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On cliques and bicliques

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Abstract

Basic definitions are given in the next paragraph. We studied second clique graphs of suspensions of graphs, $K^2(S(G))$, and characterize them, in terms of an auxiliary biclique operator B which transforms a graph G into its biclique graph $B(G)$. The characterization is then: $K^2(S(G)) \cong B(K(G))$. We found a characterization of the graphs, G, that maximize $|B(G)|$ for any given order $n = |G|$. This particular version of biclique operator is new in the literature. The main motivation to study $B(G)$ is an attempt to characterize the graphs G that maximize $|K^2(G)|$, thus mimicking a result of Moon and Moser [12] that characterizes the graphs maximizing $|K(G)|$.

The *clique graph* $K(G)$ of a a graph G is the intersection graph of the set of all (maximal) cliques of G (and $K^2(G) = K(K(G))$). The *suspension* $S(G)$ of a graph G is the graph obtained from G by adding two new vertices which are adjacent to all other vertices, but not to each other. Here, a *biclique* (X, Y) is an ordered pair of not necessarily disjoint subsets of vertices of G such that each $x \in X$ is adjacent or equal to every $y \in Y$ and such that (X, Y) is maximal under component-wise inclusion. Finally $B(G)$ is the graph whose vertices are the bicliques of G with adjacencies given by $(X, Y) \simeq (X', Y')$ if and only if $X \cap X' \neq \varnothing$ or $Y \cap Y' \neq \varnothing$.

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All our graphs are finite and simple. We usually identify induced subgraphs with their vertex set and in particular, we shall write $x \in G$, instead of $x \in V(G)$. As usual in clique graph theory, a *clique* of a graph is a maximal complete subgraph. The *clique graph* $K(G)$ of a graph G, is the intersection graph of the set of all cliques of G. Clearly, the clique operator K can be iterated: $K^{n+1}(G) = K(K^n(G))$. Clique graphs and iterated clique graphs have been studied extensively [11, 15, 19] and have found applications to the study of the Fixed Point Property of partially ordered sets [9] and to Loop Quantum Gravity [16–18].

When two vertices $x, y \in G$ are *adjacent-or-equal* we write $x \simeq y$. Let us denote by K_n, P_n, C_n and I_n the complete graph, the path graph, the cycle graph and the edgeless graph on n vertices (respectively). We also use standard notation for the *join* of two graphs, $H + G$ (also known as *Zykov sum*) and for isomorphic graphs, $G \cong H$.

Given G, let $\mathcal{B} = \{(X, Y) \in 2^G \times 2^G \mid x \simeq y, \text{ for every } x \in X \text{ and } y \in Y\}$. Define a partial order on $\mathcal{B}$ by $(X_1, Y_1) \preccurlyeq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. A *biclique* (X, Y) of G is a maximal element of $\mathcal{B}$ under $\preccurlyeq$. The *biclique graph* $B(G)$ of G is the graph whose vertices are the bicliques of G and two vertices $(X_1, Y_1), (X_2, Y_2) \in B(G)$ are adjacent if and only if $X_1 \cap X_2 \neq \varnothing$ or $Y_1 \cap Y_2 \neq \varnothing$. Observe that $|B(K_n)| = 1$, $|B(I_n)| = n + 2$ (for $n \geq 2$), $|B(P_3)| = 4$, $|B(P_n)| = 3n - 3$ (for $n \geq 4$), $|B(C_3)| = 1$, $|B(C_4)| = 16$ and $|B(C_n)| = 3n + 2$ (for $n \geq 5$). Several variations of a biclique operator have been studied in the literature by Prisner [15], Zelinka [20], Figueroa and Llano [2], and by Groshaus et al [4–8]. The variation studied here is new as far as we know.

Our main motivation comes from Moon and Moser [12] who characterized the graphs G that maximize the number of cliques $|K(G)|$: for any given order $n = |G| \geq 2$, $G = H + I_3 + I_3 + \cdots + I_3$ where $H \in \{I_2, I_3, I_2 + I_2, I_4\}$ is taken such that $|H| = |G| - 3\lfloor \frac{n-2}{3} \rfloor$. One would expect that these Moon-Moser graphs grow as fast as possible under iterated clique operators (i.e. that those G maximize $|K^n(G)|$ for all n), but experiments with small graphs show that that is not the case. The experiments (we used GAP and YAGS [1, 3]) show that, for $n \in \{2, 3, \cdots, 9\}$, $|K^2(G)|$ is maximized by $G = H + I_2 + I_2 + \cdots + I_2$, where $H \in \{I_2, I_3\}$ (although when $n \in \{4, 5\}$, there are a few other graphs that also attain the maximum $|K^2(G)|$). The experiments also suggest that $|K^2(G)|$ is bigger when G is a *suspension* than when it is not. Here, we say that G is the suspension of G_0 when $G \cong S(G_0) := G_0 + I_2$. In this work we start the study of graphs maximizing $|K^2(G)|$, by studying first the case when $G = S(G_0)$.

A few extra standard notations used here are: $N_G(x), N(x), N_G[x], N[x]$ for open and closed neighborhoods of the vertex x (in the graph G); $G-x$ for vertex removal; $\overline{G}$ for the complement graph and $G \times H$ for tensor product of graphs. We define the d-dimensional octahedral graph by $O_d = I_2+I_2+\cdots+I_2$ (d times) and the circle product by $G \circ H = \overline{\overline{G} \times \overline{H}}$. Note that in $G \circ H$, $(g_1,h_1) \simeq (g_2,h_2)$ if and only if $g_1 \simeq g_2$ in G or $h_1 \simeq h_2$ in H.

The following theorem characterizes $K^2(S(G))$ in terms of the biclique operator B.

Theorem 1 $K^2(S(G)) \cong B(K(G))$.

Proof. From the definition of $S(G)$, there are vertices $x,y \in S(G)\backslash G$, $x \not\simeq y$, such that $N(x) = N(y) = G$. Note that if $p \in K(S(G))$, then $p\backslash\{x\} \in K(G)$ or $p\backslash\{y\} \in K(G)$. Similarly, if $q \in K(G)$ then $q \cup \{x\}$, $q \cup \{y\} \in K(S(G))$. Define $\tau : B(K(G)) \to K^2(S(G))$ by

$$\tau((X,Y)) = \left(\bigcup_{q\in X}\{q \cup \{x\}\}\right) \cup \left(\bigcup_{q\in Y}\{q \cup \{y\}\}\right).$$

We claim that τ is an isomorphism of graphs. Let $P = (\bigcup_{i=1}^{r}\{q_i \cup \{x\}\}) \cup (\bigcup_{i=1}^{s}\{q'_i \cup \{y\}\})$ be a clique of $K(S(G))$, then $W = (\cup_{i=1}^{r} q_i, \cup_{i=1}^{s} q'_i)$ is a biclique of $K(G)$, it follows that $\tau(W) = P$, thus τ is surjective.

Let $W_1 = (X_1,Y_1)$ and $W_2 = (X_2,Y_2)$ be bicliques of $K(G)$, if $\tau(W_1) = \tau(W_2)$ then for $A_1 = \bigcup_{q\in X_1}\{q \cup \{x\}\}$ and $A_2 = \bigcup_{q\in X_2}\{q \cup \{x\}\}$ we have that $A_1 = A_2$, hence $X_1 = X_2$. A similar argument shows that $Y_1 = Y_2$, consequently $W_1 = W_2$ and τ is injective. Finally, the bicliques W_1 and W_2 are adjacent in $B(K(G))$ if and only if there is a clique $q \in K(G)$ such that $q \in (X_1 \cap X_2) \cup (Y_1 \cap Y_2)$, since the last statement is true if and only if $q \cup \{x\} \in \tau(W_1) \cap \tau(W_2)$ or $q \cup \{y\} \in \tau(W_1) \cap \tau(W_2)$ (i.e. when $\tau(W_1)$ and $\tau(W_2)$ are adjacent in $K^2(S(G))$), we conclude that τ preserves adjacency. □

Note that $B(G) \cong S(H)$ for some H if and only if $(\varnothing, G)$ and $(G, \varnothing)$ are bicliques of G and that this happens exactly when G is not a *cone* (i.e. G has no universal vertices). It follows from Theorem 1 that $K^2(S(G))$ is again a suspension exactly when G is not K-cone (i.e. when $K(G)$ is not a cone).

It is well known that the clique graph of a Zykov sum is a circle product, $K(G+H) \cong K(G) \circ K(H)$ [10,13,14]. Surprisingly, the biclique operator also shares this property:

Theorem 2 *For any graphs G and H, we have: $B(G+H) \cong B(G) \circ B(H)$.*

Proof. Define $\phi : B(G+H) \to B(G) \circ B(H)$ by $\phi((G_1 \cup H_1, G_2 \cup H_2)) =$

$((G_1, G_2), (H_1, H_2))$, where G_1 and G_2 are subsets of $V(G)$ and H_1 and H_2 are subsets of $V(H)$. We proceed to show that ϕ is an isomorphism of graphs.

If $T = ((G_1, G_2), (H_1, H_2)) \in B(G) \circ B(H)$ then $W = (G_1 \cup H_1, G_2 \cup H_2)$ is a biclique of $G + H$, hence $\phi(W) = T$ and ϕ is surjective. It is straightforward to verify that ϕ is injective. Now, the bicliques $W_1 = (G_1 \cup H_1, G_1' \cup H_1')$ and $W_2 = (G_2 \cup H_2, G_2' \cup H_2')$ of $G + H$ are adjacent-or-equal in $B(G + H)$ if and only if $(G_1 \cup H_1) \cap (G_2 \cup H_2) \neq \varnothing$ or $(G_1' \cup H_1') \cap (G_2' \cup H_2') \neq \varnothing$, but this last statement is true if and only if $(G_1, G_1') \simeq (G_2, G_2')$ in $B(G)$ or $(H_1, H_1') \simeq (H_2, H_2')$ in $B(H)$, therefore $\phi(W_1) \simeq \phi(W_2)$ in $B(G) \circ B(H)$ (i.e. ϕ preserves adjacency). $\square$

Given a graph G and a subset $X \subseteq G$, define $N[X] = \bigcap_{x \in X} N_G[x]$ (with $N[\varnothing] = V(G)$), and define $\beta : 2^G \to B(G)$ by $\beta(X) = (N[N[X]], N[X])$. Observe that β is surjective, and in particular, $|B(G)| \leq 2^{|G|}$. The equality holds exactly when G is an octahedral graph:

Theorem 3 *The following statements are equivalent:*

1. *β is injective.*
2. *$N[X] \neq N[X']$ for all $X, X' \subseteq G$ with $X \neq X'$.*
3. *$N[G - x] \neq N[G]$ for all $x \in G$.*
4. *For all $x \in G$, there is some $y \in G$ such that $x \not\simeq y$ and $y \simeq z$ for all $z \in G - x$.*
5. *$n = |G|$ is even and $G \cong O_d$ for $d = \frac{n}{2}$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straight forward. For (4) $\Rightarrow$ (5), observe that x is the only vertex of G satisfying $y \not\simeq x$ and hence we must have $x \simeq z$ for all $z \in G - y$. It follows that $G \cong I_2 + I_2 + \cdots + I_2$ (d times). Finally, for (5) $\Rightarrow$ (1), consider two different $X_1, X_2 \subseteq O_d$. Assume, without loss of generality, that $X_1 \setminus X_2 \neq \varnothing$, then there is a vertex $z \in X_1 \setminus X_2$ and a vertex $w \in O_d$ with $w \not\simeq z$. Note that $N[w] = G - z$, hence $w \in N[X_2]$ and $w \notin N[X_1]$. Therefore $N[X_1] \neq N[X_2]$. $\square$

By the previous theorem, in the even order case, $|G| = n = 2d$, the maximum value of $|B(G)|$ is 2^n and it is achieved exactly when $G \cong O_d$. In the odd order case, a good lower bound for the maximum value of $|B(G)|$, comes from the graph $G = I_3 + O_d$:

Lemma 4 *Let $n = 2d + 3 = |I_3 + O_d|$, then $|B(I_3 + O_d)| = \frac{5}{8} \cdot 2^n$*

Proof. By Theorem 2, $B(I_3 + O_d) \cong B(I_3) \circ B(O_d)$, hence $|B(I_3 + O_d)| = |B(I_3) \circ B(O_d)| = |B(I_3)| \cdot |B(O_d)| = 5 \cdot 2^{2d} = \frac{5}{8} \cdot 2^n$. $\square$

A full characterization of graphs G maximizing $|B(G)|$ requires more work and lies beyond the scope of this extended abstract, but it can be proved that the graphs maximizing $|B(G)|$ are precisely those considered so far:

Theorem 5 *Let G be a graph of order $n > 1$, maximizing $|B(G)|$. Then, if $n = 2d$, we have that $G \cong O_d$; otherwise, $n = 2d + 3$ and $G \cong I_3 + O_d$.*

It is known that the clique graph of an octahedral graph is again an octahedral graph (indeed: $K(O_d) \cong O_{2^{d-1}}$, see [13]) and we proved here that octahedral graphs maximize $|B(G)|$. Since $K^2(S(G)) \cong B(K(G))$, it follows that $K^2(O_d) = K^2(S(O_{d-1})) \cong B(K(O_{d-1})) \cong B(O_{2^{d-2}})$. This results, together with the experimental evidence mentioned before, suggest that the graphs maximizing $|K^2(G)|$ are the same as those maximizing $|B(G)|$, which are described in Theorem 5 (save for the few exceptions mentioned earlier when $n \in \{4, 5\}$). Certainly, if G has even order and maximizes $|K^2(G)|$, we have the following bound:

$$|K^2(G)| \geq |K^2(O_d)| = |B(O_{2^{d-2}})| = 2^{2^{d-1}} = \sqrt{2}^{\sqrt{2}^n}.$$

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