# ON THE CLIQUE BEHAVIOR OF GRAPHS WITH SMALL CONSTANT LINK

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Abstract. Given a finite simple graph $G$, we define $K(G)$ as the intersection graph of the (maximal) cliques of $G$, and inductively we define $K^n(G)$ as $G$ if $n = 0$, and as $K(K^{n-1}(G))$ if $n > 0$. We say that a graph $G$ is clique divergent if the sequence of orders $\{|K^n(G)|\}$ is unbounded, and clique convergent otherwise. Two graphs $G_1$, $G_2$ have the same clique behavior if both are clique divergent or both are clique convergent.

Given a graph $L$, if there is a finite graph $G$ such that the neighbors of any vertex of $G$ induce a graph isomorphic to $L$, then we say that $L$ is a link graph and that $G$ is a locally $L$ graph. In this paper we show that if $L$ has at most six vertices and $G_1$, $G_2$ are finite locally $L$ graphs, then $G_1$ and $G_2$ have the same clique behavior. Moreover with only one possible exception, where the clique behavior is still unknown, we show that for $|L| \leq 6$, any locally $L$ graph $G$ is clique divergent if and only if $L$ contains some induced $n$-cycle for some $n = 4, 5$ or $6$.

A broad spectrum of techniques for deciding $K$-behavior of graphs is demonstrated in this work.

## 1. Introduction

Let $\mathcal{G}$ be the class of all graphs, $\mathcal{X} \subseteq \mathcal{G}$ some graph class and $\Phi : \mathcal{G} \to \mathcal{G}$ some graph operator. Then the operator can be iterated on graphs ($\Phi^0(G) := G$; $\Phi^{n+1}(G) := \Phi(\Phi^n(G))$) and on classes of graphs ($\Phi(\mathcal{X}) := \{\Phi(G) \mid G \in \mathcal{X}\}$). This is the general setting in which the topic of *graph dynamics* delves [45]. Several natural questions arise and have been investigated here, including whether we can classify the graphs in $\Phi(\mathcal{X})$, $\Phi^2(\mathcal{X})$, ... etc. [1,2,12,13,17,22,37]; which graphs $G \in \mathcal{X}$ converge under $\Phi$ (i.e. $\Phi^n(G) \cong \Phi^m(G)$ for some $n < m$) or diverge under $\Phi$ [14,18,21,23,34,35,53–57]; or which graphs $G \in \mathcal{X}$ satisfy some graph equation $\Phi(G) \cong \Phi'(G)$ for some graph operator $\Phi'$ [3,5–7,10,11,38,49,50]. In this context, one of the most intensely studied graph operators is the *clique graph operator* $K$ (defined below) [1,3,5–7,10–14,16,19,26–35,40–43,49,51]. Applications of the theory of the clique graph operator include the fixed point property for posets [26] and Loop Quantum Gravity [46–48]. In this paper we study convergence and divergence under the clique operator $K$ for the class of *graphs with constant link* (defined below) with maximum degree at most 6.

We refer the reader to [25] for the terminology on graph theory not explicitly defined here. We identify vertex sets with their induced subgraphs. In particular we often write $v \in G$ instead of $v \in V(G)$. We denote by $|G|$ the cardinality of the set of vertices of the graph $G$.

As usual in clique graph theory, a *complete* of $G$ is a complete subgraph of $G$, whereas a *clique* of $G$ is a maximal complete of $G$.

Partially supported by SEP-CONACyT, grant 183210.

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*Key words and phrases.* clique graph, clique behavior, link graph, graph dynamics.

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The *clique graph* $K(G)$ is the intersection graph of the cliques of $G$. The *iterated clique graphs* $K^n(G)$ are defined recursively by $K^0(G) = G$ and $K^n(G) = K(K^{n-1}(G))$. If the sequence $\{|K^n(G)|\}$ is bounded (equivalently, $K^m(G) \cong K^n(G)$ for some $m > n$), we say that $G$ is *$K$-convergent*. On the other hand, $G$ is called *$K$-divergent* if the sequence $\{|K^n(G)|\}$ is unbounded. The *$K$-behavior* of $G$ can be either $K$-convergent or $K$-divergent. This is a major topic in the theory of clique graphs, and many papers have appeared providing techniques for determining the $K$-behavior (for example [14, 16, 19, 28, 30, 33–35, 40, 41]) however, it is not known whether the $K$-behavior of a graph is algorithmically computable or not [30, 36]. In this paper we write $G \approx H$ when $G$ and $H$ have the same $K$-behavior and we write $G \preccurlyeq H$ if either $G \approx H$ or $G$ is $K$-convergent and $H$ is $K$-divergent.

The *link* or *open neighborhood* $N_G(x)$ of the vertex $x$ in the graph $G$ is (the subgraph induced by) the set $\{ y \in V(G) \mid y \sim x \}$. If $L$ is a graph, we say that $G$ *is locally* $L$ if $N_G(x) \cong L$ for all $x \in G$. In this case, it is also said that $G$ is an *extension* of $L$, and that $G$ is a *graph with constant link* $L$. Not every graph $L$ has an extension, and following [4] we call such graphs *link graphs.* There is no known algorithm for deciding whether a given graph $L$ has an extension [8, 39], but several closely related variations of the problem (for example: allowing two different links $\{L_1, L_2\}$ for the sought extension or allowing the extension to be infinite) have been shown to be algorithmically undecidable [9, 39]. J. Hall managed to classify all link graphs with at most six vertices in 1985 [24] (see Figure 16), but even today, the complete list of all link graphs with at most seven vertices remains unknown. We shall use J. Hall's classification heavily in this work.

Note that even if the set of link graphs on at most six vertices is finite, the class of their extensions, i.e. the class of graphs with constant link and degree at most six, is not. This class is our central concern here. Our main goal in this paper is to prove Theorem 1.1 and Theorem 1.2:

**Theorem 1.1.** *If $|L| \leq 6$, then any two locally $L$ graphs $G_1$ and $G_2$ have the same $K$-behavior.*

Moreover, we can even characterize the $K$-behavior of all the graphs in our class, except for *one* case where the $K$-behavior of the only extension of $\mathbb{M}$ is still unknown (here, $\mathbb{M}$ is the graph $(6, 7, 6)$ in Figure 16, i.e. a pentagon glued to a triangle by an edge).

**Theorem 1.2.** *A graph $G$ with constant link $L \neq \mathbb{M}$ and $|L| \leq 6$ is $K$-divergent exactly when $L$ contains some induced $C_4, C_5$ or $C_6$.*

Theorem 1.1 follows immediately from Theorem 1.2 which will be proved throughout Sections 4 and 5. In Section 6, Theorem 6.1 shows that the requirement $|L| \leq 6$ in both theorems is necessary (but perhaps not sharp, see Problem 2), by showing two extensions of $C_5 \cup P_4$ with different $K$-behavior.

Besides the intrinsic interest of the results, this extensive exploration features a broad spectrum of techniques for deciding $K$-behavior of graphs, which could be an attractive route into the already ample literature on the topic for the reader who is interested in clique graph theory, but who is not an expert yet. Furthermore, this kind of endeavours serve to test the strength and maturity of a theory, to find its weaknesses, and to spot the places where the theory does not yet work in order to direct future work. In this case, it also served to clean up the theory a bit and thus to distill simple, easy-to-use criteria for deciding $K$-behavior that, nevertheless, are very frequently useful in practice, like those in Proposition 2.11 and Section 3.

## 2. Preliminaries

We shall start by studying the vertices of $K^2(G)$. These are the cliques of $K(G)$ and also the cliques of cliques of $G$. Cliques of cliques come in two types: stars and neckties.

If $x \in G$, the *star* of $x$ is the set $x^* = \{ q \in K(G) \mid x \in q \}$. In general $x^*$ is a complete of $K(G)$ but not always a clique of cliques. If $x^* \in K^2(G)$, we say that $x$ is a *normal vertex*. As noted in [29], if $x, y$ are normal vertices, then $x^* \sim y^*$ in $K^2(G)$ if and only if $x \sim y$ in $G$. Any clique of cliques which is not a star is called a *necktie*. Note that a clique of cliques $Q$ is a necktie if, and only if, $\cap Q = \varnothing$.

A graph $G$ is a *cone* if there is a vertex (an *apex*) which is adjacent to every other vertex of $G$. A graph $G$ is *clique-cone* if $K(G)$ is a cone.

**Proposition 2.1.** *If $G$ is locally $L$, and $L$ is not clique-cone, then all vertices of $G$ are normal.*

**Proof:** Suppose $G$ is a locally $L$ graph and $x \in G$ is not normal. Then there is $q \in K(G)$ that intersects all elements of $x^*$ but with $x \notin q$. If $q' = q \cap N_G(x)$, then $q'$ intersects every clique of $N_G(x)$ since these cliques are exactly the cliques in $x^*$ with the vertex $x$ removed. Now, extend $q'$ to a clique $q''$ of $N_G(x)$. Then $q''$ also intersects all other cliques of $N_G(x) \cong L$. Hence $L$ is clique-cone. □

A triangle $T$ in $G$ is called an *internal triangle* if it is a clique and each of its edges is contained in at least two cliques. Given an internal triangle $T$, the set $\{ q \in K(G) \mid |q \cap T| \geq 2 \}$ is a complete of $K(G)$, and when it is a clique of $K(G)$ it is called *the necktie of* $T$, denoted by $Q_T$. We will also say that $T$ is the *center* of the necktie and that any other clique in $Q_T$ is an *ear* of the necktie. Note that $Q_T$ is indeed a necktie.

Following [43], for $Q \in K^2(G)$ we define the *basement* $B(Q)$ of $Q$ as:

$$B(Q) = \bigcup_{q \in Q} q \subseteq V(G).$$

It is immediate that $x^* \sim Q$ in $K^2(G)$ if and only if $x \in B(Q)$.

Following [4], if the vertices $v$, $w$ have $n$ common neighbors in a graph $G$, we will say that the edge $vw$ is *marked* $n$. For example, if $G$ is locally $L$ and $w \in G$ has degree $n$ in $N(v) \cong L$, then the edge $vw$ is marked $n$.

**Theorem 2.2.** (Theorem 3 from [4]) *Let $G$ be a graph with constant link $L$ and $v \in V(G)$. Then for each $s \in N_G(v)$, the marks of the edges of $N(v)$ incident to $s$ must be the same (including multiplicities) as the degrees of the neighbors of some $t \in L$ with $\deg_L(t) = \deg_{N(v)}(s)$.*

This fundamental theorem of Blass, Harary and Miller will be used to show that the marks in figures 6, 11, 12 and 14 are as indicated.

We devote the rest of this section to compile the needed previously known results on clique behavior.

A graph is *Helly* if any collection of pairwise intersecting cliques of $G$ has non-empty total intersection. It is known since Escalante's seminal paper [16] that Helly graphs are $K$-convergent, as shown in the next statement which follows immediately from his Sätze 1 and 2.

**Theorem 2.3.** [16] *If $G$ is Helly, then so is $K(G)$, and $K^2(G)$ is isomorphic to an induced subgraph of $G$. In particular, any Helly graph is $K$-convergent.*

Hellyness can be determined in polynomial time thanks to the characterization of Dragan and Szwarcfiter in terms of extended triangles: If $T = \{x, y, z\}$ is a triangle in $G$, its *extended triangle* $\hat{T}$ is the induced subgraph on the vertices that are neighbors to at least two of the vertices in $T$.

**Theorem 2.4.** [15, 52] *A graph $G$ is Helly if, and only if, any extended triangle in $G$ is a cone.*

The next theorem follows immediately from Prisner's theorem characterizing hereditary Helly graphs in terms of the Hajós family of graphs. The use of the Hajós diagram comes from [32, p. 1159].

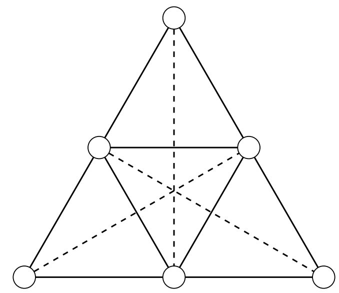

FIGURE 1. The Hajós diagram.

**Theorem 2.5.** ( [44, Theorem 2.1]) *Let $G$ be a graph such that for any subgraph as the solid line part of Figure 1, $G$ contains at least one of the dotted edges. Then any induced subgraph of $G$ is Helly and, in particular, $G$ is Helly.*

The *girth* of a graph is the length of its shortest cycle. The *local girth* of a graph $G$ is the minimum of the girths of $N_G(x)$, where $x$ varies among the vertices of $G$.

**Theorem 2.6.** (Theorem 8 from [29]) *If the graph $G$ has local girth at least 7, then $K(G)$ is Helly.*

The $n$-th octahedron $O_n$ is defined as $\overline{nK_2}$ (the complement of the disjoint union of $n$ copies of $K_2$).

**Theorem 2.7.** [40] *If $n \geq 1$, $K(O_n) = O_{2^{n-1}}$. In particular, $O_n$ is $K$-divergent for $n \geq 3$.*

If $G$, $H$ are graphs, then a *graph morphism* $f\colon G \to H$ is a map $f$ from the vertex set of $G$ to the vertex set of $H$, such that if $x$ is adjacent to $y$ in $G$, then either $f(x)$ is adjacent to $f(y)$ in $H$ or $f(x)$ is equal to $f(y)$. If $H$ is a subgraph of $G$, then the morphism $f\colon G \to H$ is a *retraction* if $f(h) = h$ for all $h \in H$. The following is NEUMANN-LARA's celebrated Retraction Theorem.

**Theorem 2.8.** [40] *If $f\colon G \to H$ is a retraction, and $H$ is $K$-divergent, then $G$ is $K$-divergent.*

An edge $uv$ is a *local bridge* if it is a bridge of the subgraph induced by $N[u] \cup N[v]$. As usual in the literature we denote by $G/uv$ and $G\backslash uv$ the operations of edge contraction and edge removal. The next two results follow trivially from theorems 6.1 and 6.2 in [19], since that paper uses a finer notion of clique behavior.

**Theorem 2.9.** [19] *Let $G$ be a graph and let $uv \in E(G)$ be a local bridge. Then $G \approx G/uv$.*

**Theorem 2.10.** [19] *Let $G$ be a graph and let $uv \in E(G)$ be in no triangle. Then $G\backslash uv \preccurlyeq G$.*

The previous two results will be used here in the following manner. Assume that $w \in G$ is a *local cutpoint*, which means that the removal of $w$ disconnects its closed neighborhood $N[w]$ (see Figure 2). Then $N(w)$ is the disjoint union of two (not necessarily connected) graphs $H$ and $J$. Now let us split the vertex $w$ into an edge $uv$ such that $N(u) = H \cup \{v\}$ and $N(v) = J \cup \{u\}$ to obtain the graph $G'$. Then the edge $uv$ is necessarily a local bridge of $G'$ and $G \approx G'$ by Theorem 2.9. If we now obtain $G''$ by removing the edge $uv$ from $G'$ we get $G' \succcurlyeq G''$ by Theorem 2.10. Hence we have:

**Proposition 2.11.** *Let us cut $G$ through a local cutpoint to obtain $G''$. Then we have $G \succcurlyeq G''$.*

We shall use the standard notation $C_n(a_1, a_2, \ldots, a_r)$ for the circulant on $n$ vertices and set of jumps $\{a_1, a_2, \ldots, a_r\}$.

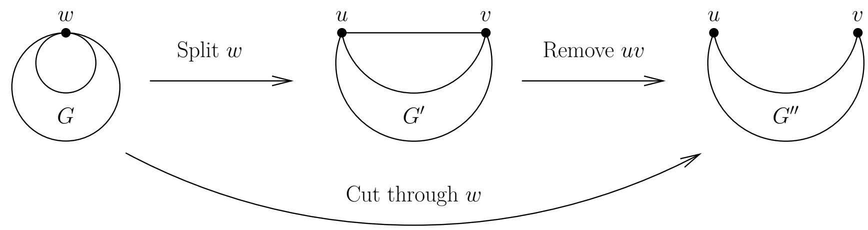


FIGURE 2. Cutting through a local cutpoint.

## 3. General propositions

**Link graphs where all edges have a dominating extreme**

Given two vertices, $x, y$, we say that $x$ *dominates* $y$ if $N[y] \subseteq N[x]$.

**Proposition 3.1.** *Let $L$ be a graph such that in any edge of $L$ one of the vertices dominates the other. Then any locally $L$ graph $G$ is Helly.*

**Proof:** By Theorem 2.4 it will suffice to prove that any extended triangle $\hat{T}$ of $G$ is a cone. Let $T = \{x, y, z\}$. Without loss of generality, $y$ dominates $z$ in $N(x) \cong L$, so $\hat{T}$ is a cone with apex $y$. □

**Separators**

We consider now the case where $G$ is locally $L$, and the link graph $L$ is a disjoint union $L = H \cup J$. We denote by $A(G)$ the set of arrows of $G$, that is, $A(G)$ contains two arrows in opposite directions for each edge of $G$.

**Definition 3.2.** We say that a map $\phi\colon A(G) \to \{H, J\}$ is a *separator* if for every $x \in G$ we have that $G[\{z \mid \phi(xz) = H\}] \cong H$, $G[\{z \mid \phi(xz) = J\}] \cong J$. In this case, we will use the notation $H_x = \{z \mid \phi(xz) = H\}$, $J_x = \{z \mid \phi(xz) = J\}$. We say that the separator $\phi$ is *symmetric* if $\phi(xy) = \phi(yx)$ for any arrow $xy \in A(G)$. The graph $G$ is called *separable* if it has a symmetric separator.

We will consider each locally $H \cup J$ graph as coming with some separator $\phi$. We note that if $G$ is separable, then its subgraph generated by all edges $xy$ such that $\phi(xy) = H$ is locally $H$.

**Lemma 3.3.** *If $\{x, y, z\}$ is a triangle in $G$, then $\phi(xy) = \phi(xz)$.*

**Proof:** The union $H_x \cup J_x$ is disjoint and $yz$ is an edge in $G$. □

**Theorem 3.4.** *If all locally $H$ graphs are $K$-divergent, then every separable locally $H \cup J$ graph is $K$-divergent.*

**Proof:** Let $G$ be a separable locally $H \cup J$ graph. Observe that every vertex in $G$ is a local cutpoint, hence repeatedly cutting through them as in Proposition 2.11 we get a disjoint union of locally $H$ and locally $J$ graphs, and the result follows. □

**Proposition 3.5.** *Let $H$, $J$ be two graphs such that $N_H[x]$ is not isomorphic to an induced subgraph of $J$ for all $x \in H$. Then any locally $L = H \cup J$ graph is separable.*

**Proof:** If $\phi$ is not symmetric, there is an edge $xy \in G$ such that $\phi(xy) = J$ and $\phi(yx) = H$, so $x \in H_y$ and $y \in J_x$. We will show that $N_{H_y}[x]$ is isomorphic to a subgraph of $J_x$, contradicting our hypothesis. Take $z \in N_{H_y}(x)$, then $\{x, y, z\}$ is a triangle and by Lemma 3.3 $\phi(xz) = J$, so $N_{H_y}(x) \subseteq J_x$. Since both $x$ and $y$ are adjacent to all of $N_{H_y}(x)$, we have $N_{H_y}[x] = N_{H_y}(x) + x \cong N_{H_y}(x) + y \subseteq J_x$. □

**Special retractions**

**Definition 3.6.** Let $H$ be a subgraph of $G$. We say that a retraction $\rho\colon G \to H$ is a *special retraction* if for every pair $\{v_1, v_2\}$ of non-adjacent vertices of $H$ it follows that $|\rho^{-1}(v_i)| = 1$ for some $i \in \{1, 2\}$.

**Theorem 3.7.** *The following statements about a graph $G$ with a subgraph $H$ isomorphic to $O_n$ are equivalent:*

(1) *$G$ has a special retraction to $H$.*
(2) *There is a clique of $H$ that is also a clique of $G$.*

**Proof:** Suppose that $H$ has vertices $\{1, 2, \ldots, 2n\}$ and non-edges $\{1, n+1\}$, $\{2, n+2\}$, …, $\{n, 2n\}$. Then $\{1, 2, \ldots, n\}$ and $\{n+1, n+2, \ldots, 2n\}$ are cliques of $H$ and completes of $G$.

We prove that (1) implies (2). Let $\rho\colon G \to H$ be a special retraction. Without loss of generality, we can then assume that $|\rho^{-1}(1)| = |\rho^{-1}(2)| = \cdots = |\rho^{-1}(n)| = 1$. Suppose now that we had $x \in G \setminus H$ with $x \sim i$ for all $i \in \{1, 2, \ldots, n\}$. Then $\rho(x)$ is adjacent or equal to $i$ for all $i \in \{1, 2, \ldots, n\}$. Since $\rho(x) \in H$ and $\{1, 2, \ldots, n\}$ is a clique of $H$ we must have then $\rho(x) \in \{1, 2, \ldots, n\}$. But this contradicts that $|\rho^{-1}(i)| = 1$ for $i = 1, 2, \ldots, n$. This proves that $\{1, 2, \ldots, n\}$ is a clique of $G$.

Now we prove that (2) implies (1). We can assume that $\{1, 2, \ldots, n\}$ is a clique of $G$. Thus $H$ is an induced subgraph of $G$ (if a non-edge of $H$ were an edge of $G$, the latter set would not be a clique).

We define the morphism $\rho\colon G \to H$ by the following set of rules (use always the first applicable rule):

$$\rho(x) = \begin{cases} x & \text{if } x \in H, \\ n+1 & \text{if } x \not\sim 1, \\ n+2 & \text{if } x \not\sim 2, \\ \cdots & \\ 2n & \text{if } x \not\sim n. \end{cases} \tag{1}$$

Since $\{1, 2, \ldots, n\}$ is a clique of $G$, we have that $\rho(x)$ is defined for all $x \in G$. We prove now that $\rho$ is a graph morphism. Suppose $x \sim y$ in $G$.

If $x, y \in H$, then $\rho(x) = x \sim y = \rho(y)$ in $H$ (induced in $G$). If $x, y \notin H$, then we must have $\rho(x), \rho(y) \in \{n+1, n+2, \ldots, 2n\}$ which is a clique, so $\rho(x)$ and $\rho(y)$ are adjacent or equal.

Now assume $x \notin H$ and $y \in H$. If $y \in \{1, 2, \ldots, n\}$, then the definition of $\rho$ forbids that $\rho(x) = n + y$ since $x$ and $y$ are adjacent. Since $n + y$ is the only vertex in $H$ which is not adjacent (or equal) to $y = \rho(y)$, it follows that $\rho(x)$ and $\rho(y)$ are adjacent. Else, if $y \in \{n+1, n+2, \ldots, 2n\}$, then both $\rho(x)$ and $y = \rho(y)$ are in this latter set, as before so $\rho(x)$ and $\rho(y)$ are adjacent or equal. □

Combining Theorems 2.7, 2.8 and 3.7, we have the following handy result:

**Theorem 3.8.** *If $O_n$ is a subgraph of $G$ for some $n \geq 3$ and some clique of $O_n$ is also a clique of $G$, then $G$ is $K$-divergent.* □

## 4. The first 61 cases

According to Hall [24], there are 65 link graphs with at most six vertices, which are shown in Figure 16. We denote by $P_n$ the path on $n$ vertices (note that Hall denotes this by $P_{n-1}$), and by $\boxed{(p,q,r)}$ the $r$-th graph with order $p$ and $q$ edges in the list of Appendix 1 to the book of Harary [25], whenever there is not a more immediate notation.

As we shall see in this section, most of the 65 link graphs can be treated by general methods like those in our previous section and by previously known results.

4.1. **Link graphs where in any edge one vertex dominates the other.** By inspection, the following 34 link graphs $L$ are Helly by Proposition 3.1, hence any extension of any of them is $K$-convergent by Theorem 2.3. $K_1$, $2K_1$, $K_2$, $3K_1$, $K_2 \cup K_1$, $K_3$, $4K_1$, $K_2 \cup 2K_1$, $2K_2$, $K_3 \cup K_1$, $K_4$, $5K_1$, $K_2 \cup 3K_1$, $2K_2 \cup K_1$, $K_3 \cup 2K_1$, $P_3 \cup K_2$, $K_3 \cup K_2$, the graph $(5,6,1)$, $K_4 \cup K_1$, $K_5$, $6K_1$, $K_2 \cup 4K_1$, $2K_2 \cup 2K_1$, $K_3 \cup 3K_1$, $P_3 \cup K_2 \cup K_1$, $3K_2$, $K_3 \cup K_2 \cup K_1$, the graph $(6,5,15)$, $K_4 \cup 2K_1$, the graph $(6,6,3)$, $2K_3$, $K_4 \cup K_2$, $K_5 \cup K_1$, $K_6$.

4.2. **Link graphs with large girth.** The next 7 link graphs have infinite girth: $P_4$, $P_4 \cup K_1$, $P_5$, $P_4 \cup 2K_1$, $P_5 \cup K_1$, $P_4 \cup K_2$, $P_6$. Since their extensions have infinite local girth, their clique graphs are Helly by Theorem 2.6 and hence these extensions are $K$-convergent by Theorem 2.3.

4.3. **Link graphs with unique extension.** Besides the already considered $K_1, \ldots, K_6$, there are 10 link graphs $L$ with a unique extension (see Table I or 4.3, 4.14 and the corollary to Theorem R in [24]). Therefore Theorem 1.1 is trivially true for these links graphs. Nevertheless, with only one exception whose behavior is still unknown (namely the extension of $L = \mathbb{M} = (6,7,6)$), we can easily prove that all these extensions are $K$-divergent as required by Theorem 1.2:

- $C_4$**:** The only locally $C_4$ graph is $O_3$, and we know that it is $K$-divergent by Theorem 2.7.
- $C_5$**:** The only locally $C_5$ graph is the icosahedron, and it was proven in [42] that it is $K$-divergent.
- **the graph** $(5,6,4)$**:** In this case the only locally $L$ graph is $\overline{C_8}$, which was proven in [31] to be $K$-divergent.
- **the graph** $\mathbb{M} = (6,7,6)$**:** Here the only locally $L$ graph $G$ is described in Section 4.14 from [24]. It can also be described (Isabel Hubard, private communication) as the graph obtained from the graph of the *snub cube* (Figure 3) adding both diagonals to each of the six square faces. We have been unable to determine the $K$-behavior of $G$, but we conjecture it is $K$-divergent.
- **the graph** $(6,8,5)$**:** The only locally $L$ graph $G$ is the line graph of $O_3$. It can be readily checked by computer (we used GAP [20]) that $K^3(G)$ is isomorphic to $O_8$, hence it is $K$-divergent by Theorem 2.7.
- **the graph** $(6,9,7)$**:** The only locally $L$ graph $G$ is the line graph of $K_5$. The six black vertices in Figure 4 induce an $O_3$, and there is a clique of $O_3$ (e.g., the upper leftmost black triangle) which is a clique of $G$, hence Theorem 3.8 applies and $G$ is $K$-divergent.
- **the graph** $(6,9,16)$**:** The only locally $L$ graph $G$ is $C_{10}(1,2,4)$. The clique graph of $G$ is isomorphic to the suspension of $C_{10}(1,2)$ (that is, the Zykov sum of $C_{10}(1,2)$ and $2K_1$), hence $K(G)$ is divergent by [31, Theorem 4.6].
- $K_{3,3}$**:** The only locally $L$ graph $G$ is $K_{3,3,3}$, as shown in [40], this graph retracts to $K_{2,2,2} = O_3$, hence it is $K$-divergent. See also [31, Theorem 5.2].
- **the graph** $(6,10,7)$**:** In this case the only locally $L$ graph is $\overline{C_9}$, which was proven in [31] to be $K$-divergent.
- $O_3$**:** The only locally $O_3$ graph is $O_4$, and this is $K$-divergent by Theorem 2.7.

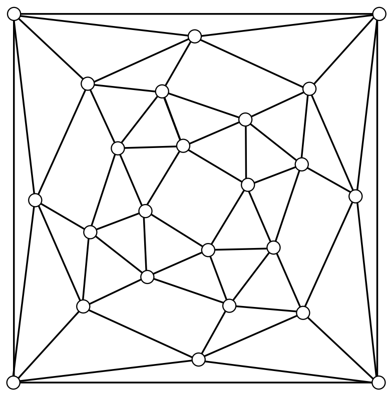

FIGURE 3. Snub Cube.

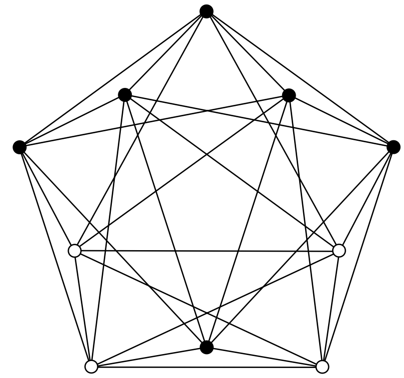

FIGURE 4. The line graph of $K_5$ and an octahedron (black vertices) in it.

4.4. **Link graphs with separable extensions.** All the extensions of the following 5 link graphs are separable by Proposition 3.5: $C_4 \cup K_1$, $C_4 \cup 2K_1$, $C_5 \cup K_1$, $C_4 \cup K_2$, and the graph $(6,6,2)$ (which is the disjoint union of the graph $(5,6,4)$ with $K_1$). By Theorem 3.4 all these extensions are $K$-divergent since we already know by subsection 4.3 that the extensions of $C_4$, $C_5$, and the graph $(5,6,4)$ are so.

4.5. **The hexagonal link graph.** The collection of locally $C_6$ graphs is infinite, and it was proven in [28] that all of them are $K$-divergent.

4.6. **The link graph** $(6,6,11)$**.** In this case, by [24, Section 4.10] all extensions of $L$ have the form $C_n(1,2,4)$, where $n = 11$ or $n \geq 13$. If $C_n(1,2,4), C_m(1,2,4)$ are two graphs of such form, then it is immediate to see that there is a *triangular covering map* from $C_{mn}(1,2,4)$ to each of $C_n(1,2,4), C_m(1,2,4)$ (see [28] for the theory of covering maps). It follows from [28, Corollary 2.3] that $C_n(1,2,4), C_m(1,2,4)$ have the same $K$-behavior. It is proven in [33] that such $K$-behavior is actually $K$-divergence.

4.7. **The link graph** $(6,7,5)$**.** From Hall's description in [24, p. 428-429], the collection of all locally $(6,7,5)$ graphs $G$ consists of an infinite family of graphs ("necklaces" made of at least four octahedra) and one sporadic case: $C_{12}(1,2,4)$ which has two octahedra. In both cases every triangle is a clique of $G$ (since $(6,7,5)$ has no triangles). Hence, by Theorem 3.8, $G$ is $K$-divergent.

4.8. **The link graph** $(6,7,13)$**.** Let $L$ be the graph (6,7,13), and $G$ be a locally $L$ graph. Let $x \in G$, and consider $N(x) \cup \{x\}$ as in Figure 5:

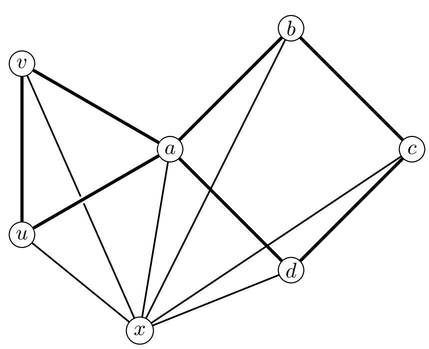


FIGURE 5. $N(x) \cup \{x\}$ in a locally $(6,7,13)$ graph

Observe that vertex $a$ has 5 of its 6 neighbors already drawn in Figure 5 and that the only way for its neighborhood to be isomorphic to $L$ is with a new vertex $y \notin N[x]$ which must be adjacent to $\{a, b, d\}$. All the neighbors of $a$ and $x$ are determined at this time. Now consider the 4-path $y \sim a \sim x \sim c$ in $N_G(b)$, if this 4-path is not completely contained in the 4-cycle of $N_G(b) \cong L$, then we would have that either $\{a, y\}$ or $\{x, c\}$ would be contained in the triangle of $N_G(b) \cong L$, necessarily ruining the already determined neighborhoods of either $a$ or $x$. Hence, the 4-path is completely contained in the 4-cycle of $N_G(b)$, then $y \sim c$ and $\{a, b, c, d, x, y\}$ induces an octahedron $O_3$ in $G$ such that $\{a, b, x\}$ is a clique of $G$. Hence, $G$ is divergent by Theorem 3.8.

4.9. **The link graph** $(6, 9, 11)$**.** Any extension of this graph is isomorphic to $C_n(1, 2, 3)$ ( [24, Section 4.11]) for some $n \geq 10$. Any such graph is self-clique (see [27, Lemma 1]). A graph $G$ is *self-clique* if $K(G)$ is isomorphic to $G$, hence any such graph is $K$-convergent.

## 5. The remaining 4 cases

There are only four remaining link graphs $L$ to consider: $(6, 6, 10), (6, 6, 13), (6, 6, 14), (6, 7, 23)$. All the extensions of these link graphs are $K$-convergent and will be considered by ad hoc methods in this section. These methods, however, have some common traits: they all use marked edges as in Theorem 2.2, and they all end up proving that some graph $X$ is Helly. If $G$ is the corresponding extension of the link graphs above, we shall prove that $G$ is Helly in the cases $L = (6, 6, 13), (6, 6, 14)$, that $K(G)$ is Helly in the case $L = (6, 6, 10)$ and that $K^2(G)$ is Helly in the case $L = (6, 7, 23)$.

5.1. **The link graph** $(6, 6, 10)$**.** Let $G$ be a locally (6,6,10) graph. Our goal in this subsection is to show that $K(G)$ is Helly.

We first note the following easy facts:

(1) Since (6,6,10) is not clique-cone, then all stars $x^*$ of $G$ are cliques of cliques, by Proposition 2.1.
(2) As the cliques of (6,6,10) are triangles and edges, the cliques of $G$ are tetrahedra and triangles.

Let $x \in G$ and consider $N[x]$ as in Figure 6.

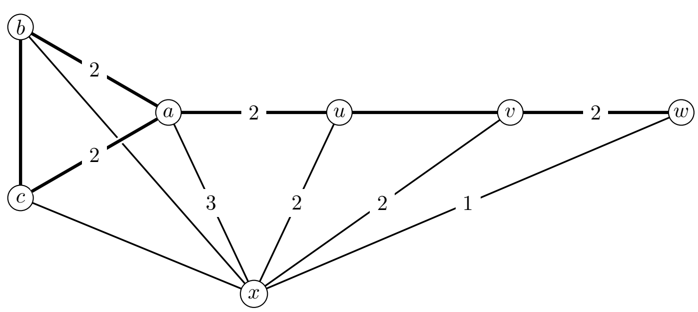


FIGURE 6. $N[x]$ in a locally $(6, 6, 10)$ graph

We shall use Theorem 2.2 and the definition of marked edges above it. Since all the neighbors of $x$ are already present in Figure 6, we can readily know the marks of edges incident to $x$, which are the number of common neighbors of the vertices in each edge. In particular, edges $ax$, $ux$, $vx$, $wx$ are marked as shown. We can not know directly the marks of the other edges since there may be other common neighbors in $G$ not shown in the drawing, hence we use Theorem 2.2 for that purpose: Since $a$ is the only vertex in $N_G(x) \cong L$ with degree 3, it follows that the marks of the edges in $N_G(x)$

incident to $a$ must be the same (including multiplicities) as the degrees in $L$ of the neighbors of (the only vertex in $L$ with the same degree as) $a$. Hence, the edges $ab$, $ac$, $au$ are marked 2. Similarly, $w$ is the only vertex of $N_G(x) \cong L$ of degree 1 and hence the only edge in $N_G(x)$ incident to $w$ must be marked with the degree of only neighbor of (the only vertex in $L$ with the same degree as) $w$. Hence $wv$ is also marked 2.

To explain further the use of Theorem 2.2 let us note that there are 4 vertices of degree 2 in $N_G(x) \cong L$, three of them have neighbors of degree 2 and 3 and one of them has neighbors of degrees 2 and 1. Hence, the edges incident to these vertices of degree 2 may be marked either 2 and 3 or 2 and 1. Therefore the marks of the edges $bc$ and $uv$ may be marked either 3 or 1, but Theorem 2.2 does not determine these marks completely.

From the marks shown in Figure 6 we have:

**Lemma 5.1.** *If a triangular clique in $G$ shares an edge with a tetrahedron, then such triangular clique is an internal triangle.*

**Proof:** Say that the triangular clique $\{x, y, z\}$ is such that the edge $xy$ is also in a tetrahedron. Considering $N(x)$, we obtain that the edge $yz$ is marked 2, and the edge $xz$ is marked 2. □

**Lemma 5.2.** *Any vertex in $G$ is contained in exactly one tetrahedron and three triangles that are cliques, where at least one of such triangles is internal and at least one of those triangles is not internal. In particular, no two tetrahedra in $G$ intersect, no two internal triangles intersect in exactly one vertex, and no three internal triangles intersect.*

**Proof:** Since the cliques of $L$ are a triangle and three edges, it follows that the cliques of $G$ containing any given vertex is one tetrahedron and three triangles. From Figure 6 one obtains that the triangle $\{x, a, u\}$ is internal (since each of its edges is marked at least 2), and the triangle $\{x, v, w\}$ is not internal (since one edge is marked 1). The other claims are straightforward. □

**Lemma 5.3.** *Let $Q$ be a necktie of $G$. Then in $Q$ there is exactly one tetrahedron.*

**Proof:** We have already observed that there cannot be more than one tetrahedron in $Q$, so let us assume that $Q$ consists only of triangular cliques. Let $\mathcal{C} = \{t_1, t_2, \ldots, t_s\} \subseteq Q$ be a minimal collection of triangles that intersect pairwise but have empty total intersection. Then $s \geq 3$, and by minimality we can take $x_i \in \cap(\mathcal{C} - \{t_i\})$ for $i = 1, 2, 3$. Now let $y_1, y_2, y_3 \in G$ be such that $t_1 = \{x_2, x_3, y_1\}$, $t_2 = \{x_1, x_3, y_2\}$ and $t_3 = \{x_1, x_2, y_3\}$.

The complete $\{x_1, x_2, x_3\}$ is actually a clique: assume to the contrary there is $u \notin \{x_1, x_2, x_3\}$ such that $u \sim x_i$ for $i = 1, 2, 3$. Now $x_2$ has neighbors $x_3, y_3, u$ in $N(x_1) \cong L$, and similarly $x_3$ would also have degree at least 3 in $N(x_1)$, which is a contradiction.

One of the edges $x_1y_2$ or $x_1y_3$ is contained in a tetrahedron (since the vertex $x_1$ has to be contained in a tetrahedron), without loss, assume such edge is $x_1y_2$. Then the edge $x_3y_1$ must also be contained in some tetrahedron. Considering $N(x_1)$, we obtain that the edge $y_2x_3$ has to be marked 2. But none of the neighbors of $x_3$, besides $x_1$, can also be a neighbor of $y_2$. Hence $Q$ cannot consist only of triangular cliques. □

**Lemma 5.4.** *Let $Q$ be a necktie of $G$. Then $Q = Q_T$ for some triangular clique $T$. The ears of $Q_T$ are exactly one tetrahedron and two triangular cliques.*

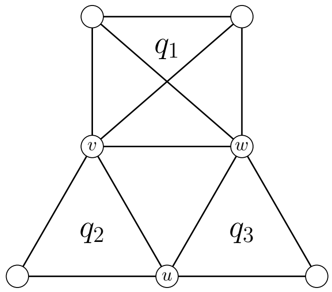


FIGURE 7. No triangle in $Q$ intersects the tetrahedron at an edge

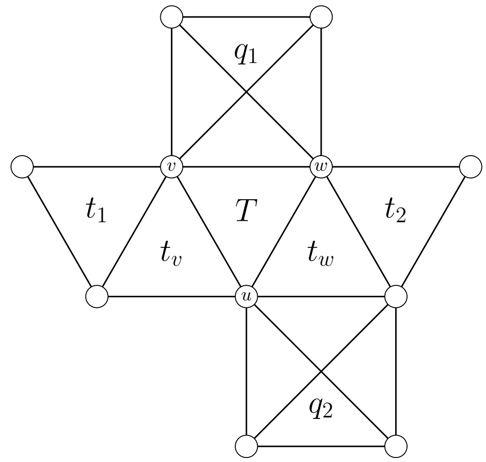


FIGURE 8. Neighborhood of $T$ in $K(G)$

**Proof:** Let $Q$ be a necktie of $G$. By Lemma 5.3, we may assume that there is a tetrahedron $q_1 \in Q$ and all other cliques in $Q$ are triangles.

We claim that there is a $T \in Q$ such that $|q_1 \cap T| = 2$. Otherwise, any other clique $q_2 \in Q$ meets $q_1$ in exactly one vertex and, since $\cap Q = \varnothing$, there must be some other clique $q_3 \in Q$ as in Figure 7.

There, $T = \{u, v, w\}$ is a clique in $G$. Since $|T \cap q_1| = 2$, $T \notin Q$, therefore not every clique in $Q$ meets $T$, so there is a clique $q_4 \in Q$ which meets $q_1$, $q_2$ and $q_3$, but not $T$. Now $q_4$ must meet $q_2$ and $q_3$ in the unlabeled vertices of the figure, but then there would be a 4-cycle in $N(u) \cong L$ which is absurd. Hence our claim follows, and there is a clique $T \in Q$ such that $T \cap q_1 = \{v, w\}$.

Let $u \in G$ be such that $T = \{u, v, w\}$. By Lemma 5.1, $T$ is an internal triangle. Let $t_v \neq T$ be the triangle of $G$ sharing $u, v$ with $T$, and let $t_w \neq T$ be the triangle of $G$ sharing $u, w$ with $T$. Now, the only way to complete the neighborhoods of $v$ and $w$ is by adding the triangles $t_1$ and $t_2$ as depicted in Figure 8. Note that the subgraph induced by $T \cup t_v \cup t_w$ must not have any additional edge other than those depicted, since that would ruin the neighborhoods of $v$, $w$ or $u$. Finally, completing the neighborhood of the vertex $u$ requires a tetrahedron $q_2$ which, by the previous restrictions, can only be drawn either sharing and edge with $t_w$ (as depicted) or, symmetrically, sharing and edge with $t_v$. Hence, without loss of generality, we may assume that $q_2$ is as depicted. It follows that $N_{K(G)}[T] = \{q_1, T, t_v, t_w, t_1, t_2, q_2\}$.

Since $T \in Q$, we have that $Q \subseteq N_{K(G)}[T]$. There are exactly 5 cliques of cliques of $G$ contained in $N_{K(G)}[T]$: $u^*, v^*, w^*, Q_{t_w}$ and $Q_T$. Hence $Q = Q_T$ is the only one satisfying the required conditions $\cap Q = \varnothing$ and $q_1 \in Q$. Therefore $Q = Q_T = \{q_1, T, t_v, t_w\}$, and the ears of $Q_T$ are $q_1$, $t_v$ and $t_w$ as required. □

We know now that the cliques of $K(G)$ correspond to vertices and internal triangles of $G$. The following Lemma describes the adjacencies in $K^2(G)$.

**Lemma 5.5.** *Let $x, y \in G$, $x \neq y$, and $T, T'$ be internal triangles in $G$, $T \neq T'$. Then*

(1) *$x^* \sim y^*$ in $K^2(G)$ if and only if $x \sim y$ in $G$,*
(2) *$x^* \sim Q_T$ in $K^2(G)$ if and only if $x$ is a neighbor of two vertices of $T$,*
(3) *$Q_T \sim Q_{T'}$ in $K^2(G)$ if and only if $|T \cap T'| = 2$ or two vertices of $T$, together with two vertices of $T'$, induce a tetrahedron.*

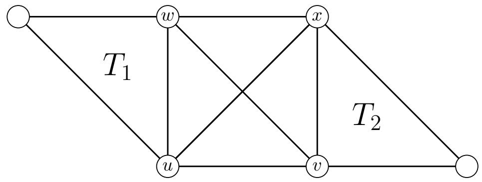


FIGURE 9. A $K_4$ in two neckties

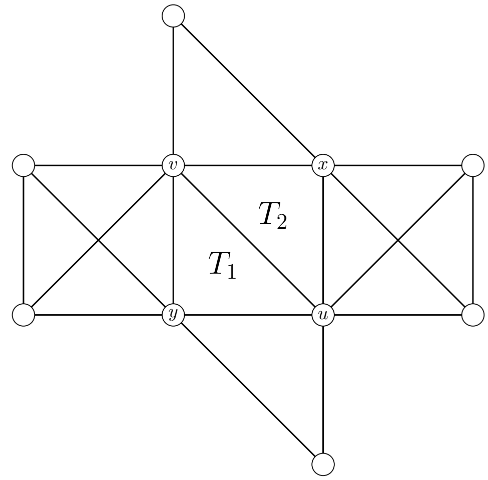


FIGURE 10. Centers sharing an edge

**Proof:** The first statement was noted in section 2. For the second: we also noted in that section that $x^* \sim Q_T$ if and only if $x \in B(Q_T)$, and then the claim follows since the basement $B(Q_T)$ contains exactly the vertices that are neighbors of two vertices of $T$.

For the third statement: Let $q \in Q_T \cap Q_{T'}$. By definition, $|q \cap T|$, $|q \cap T'| \geq 2$. If $T \cap T' = \emptyset$, then $q$ has to be a tetrahedron. Since $|T \cap T'| = 1$ is ruled out by Lemma 5.2, the only other possibility is that $|T \cap T'| = 2$. Conversely, if $|T \cap T'| = 2$, then $T \in Q_T \cap Q_{T'}$, and if two vertices of $T$, together with two vertices of $T'$, induce a tetrahedron, then such tetrahedron is in $Q_T \cap Q_{T'}$. □

**Theorem 5.6.** *If $G$ is a locally $(6, 6, 10)$ graph, then $K(G)$ is Helly (hence $G$ is $K$-convergent).*

**Proof:** Let $\mathcal{C} = \{x_1^*, x_2^*, \ldots, x_r^*, Q_{T_1}, \ldots, Q_{T_s}\}$ be a collection of mutually intersecting cliques of $K(G)$, we shall prove that $\cap\mathcal{C} \neq \emptyset$.

Assume first that $s \geq 2$ and that a tetrahedron lies in $Q_{T_1} \cap Q_{T_2}$, see Figure 9. Given the adjacencies described in Lemma 5.5, it follows that $s = 2$, that $r \leq 4$ and that $\{x_1, \ldots, x_r\} \subseteq \{u, v, w, x\} = B(Q_{T_1}) \cap B(Q_{T_2})$. In any case, the tetrahedron is contained in $\cap\mathcal{C}$.

Now, suppose that $T_1$, $T_2$ intersect in an edge, as in Figure 10. In this case, again we obtain that $s = 2$, and we have that $\{x_1, \ldots, x_r\}$ is a complete contained in $B(Q_{T_1}) \cap B(Q_{T_2}) = \{u, v, x, y\}$. Hence $r \leq 3$ and $\{x_1, \ldots, x_r\}$ is contained in $\{u, v, z\}$, where $z$ is either $x$ or $y$. Depending on $z$, we have that either $T_1 \in \cap\mathcal{C}$ or $T_2 \in \cap\mathcal{C}$.

If $s = 1$, since $Q_{T_1}$ induces a subgraph as in Figure 7 we have that $\{x_1, \ldots, x_r\}$ is a complete contained in a clique $q$ of $Q_{T_1}$. It follows that $q \in \cap\mathcal{C}$.

If $s = 0$, then extending $\{x_1, \ldots, x_r\}$ to a clique $q$, we get $q \in \cap\mathcal{C}$. □

5.2. **The link graphs** $(6, 6, 13)$ **and** $(6, 6, 14)$**.** We may treat these two cases simultaneously.

Let $L$ be $(6, 6, 13)$ or $(6, 6, 14)$. We claim that, for any $x \in G$ the edges in $N(x)$ are marked as shown in figures 11 and 12. Let us see why.

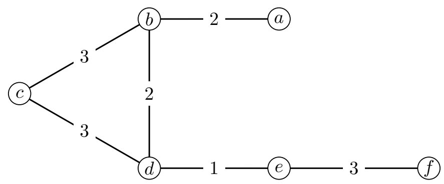


FIGURE 11. $(6, 6, 13)$ marked

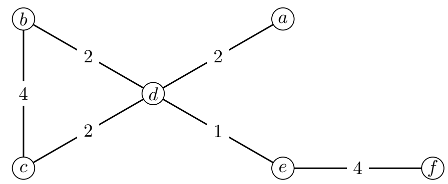


FIGURE 12. $(6, 6, 14)$ marked

Consider Figure 11. There are 2 vertices of degree 3 in $N(x) \cong L$ ($b$ and $d$), and the degrees of their neighbors are respectively 1,2,3 and 2,2,3. It follows by Theorem 2.2 that the edges incident with $b$ or $d$, which are $ba, bc, bd$ and $db, dc, de$, must be marked either 1,2,3 or 2,2,3. Similarly, edges $ef, ed$ and $cb, cd$ must be marked either 1,3 or 3,3 and the edges $fe$ and $ab$ must be marked either 2 or 3. Note then that since the edge $ef$ must be marked (1 or 3) *and* (2 or 3) its mark must be 3. Now edges $cb$ and $cd$ must be marked either 1 or 3, but $c$ and $b$ have at least 2 common neighbors, namely $d$ and $x$, hence its mark must be 3 (see the definition of marks above Theorem 2.2). For the same reasons, the mark of $cd$ must also be 3. Now the remaining edges ($ba, bd$) incident to $b$ can not use the mark 3 already used in $bc$, and hence $ba, bd$ must be marked either 1,2 or 2,2. but we already said that the mark of $ba$ must be 2 or 3, hence the mark of $ba$ is 2. For the same reasons, edges $de, db$ can not use the mark 3 already used in $dc$, hence $de, db$ must be marked 1,2 or 2,2. But we already said that $de$ must be marked 3 or 1, hence $de$ is marked 1. Finally since the edges $db, dc, de$ must be marked 1,2,3 or 2,2,3 and the marks 1 and 3 are already used by $de$ and $dc$, the mark of $db$ is 2.

The same kind of arguments apply for Figure 12: Edges $da, db, dc, de$ must be marked 1,2,2,2. Edges $bd, bc$; $cb, cd$ and $ed, ef$ must be marked either 1,4 or 2,4. And edges $ad$ and $fe$ must be marked 2 or 4. Hence $ad$ is marked 2. The mark of $bc$ is at least 2 since $b$ and $c$ have at least two common neighbors in $G$ ($d$ and $x$). For the same reason, $dc$ and $db$ are also marked at least 2. Hence $de$ is marked 1, $db$ and $dc$ are marked 2, and $bc$ and $ef$ are marked 4.

**Lemma 5.7.** *If $G$ is a locally $(6, 6, 13)$ or $(6, 6, 14)$ graph, then $G$ has no internal triangles.*

**Proof:** Considering the marks shown in figures 11 and 12, and that for any $x$, the edge joining $x$ to some vertex $y \in N_G(x)$ is marked with $\deg_{N(x)}(y)$, it follows that in any triangle that is a clique there is at least one edge marked 1. Hence the claim holds. □

**Theorem 5.8.** *If $G$ is a locally $(6, 6, 13)$ or a locally $(6, 6, 14)$ graph, then $G$ is Helly, hence $K$-convergent.*

**Proof:** We will use Theorem 2.5. By way of contradiction, assume that $G$ has a subgraph $H$ as in Figure 13 and $G$ has none of the dotted edges. Therefore, each edge of the triangle $T = \{u, v, w\}$ is contained in at least two cliques. Thus, since the triangle $T$ is not internal (by Lemma 5.7) $T$ is not a clique and there is some extra vertex $x \notin H$ which extends $T$ to a tetrahedron. But then each edge of $T$ has a mark of at least 3, which contradicts the labeling of the edges in the triangles in figures 11 and 12. □

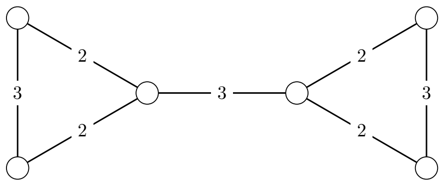


FIGURE 14. $(6, 7, 23)$ marked

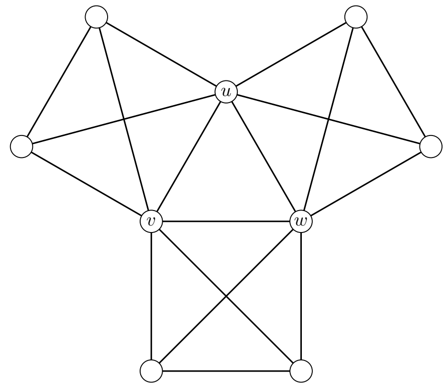


FIGURE 15. Three mutually intersecting tetrahedra

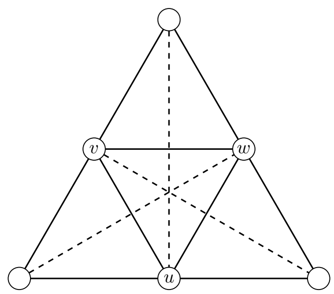


FIGURE 13. Hypothetical subgraph $H$ of $G$

5.3. **The link graph** $(6, 7, 23)$**.** Let $G$ be a locally $(6, 7, 23)$ graph. Our goal is to show that $K^2(G)$ is Helly.

The cliques of $G$ are triangles and tetrahedra. Each vertex is contained in exactly one triangular clique and two tetrahedra. In particular, neither two internal triangles nor three tetrahedra intersect. Two tetrahedra can only meet at a vertex. Also, a triangular clique and a tetrahedron can only meet at an edge.

Let $x \in G$. Using again Theorem 2.2 and the same kind of arguments as in the previous two subsections, one obtains that the marks of the edges of $N_G(x) \cong (6, 7, 23)$ are as shown in Figure 14. From there, it follows that all triangular cliques are internal.

**Lemma 5.9.** *No vertex $x \in G$ is normal.*

**Proof:** Let $x \in G$ and $u, v$ the vertices of degree 3 in $N(x) \cong L$. From Figure 14, it follows that the edge $uv$ is marked 3, and so it is contained in a triangle $T' \neq \{x, u, v\}$. Extending $T'$ to a clique $q$, we obtain that $q \notin x^*$, but $q$ intersects all three cliques in $x^*$. Hence $x^*$ is not a clique. □

**Lemma 5.10.** *Let $Q$ be a clique of $K(G)$. Then $Q = Q_T$ for some triangular clique $T$. The ears of $Q_T$ are exactly three tetrahedra.*

**Proof:** From previous observations, it follows that $Q$ can contain at most one triangular clique. Suppose there were no triangular cliques in $Q$. Since $Q$ is not a star, there are three tetrahedra in $Q$ that pairwise intersect but with empty intersection, and so we have that the graph in Figure 15 is a subgraph of $G$. These three tetrahedra cannot be the whole of $Q$, since the triangle $\{u, v, w\}$ intersects the three. However, since the neighborhood of any of $u, v, w$ is already drawn in Figure 15, there cannot be any further adjacencies between any two vertices in Figure 15, which would contradict that there is at least one more tetrahedron in $Q$. It follows that $Q$ contains exactly one triangular clique $T$. Since the neighborhood of $T$ in $K(G)$ is precisely $Q_T$, and this is a complete, it follows that $Q_T$ is a clique. Since no two triangular cliques intersect, the ears of $Q_T$ have to be three tetrahedra. □

**Lemma 5.11.** *Let $T, T'$ be triangular cliques. Then $Q_T \sim Q_{T'}$ if and only if two vertices of $T$, together with two vertices of $T'$, induce a tetrahedron.*

**Proof:** Suppose $T, T'$ are internal triangles and $q \in Q_T \cap Q_{T'}$. By definition, $|q \cap T| = |q \cap T'| = 2$. Since we have observed that $T \cap T' = \emptyset$, then $q$ has to be a tetrahedron. Now, if two vertices of $T$, together with two vertices of $T'$, induce a tetrahedron, then such tetrahedron is in $Q_T \cap Q_{T'}$. □

**Theorem 5.12.** *If $G$ is a locally $(6, 7, 23)$ graph, then $K^2(G)$ is Helly (hence $G$ is $K$-convergent).*

**Proof:** Using the adjacency rule shown in Lemma 5.11, it is immediate that if there was a triangle in $K^2(G)$, then there would be a vertex $x \in G$ with a cycle of order 4 in $N_G(x)$. Since $K^2(G)$ has no triangles, then it is Helly by Theorem 2.4. □

This concludes the proof of our Main Theorems 1.1 and 1.2.

## 6. An Example

In this paper, we have shown that if $|L| \leq 6$, then any two finite locally $L$ graphs have the same $K$-behavior. However, this is not true for $|L| = 9$.

**Theorem 6.1.** *There are graphs $G_1$ and $G_2$ which are both locally $C_5 \cup P_4$ such that $G_1$ is $K$-divergent and $G_2$ is $K$-convergent.*

**Proof:** Let $I$ be the icosahedron, $D = I - v$ (the icosahedron with a vertex removed) and $C = C_{10}(1, 2)$. Observe that $I$ is locally $C_5$, that $C$ is locally $P_4$ and that $D$ has 6 vertices with link $C_5$ and 5 vertices with link $P_4$.

In general, given two graphs $T_1, T_2$ which are locally $L_1, L_2$ (respectively), their Cartesian product $T = T_1 \square T_2$ is locally $L_1 \cup L_2$. Besides, the resulting graph is always separable and cutting through all vertices of $T$ gives a disjoint union of several copies of $T_1$ and $T_2$. Hence if either $T_1$ or $T_2$ is $K$-divergent, so is $T$ by Proposition 2.11. Therefore $G_1 = I \square C$ is locally $C_5 \cup P_4$ and $K$-divergent since the icosahedron es known to be $K$-divergent [42].

The idea for constructing $G_2$ is to take several copies of $D$ and $C$ and bijectively identify each vertex of link $C_5$ with some vertex of link $P_4$ in such a way that all the resulting vertices are local cutpoints with link $C_5 \cup P_4$ (as it turns out, this can be done with 10 copies of $D$ and one copy of $C$). Any such graph must be $K$-convergent thanks to the theory in [19] (since all the vertices in $G_2$ are *persistent* and since the *marked graphs* $(D, V(D))$ and $(C, V(C))$ are easily shown to be $\xi$-*convergent*). Alternatively, it can be directly checked that the following explicit example satisfies the required conditions:

Take the permutation $\pi = (1\ 2\ \cdots\ 60)^6 = (1\ 7\ \cdots\ 55)(2\ 8\ \cdots\ 56)(3\ 9\ \cdots\ 57)\cdots(6\ 12\ \cdots\ 60)$ and let $G_2$ be the (minimal) $\pi$-invariant graph where the following relations hold: $N(1) \supseteq \{2, 3, 4, 5, 6, 7, 13\}$, $N(2) \supseteq \{3, 6, 9, 10, 18, 34, 45, 51\}$, $N(3) \supseteq \{4, 21, 52\}$, $N(4) \supseteq \{5, 12, 36, 53\}$, $N(5) \supseteq \{6, 12, 23, 48, 54\}$.

Then the just defined $G_2$ is locally $C_5 \cup P_4$ and $K$-convergent. The orders of its iterated clique graphs are $\big(|K^n(G_2)|\big)_{n=0}^{\infty} = \big(60, 160, 160, 370, 810, 2880, 620, 420, 370, 420, \dots\big)$ with $K^7(G_2) \cong K^9(G_2)$. □

Our results beg for the following questions:

**Problem 1.** *Is it true that the only locally* $\mathbb{M} = (6, 7, 6)$ *graph is* $K$*-divergent?*

**Problem 2.** *Which is the greatest number* $n$ *such that for any* $L$ *with* $|L| \leq n$*, any two locally* $L$ *graphs have the same* $K$*-behavior. The only possibilities are* $n = 6, 7$ *or* $8$.

**Problem 3.** *Is there a* connected *graph* $L$*, such that there are two locally* $L$ *graphs with different* $K$*-behavior?*

**Problem 4.** *Is it true that for all* $n \geq 9$*, there is some* $L$*, with* $|L| = n$*, having a pair of extensions* $G_1, G_2$ *with different* $K$*-behavior?*

**Acknowledgements** We are grateful to an anonymous referee for several comments which improved the presentation of this paper.

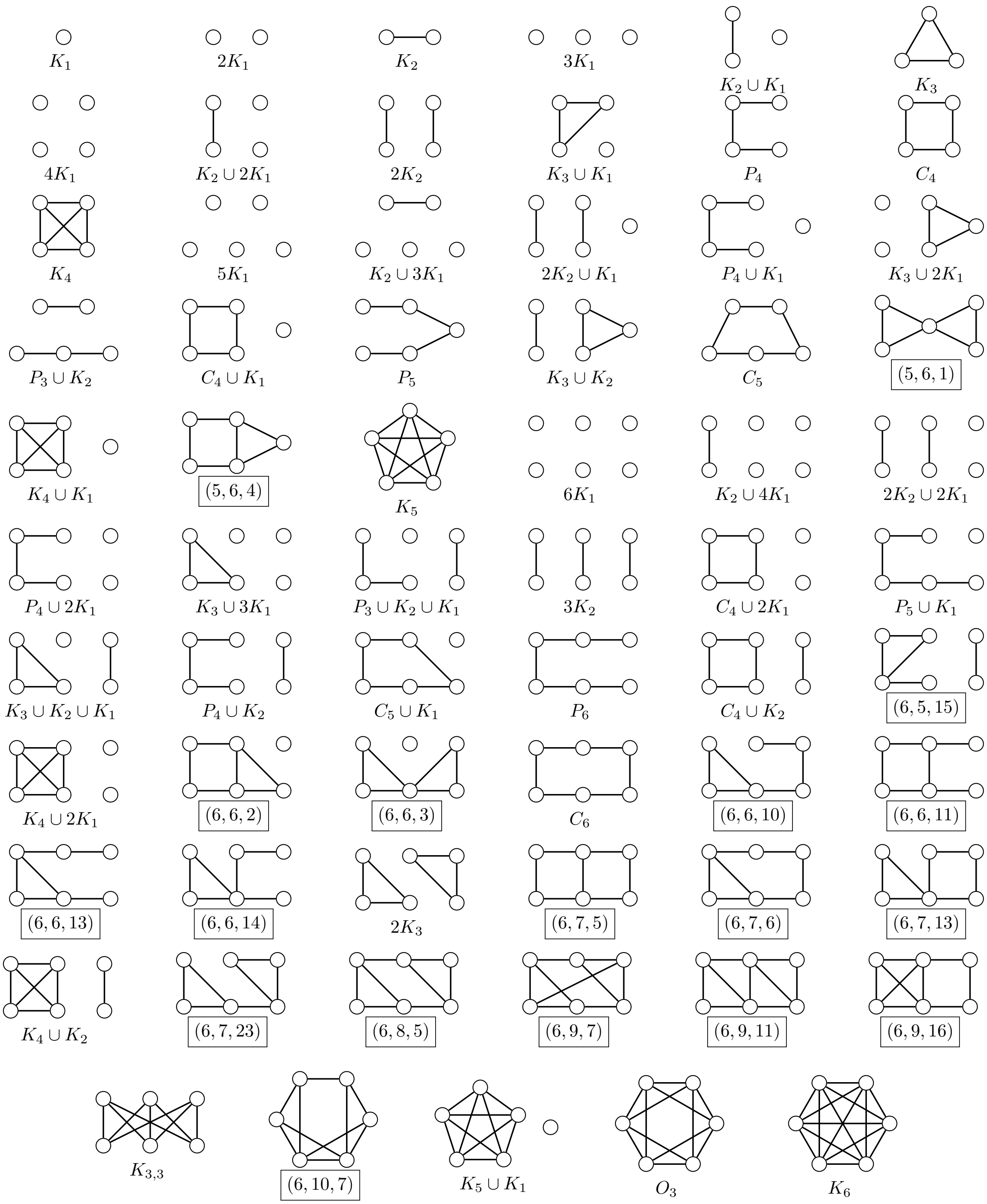


Figure 16. The 65 link graphs with at most six vertices

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