SMALL LOCALLY nK_2 GRAPHS

F. LARRIÓN, M. A. PIZAÑA, AND R. VILLARROEL-FLORES

Abstract. A locally nK_2 graph G is a graph such that the set of neighbors of any vertex of G induces a subgraph isomorphic to nK_2. We show that a locally nK_2 graph G must have at least $6n-3$ vertices, and that a locally nK_2 graph with $6n-3$ vertices exists if only if $n \in \{1,2,3,5\}$, and in these cases the graph is unique up to isomorphism. The case $n=5$ is surprisingly connected to a classic theorem of algebraic geometry: The adjacency relation among the vertices of the only locally $5K_2$ graph on $6n-3=27$ vertices is the same as the incidence relation among the 27 straight lines on any nonsingular complex, projective cubic surface.

1. Introduction

All our graphs are finite, simple and connected. If x is a vertex of the graph G, we denote by $N_G(x)$ the subgraph of G induced by the neighbors of x in G. A graph G is called *locally homogeneous* if there is a graph H such that $N_G(x) \cong H$ for all $x \in G$. There is an ample literature on locally homogeneous graphs, see for example: [1, 2, 3, 4, 5, 6, 7], where the problems of realization (given H, find a (finite) graph G which is locally H) and characterization (given H, characterize all locally H graphs) are addressed.

The graph nK_2 is the disjoint union of n copies of the complete graph K_2 on two vertices. For $n=1$ it is immediate that the only locally nK_2 is K_3. For each $n \geq 2$ there is an infinite number of finite graphs which are locally nK_2, see the construction techniques of [3, 6, 7]. We will show in section 2 that the number of vertices for a locally nK_2 graph is bounded below by $6n-3$, and then our main result:

Theorem. *A locally nK_2 graph G with $6n-3$ vertices exists if, and only if, $n \in \{1,2,3,5\}$, and in those cases, it is unique up to isomorphism.*

We give a simple description of the three graphs, besides K_3, that have this extremal property.

In this paper, a maximal complete subgraph is called a *clique*. We identify each induced subgraph of G with its vertex set. The adjacency relation will be denoted as $\sim$.

Key words and phrases. graphs with constant link, locally homogeneous graphs, locally matching graphs.

2. Proof of the theorem

From now on, assume G to be a locally nK_2 graph and $n \geq 2$. First thing we note is that the cliques of G are exactly its triangles, and no edge of G is in two cliques. We will use this property extensively.

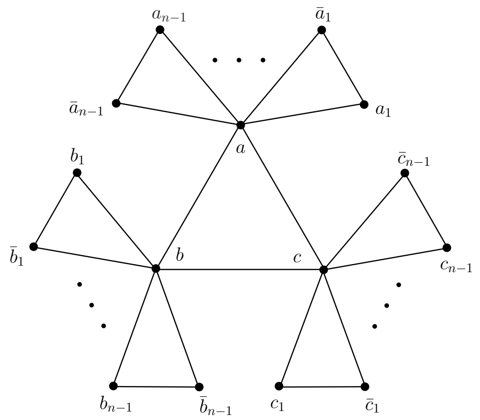

Figure 1. A triangle in G and its neighbours.

Let $T = \{a, b, c\}$ be a triangle of G (cf. Figure 1). If there is some $x \in N(a) \cap N(b)$ with $x \neq c$, vertex b would have a path on three vertices in its neighbourhood. Hence $N(a) \cap N(b) = \{c\}$ and since our situation is symmetrical on a, b and c, it follows that $|G| \geq |N(a) \cup N(b) \cup N(c)| = 6n - 3$. Hence:

Lemma 1. [8] *A locally nK_2 graph G has at least $6n - 3$ vertices.* □

Assume from now on that G has $6n - 3$ vertices. Fix a triangle $T = \{a, b, c\}$ of G. By the previous argument, $V(G) = N(a) \cup N(b) \cup N(c)$. Let's label the vertices of G as in Figure 1, that is, we set $N(a) = \{b, c\} \cup \{a_1, a_2, \ldots, a_{n-1}\} \cup \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-1}\}$ with $a_i \sim \bar{a}_i$; and we also label the neighbors of b and c in a similar way. We will use letters x, y, z to refer to (generic) elements in $\{a, b, c\}$. Likewise, (say) x_i and $\bar{x}_i$ are generic elements in $V(G) - T$, but we must always have that $\{x, x_i, \bar{x}_i\}$ is a triangle. Unless otherwise stated we assume that no two of x, y and z are equal.

Since $N_G(x) \cong nK_2$, it follows that the $2n-2$ vertices of $N_G(x_i)$ besides x and $\bar{x}_i$ are exactly half of the $4n - 4$ vertices in $G - N[x] = \{y_1, \bar{y}_1, \ldots, y_{n-1}, \bar{y}_{n-1}\} \cup \{z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}\}$. But we can not have both $x_i \sim y_j$ and $x_i \sim \bar{y}_j$, for otherwise, the edge $y_j\bar{y}_j$ would be in two triangles. Likewise, we could not have both $x_i \sim y_j$ and $\bar{x}_i \sim y_j$. Hence, it follows that for all j, the vertex x_i is adjacent to exactly one of y_j, $\bar{y}_j$ and exactly one of z_j, $\bar{z}_j$ and that $\bar{x}_i$ is adjacent to exactly the other two. So we just proved that:

Lemma 2. *The edges connecting $\{x_i, \bar{x}_i\}$ to $\{y_j, \bar{y}_j\}$ form a perfect matching.* □

Therefore, we know that the connections from $\{x_i, \bar{x}_i\}$ to $\{y_j, \bar{y}_j\}$ can only be either *straight* ($x_i \sim y_j$ and $\bar{x}_i \sim \bar{y}_j$) or *twisted* ($x_i \sim \bar{y}_j$ and $\bar{x}_i \sim y_j$). This allows us to use an auxiliary representation for those connections of G: Set $A_i = \{a_i, \bar{a}_i\}$, $B_i = \{b_i, \bar{b}_i\}$ and $C_i = \{c_i, \bar{c}_i\}$. Let H_n be the complete tripartite graph on $3n-3$ vertices with $n-1$ vertices in each part. We assume without loss that the vertices of H_n are precisely the aforementioned A_i, B_i and C_i for $i \in I := \{1, \dots, n-1\}$. Moreover we set the parts of H_n to be precisely $\{A_i\}_{i \in I}$, $\{B_i\}_{i \in I}$ and $\{C_i\}_{i \in I}$. Now, whenever the connections from $X_i := \{x_i, \bar{x}_i\}$ to $Y_j := \{y_i, \bar{y}_i\}$ are straight in G, we paint the edge X_iY_j blue in H_n, otherwise (when the connections are twisted) we paint it red. It should be clear that the possible adjacencies of G (compatible with Lemma 2) are in bijection with the edge-colorings of H_n.

Now, for any triangle $A_iB_jC_k$ of H_n, we observe that $A_i \cup B_j \cup C_k$ induces a disjoint union of two triangles in G if an only if $A_iB_jC_k$ has an even number of red edges. In such case, we say that $A_iB_jC_k$ is a *good* triangle. We say that an edge-coloring of H_n with colors blue and red is *valid* if every edge of H_n is contained in exactly one good triangle. Since every edge of G is contained in exactly one triangle, we have:

Lemma 3. *Any locally nK_2 graph G with $6n-3$ vertices determines a valid edge-coloring of H_n and conversely, a valid edge-coloring leads to a locally nK_2 graph.* □

However, different colorings can lead to isomorphic locally nK_2 graphs, and in this case, we say that the colorings are *equivalent*. For example, interchanging the names of the two vertices of a pair $X_i = \{x_i, \bar{x}_i\}$ in G would mean that all edges of H_n incident to vertex X_i would switch colors. In this case, we say that we applied *a twist* to the vertex X_i. We can also reorder some vertices of one of the parts of H_n and obtain an equivalent coloring. Twists and reorderings (of the kind described) will be the only two operations that we will use to reduce any edge-coloring of H_n to one, specific, canonical one.

Lemma 4. *For each $n \geq 2$ any two valid edge-colorings of H_n are equivalent. In particular, for each $n \geq 2$ there is, up to isomorphism, at most one locally nK_2 graph on $6n-3$ vertices.*

Proof: We start, then, with a valid coloring of H_n.

Step 1. The edges A_1B_i for $i = 1, \dots, n-1$ can be assumed to be all blue, by applying a twist to some of the B_i if necessary. Similarly, all edges of the form A_1C_i for $i = 1, \dots, n-1$ and A_iB_1 for $i = 2, \dots, n-1$ can be assumed to be blue.

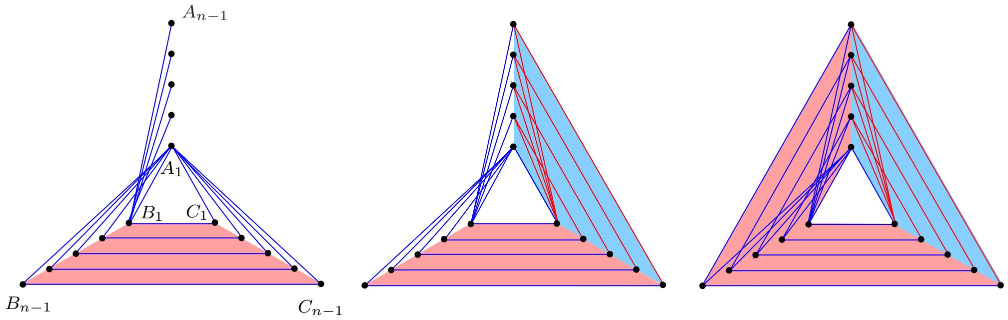

FIGURE 2. Edge colorings of the tripartite graph H_n

Step 2. Since each edge of the form A_1B_i for $i = 1, \dots, n-1$ is in exactly one good triangle, for each vertex in $\{B_1, \dots, B_{n-1}\}$ there must be exactly a blue edge joining it to a vertex in $\{C_1, \dots, C_{n-1}\}$. Moreover, all such blue edges form a perfect matching, since otherwise there would be an edge in two good triangles. By reordering $\{C_1, \dots, C_{n-1}\}$ if necessary, we can assume that the edges B_iC_i for $i = 1, \dots, n-1$ are blue, and so all edges B_iC_j with $i \neq j$ are red, see Fig. 2 (left).

Note that this settles the case $n = 2$, so we can assume $n \geq 3$ in what follows.

Step 3. The edges C_1A_i for $i = 2, \dots n-1$ are now forced to be red, for otherwise the edge B_1C_1 would be in two good triangles.

Step 4. Since each edge of the form A_iB_1 for $i = 2, \dots, n-1$ is in exactly one good triangle, for each vertex in $\{C_2, \dots, C_{n-1}\}$ there must be exactly a red edge joining it to a vertex in $\{A_2, \dots, A_{n-1}\}$, and in fact, these red edges form a perfect matching between these two sets (if, for example, the edges C_2A_2 and C_3A_2 were red, the edge A_2B_1 would be in two good triangles), so by reordering $\{A_2, \dots, A_{n-1}\}$ we can assume without loss that the edges C_iA_i for $i = 2, \dots, n-1$ are red, and so all edges C_iA_j with $2 \leq i, j \leq n-1$, $i \neq j$ are blue.

Step 5. If $n = 3$, then the only edge we still have to consider is A_2B_2, and this has to be blue for C_1A_2 to be in a good triangle. So assume $n \geq 4$. An edge A_iB_j for $2 \leq i, j \leq n-1$, $i \neq j$ has to be red, since if it were blue it would be in the good triangles $A_iB_jC_1$ and $A_iB_jC_j$. This forces, for each $i = 2, \dots, n-1$, that the edge A_iB_i is blue, since otherwise the edge C_1A_i is in no good triangle.

Hence, any valid edge-coloring of H_n is equivalent to the specific coloring obtained. □

However, we claim that if $n = 4$ or $n \geq 6$, the coloring just described is not valid. Consider the edge A_3B_2, which is painted red. If $n = 4$, it is not contained in any good triangle (the triangles which contain such edge have the other two edges both red or both blue). If $n \geq 6$, it is contained in at least two good triangles ($A_3B_2C_4$ and $A_3B_2C_5$).

It only remains to be shown that in the remaining cases $n = 2, 3, 5$, the described coloring is valid, thus producing the required locally nK_2 graphs. Of course, this can be done by carefully checking the coloring or the associated graph H_n, but it is also enough to show for each n *any* such locally nK_2 graph G_n on $6n-3$ vertices. This latter approach happens to be more direct: Using Nešetril's notation for products, we obtain $G_2 := K_3 \square K_3 \cong K_3 \times K_3$. Also, $G_3 := \overline{L(K_6)}$ is the the complement of the line graph of K_6, also known as the Kneser graph $KG_{6,2}$, its vertices are $\{\, A \subseteq \{1,2,3,4,5,6\} \mid |A| = 2 \,\}$ and AA' is an edge whenever $A \cap A' = \varnothing$. The remaining graph G_5 can be described combinatorially by its triangles:

$\{1,2,25\}$, $\{1,9,17\}$, $\{1,11,19\}$, $\{1,13,21\}$, $\{1,15,23\}$, $\{2,10,18\}$, $\{2,12,20\}$, $\{2,14,22\}$, $\{2,16,24\}$, $\{3,4,25\}$, $\{3,9,20\}$, $\{3,11,18\}$, $\{3,14,23\}$, $\{3,16,21\}$, $\{4,10,19\}$, $\{4,12,17\}$, $\{4,13,24\}$, $\{4,15,22\}$, $\{5,6,25\}$, $\{5,9,22\}$, $\{5,12,23\}$, $\{5,13,18\}$, $\{5,16,19\}$, $\{6,10,21\}$, $\{6,11,24\}$, $\{6,14,17\}$, $\{6,15,20\}$, $\{7,8,25\}$, $\{7,9,24\}$, $\{7,12,21\}$, $\{7,14,19\}$, $\{7,15,18\}$, $\{8,10,23\}$, $\{8,11,22\}$, $\{8,13,20\}$, $\{8,16,17\}$, $\{9,10,26\}$, $\{11,12,26\}$, $\{13,14,26\}$, $\{15,16,26\}$, $\{17,18,27\}$, $\{19,20,27\}$, $\{21,22,27\}$, $\{23,24,27\}$, $\{25,26,27\}$.

It can also be described as the complement of the Schläfli graph. But, perhaps the easiest way to verify the existence of such graph is just to recall the classical 27 straight line theorem from algebraic geometry, which says (see for instance Proposition 7.3 in [9]) that any nonsingular cubic surface $S \subset \mathbb{C}P^3$ contains exactly 27 lines and that any of them intersects exactly 10 of the other lines, which in turn intersect each other in pairs. Hence our G_5 is simply the incidence graph of these straight lines.

Quod erat demonstrandum.

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(Larrión) Instituto de Matemáticas., Universidad Nacional Autónoma de México., México, D.F. C.P. 04510.

E-mail address: paco@math.unam.mx

(Pizaña) Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica., Av. San Rafael Atlixco 186. Col. Vicentina., 09340 México, D.F. MEXICO.

E-mail address: map@xanum.uam.mx, http://xamanek.izt.uam.mx/map

(Villarroel-Flores) Centro de Investigación en Matemáticas,, Universidad Autónoma del Estado de Hidalgo,, Carr. Pachuca-Tulancingo km. 4.5, Pachuca Hgo. 42184, MEXICO.

E-mail address: rafaelv@uaeh.edu.mx