrow{\#_t} K^{2t}(X)$.*

Proof. The base case $t = 1$ is just 6.5. Assuming that $X \xrightarrow{\#_{t+1}} Y$, we have $X \xrightarrow{\#_1} Z \xrightarrow{\#_t} Y$ for some Z. Applying t times 6.4 to $Z \xrightarrow{\#_t} Y$ we get $KC(Z) \xrightarrow{\#_t} KC(Y)$ so, by the inductive hypothesis, $(KC)^{t+1}(Y) = (KC)^t KC(Y) \xrightarrow{\#_t} K^{2t} KC(Z)$. On the other hand, from $X \xrightarrow{\#_1} Z$ we obtain $KC(Z) \xrightarrow{\#_1} K^2(X)$ by the base case and applying 6.2 to this $2t$ times we get $K^{2t} KC(Z) \xrightarrow{\#_1} K^{2t}(K^2(X)) = K^{2(t+1)}(X)$. Therefore, $(KC)^{t+1}(Y) \xrightarrow{\#_{t+1}} K^{2(t+1)}(X)$. □

6.2. **h-self-cliqueness and pi$K$-divergence.** In $\mathcal{G}$ a graph A is *self-clique* if $K(A) \cong A$. This particular case of K-convergence appeared in [14] and has been much studied, see [32].

In [33], using the topology of the flag complex, A was called *homotopy K-invariant* if $|\Delta(A)|$ and $|\Delta(K(A))|$ have the same homotopy type, and A was called *homotopy K-permanent* if $|\Delta(A)| \simeq |\Delta(K^n(A))|$ for all $n \geq 0$. The distinction was necessary because there are homotopy K-invariant graphs which are not homotopy K-permanent, see Proposition 3.2 of [33].

For our (flagless) graph homotopy the situation is better. We call A *homotopy self-clique*, or *h-self-clique* if A is self-clique in $h\mathcal{G}$, i.e. if $K(A) \cong A$ in $h\mathcal{G}$. In other words, A is h-self-clique if $K(A) \simeq A$ in $\mathcal{G}$, i.e. if A is self-clique *up to homotopy.* This new *hK-invariance* is always "permanent," somehow like the derivability of complex functions as opposed to real ones:

Lemma 6.7. *If $A \simeq K(A)$, then $A \simeq K^n(A)$ for all $n \geq 0$.*

Proof. Just apply 5.2 iteratively: $A \simeq K(A) \simeq K^2(A) \simeq K^3(A) \simeq \cdots \simeq K^n(A) \simeq \cdots$ □

Another kind of K-convergence that has been considered in $\mathcal{G}$ is the following: A is *K-periodic* of *period p* if $K^p(A) \cong A$ and $p \geq 1$ is minimal. For period $p = 1$ this reduces to self-cliqueness. As recalled in §5.3, any K-convergent graph A is eventually K-periodic: $K^n(A)$ is K-periodic for some $n \geq 0$. Examples of K-periodic graphs of any period were given already in [14]. The adaptation to $h\mathcal{G}$ is immediate: A is *hK-periodic* of *period p* if $K^p(X) \cong X$ in $h\mathcal{G}$ (or, equivalently, $K^p(X) \simeq X$ in $\mathcal{G}$), where $p \geq 1$ is minimal.

Lemma 6.8 ([47], [26])**.** *$K(X \times Y) \cong K(X) \times K(Y)$ for all $X, Y \in \mathcal{G}$.* □

Lemma 6.9. *Let A be hK-periodic of period p and take the product Y of the K-orbit of A, i.e. $Y = A \times K(A) \times K^2(A) \times \ldots \times K^{p-1}(A)$. Then Y is hK-invariant.*

Proof. Let us calculate in $\mathcal{G}$. Since we have that $K^p(A) \simeq A$, then also $K^p(A) \times B \simeq A \times B$ for any graph B: Indeed, if $f : K^p(A) \to A$ and $g : A \to K^p(A)$ are homotopy inverse to each other, then so are $f \times 1_B : K^p(A) \times B \to A \times B$ and $g \times 1_B : A \times B \to K^p(A) \times B$. Let us now define $B = K(A) \times K^2(A) \times \ldots \times K^{p-1}(A)$, so $Y = A \times B$. Then, by 6.8 and symmetry of $\times$, $K(Y) \cong K(A) \times K(B) \cong K(A) \times K^2(A) \times \ldots \times K^{p-1}(A) \times K^p(A) = B \times K^p(A) \cong K^p(A) \times B$. But then $K(Y) \cong K^p(A) \times B \simeq A \times B = Y$ in $\mathcal{G}$, which means that $K(Y) \cong Y$ in $h\mathcal{G}$. □

The graph Y is pi*K-divergent* if $\mathrm{pi}(K^n(Y)) \to \infty$ as $n \to \infty$. Not only $|K^n(Y)| \to \infty$, but even the minimal length $l(n)$ of a chain of subgraphs $K^n(Y) \succcurlyeq X_1 \succcurlyeq X_2 \succcurlyeq \ldots \succcurlyeq X_{l(n)}$, each of them dominant in the previous one and ending in a stiff $X_{l(n)}$ tends to infinity with n.

Theorem 6.10. *Let Z be stiff, K-divergent, and hK-invariant. Then Z is* pi*K-divergent.*

Proof. Since Z is stiff, $P^\infty(Z) = Z$ by 2.17. Since Z is hK-invariant, by 6.7 $K^n(Z) \simeq Z$ for all n, so $P^\infty(K^n Z) \cong P^\infty(Z) = Z$ by 4.16 and then we have $K^n(Z) \xrightarrow{\#_{t(n)}} Z$ for each $n \geq 0$, where $t(n) = \mathrm{pi}(K^n(Z))$. By way of contradiction, assume $t(n) \not\to \infty$ as $n \to \infty$. Then there exists an infinite sequence $0 \leq n_0 < n_1 < n_2 < \ldots$ such that the set $\{t(n_i) \mid i \in \mathbb{N}\}$ is bounded. Put $m = \max\{t(n_i) \mid i \in \mathbb{N}\}$, so by 2.18 we have $K^{n_i}(Y) \xrightarrow{\#_m} Y$ for each $i \in \mathbb{N}$. By 6.6 we have $(KC)^m(Y) \xrightarrow{\#_m} K^{2m+n_i}(Y)$ for all $i \in \mathbb{N}$. Since $(KC)^m(Y)$ is finite it only has a finite number of dismantlings up to isomorphism, so we must have $K^{2m+n_i}(Y) \cong K^{2m+n_j}(Y)$ for some $i < j$, and hence Y is K-convergent in $\mathcal{G}$, a contradiction. □

Theorem 6.11. *There is a graph X which is K-divergent in $\mathcal{G}$ but K-convergent in $h\mathcal{G}$ if and only in there is a stiff graph Z which is hK-invariant and* pi*K-divergent.*

Proof. Let X be K-divergent and suppose that $K^n(X) \simeq K^m(X)$ for some $m > n \geq 0$, where $p = m - n$ is minimal. Let $A = K^n(X)$, which is hK-periodic of period p. Since all $K^t(X)$ are K-divergent, so are A and all $K^t(A)$. Let $Y = A \times K(A) \times K^2(A) \times \ldots \times K^{p-1}(A)$. Then Y is K-divergent by 6.8, and it is hK-invariant by 6.9. Now take $Z = P^\infty(Y)$, which is a stiff graph by 2.21. Since Y dismantles to Z, we have that $Z \simeq Y$ by 4.13. Then Z is K-divergent by 5.6, and $K(Z) \simeq K(Y)$ by 5.2. Since $K(Y) \simeq Y$ we arrive at $K(Z) \simeq Y \simeq Z$, so Z is hK-invariant. Now 6.10 ensures that Z is pi$K$-divergent. The converse is obvious. □

Problem 1. *Is there a K-divergent graph which is hK-convergent?*

By 6.11 there would be more than meets the eye to an affirmative answer to Problem 1.

Appendix A. Self-maps of finite sets

In this appendix X is a finite set and $f : X \to X$ a function, or a *self-map* of X. For ease of reference we record here some trivial statements about self-maps. The *digraph of f* with vertex set X and arc set $\{(x, f(x)) \mid x \in X\}$ helps to visualize what follows (see Figure 6).

As $X = f^0(X) \supseteq f(X) \supseteq f^2(X) \supseteq \cdots$ there must be a *minimal* $k \geq 0$ with $f^k(X) = f^{k+1}(X)$. We call $f^k(X)$ the *stable image* of f, and k the *stability index* of f, denoted by $s(f) = k$.

Denote the stable image of f by $X_0 = f^k(X)$. Note that $f|_{|} : X_0 \to X_0$ is bijective as $f(X_0) = X_0$. Define $f_b : X \to X$ by $f_b(x) = f(x)$ for $x \in X_0$ and $f_b(x) = x$ for $x \notin X_0$. Thus f_b is bijective, acts like f over X_0, and fixes each element of $X \smallsetminus X_0$. We call f_b *the bijective part* of f. Note that f is bijective $\Leftrightarrow k = 0 \Leftrightarrow f = f_b$. Note also that $s(f_b) = 0$ and that the digraph of f_b is obtained from that of f by replacing each arc in $X \smallsetminus X_0$ by a loop.

Define now $f_a : X \to X$ by $f_a(x) = f(x)$ if $x \notin X_0$ and $f_a(x) = x$ if $x \in X_0$. This f_a is acyclic (see below), acts like f over $X \smallsetminus X_0$, and fixes each element of X_0. We call f_a *the acyclic part* of f. Notice that $s(f_a) = s(f)$ and that the digraph of f_a is that of f with each arc in X_0 replaced by a loop. Clearly f is the product of its acyclic and bijective parts: $f = f_a \circ f_b$.

If $k > 0$, let $X_1 = f^{-1}(X_0) \smallsetminus X_0$ and, if $k > 1$, define recursively $X_j = f^{-1}(X_{j-1})$ for $j > 1$. Then $\{X_0, X_1, \ldots, X_k\}$ is a partition of X, and $\{X_1, X_2, \ldots, X_k\}$ is a partition of $X \smallsetminus X_0$. If $i \in \{1, \ldots, k\}$, note that $f(X_i) \subseteq X_{i-1}$ and define $e_i : X \to X$ by by $e_i(x) = f(x)$ for $x \in X_i$ and $e_i(x) = x$ for $x \notin X_i$. Thus e_i only moves the vertices of X_i and sends them into X_{i-1}, so e_i is idempotent ($e_i^2 = e_i$), acts like f over X_i, and fixes $X \smallsetminus X_i$ pointwise. Notice that the acyclic part of f is the product of the idempotents e_i: we have that $f_a = e_k \circ \cdots \circ e_2 \circ e_1$.

A function $f : X \to X$ is said to be *acyclic* if its only cycles are trivial: $m > 0$ and $f^m(x) = x$ imply $f(x) = x$. Any idempotent $e : X \to X$ is acyclic, since $m > 0$ and $e^m(x) = x$ imply $e(x) = e^m(x) = x$. Since $f_a = e_k \circ \cdots \circ e_2 \circ e_1$ and $e_i(X_i) \subseteq X_{i-1}$ for $i = 1, \ldots, k$, f_a is acyclic.

Both the bijective part f_b and the idempotents e_i can be factorized further. One knows that f_b is a product of cyclic permutations which are, in turn, products of transpositions $t_i = (a_i, b_i) : X \to X$. In fact, each t_i can be taken to be of the form $t_i = (a_i, f(a_i))$ using cycle decompositions of the type $(a, b, c, d) = (a, b)(b, c)(c, d)$. The cycles of f_b are precisely the cycles of f, so we have that f is acyclic $\Leftrightarrow f_b = 1_X \Leftrightarrow f = f_a$. It is easily seen that we also have that f is bijective $\Leftrightarrow k = 0 \Leftrightarrow f = f_b \Leftrightarrow f_a = 1_X$.

To decompose further the acyclic part f_a of f we factorize the idempotents e_i into pinches. The *pinch* of a to b is the function $p = [a,b] : X \to X$ given by $p(a) = b$ and $p(x) = x$ for all $x \in X \smallsetminus \{a\}$. In the trivial case $a = b$, $(a,b) = [a,b] = 1_X$. If $a \neq b$ the transposition $t = (a,b)$ is bijective but the pinch $p = [a,b]$ is not bijective, as $p(X) = X \smallsetminus \{a\}$. Each idempotent e_i will turn out to be be a product of pinches that commute with each other.

Lemma A.1. *The non-trivial and different pinches $p = [a,b]$ and $q = [x,y]$ commute with each other if and only if $b \neq x \neq a \neq y$. In other words, $p \circ q = q \circ p$ if and only if p and q are either* disjoint *(i.e. $|\{a,b,x,y\}| = 4$) or* convergent *(i.e. $|\{a,b,x,y\}| = 3$ and $b = y$).* □

Lemma A.2. *If $f : X \to X$ is idempotent, $f = \prod_{x \in X \smallsetminus f(X)} [x, f(x)]$ is a decomposition of f as a well-defined product of mutually commuting pinches. Conversely, any product $e = \prod e_i$ of mutually commuting idempotent self-maps $e_i : X \to X$ is idempotent.*

Proof. If $f \neq 1_X$, $s(f) = 1$, the partition of f is given by $X_0 = f(X)$, $X_1 = X \smallsetminus f(X)$, and f sends X_1 to X_0 and fixes X_0. The pinches $[x, f(x)]$ with $x \in X_1$ commute with each other by A.1 and then we clearly have $f = \prod_{x \in X_1} [x, f(x)]$. For the converse, by the commutativity hypothesis, $e^2 = (\prod e_i)^2 = \prod e_i^2 = \prod e_i = e$, and thus e is idempotent. □

Lemma A.3. *Let $f : X \to X$ be acyclic, $k = s(f)$, $Y = f(X)$ and $g = f|^{|}_{|} : Y \to Y$. Then:*

(1) $k \leq 1 \Leftrightarrow f$ is idempotent. Also $k = 0 \Leftrightarrow f = 1_X$.

Now assume that $k > 0$ for items (2) and (3):

(2) The partition of g is $\{Y_i = X_i \cap Y \mid 0 \leq i < k\}$. Thus g is acyclic with $s(g) = k - 1$.

(3) $1 \leq j \leq k \Rightarrow s(f^j) = \left\lceil \frac{k}{j} \right\rceil$. Therefore $s(f^k) = 1$ when $k \geq 1$. □

We close this Appendix with an example of a concrete self-map $f : X \to X$:

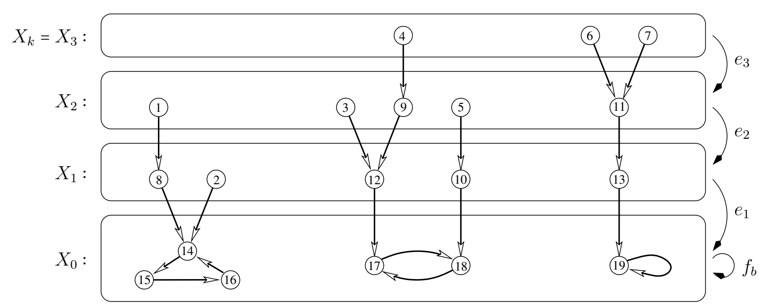


FIGURE 6. The digraph, the idempotents $e_i : X \to X$, and the partition $\{X_i\}_{i=0}^k$ of the set $X = \{1, 2, \ldots, 19\}$ for a self-map f of X with stability index $k = s(f) = 3$.

For the example in Figure 6 the decompositions are as follows: First, $f = f_a \circ f_b$ (acyclic and bijective parts). Then, $f_b = (14,15)(15,16)(17,18)$ (product of transpositions) and $f_a = e_3 \circ e_2 \circ e_1$ (product of idempotents). Finally, the idempotents are $e_3 = [4,9][6,11][7,11]$, $e_2 = [1,8][3,12][9,12][5,10][11,13]$, and $e_1 = [8,14][2,14][12,17][10,18][13,19]$ (products of mutually commuting pinches). □

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