

ITERATED CLIQUE GRAPHS AND BORDERED COMPACT SURFACES

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ABSTRACT. The clique graph $K(G)$ of a graph G is the intersection graph of all its (maximal) cliques. A graph G is said to be *K-divergent* if the sequence of orders of its iterated clique graphs $|K^n(G)|$ tends to infinity with n , otherwise it is *K-convergent*. *K*-divergence is not known to be computable and there is even a graph on 8 vertices whose *K*-behaviour is unknown. It has been shown that a regular Whitney triangulation of a closed surface is *K*-divergent if and only if the Euler characteristic of the surface is non-negative. Following this remarkable result, we explore here the existence of *K*-convergent and *K*-divergent (Whitney) triangulations of compact surfaces and find out that they do exist in all cases except (perhaps) where previously existing conjectures apply: It was conjectured that there is no *K*-divergent triangulation of the disk, and that there are no *K*-convergent triangulations of the sphere, the projective plane, the torus and the Klein bottle. Our results seem to suggest that the topology still determines the *K*-behaviour in these cases.

1. INTRODUCTION

Our graphs are finite and simple. A *clique* of a graph G is a maximal complete subgraph of G , or just its set of vertices, as we identify induced subgraphs with their vertex sets. The *clique graph* $K(G)$ is the intersection graph of the cliques of G , this construction is similar to that of the line graph, indeed the clique graph coincides with the line graph whenever G is triangle-free with minimum degree at least 1. Here, we are interested in the dynamical behaviour of G under the iteration of the clique operator K . It is known that the clique operator exhibits a much richer dynamics than the line graph [28]. The *iterated clique graphs* $K^n(G)$ are defined by $K^0(G) = G$, $K^{n+1}(G) = K(K^n(G))$. There are two main types of *clique behaviour*: If the sequence $G, K(G), K^2(G), \dots$ has only a finite number of non-isomorphic graphs (equivalently, there are $0 \leq m < n$ such that $K^m(G) \cong K^n(G)$), we say that G is *K-convergent*, otherwise G is *K-divergent*: the order of $K^n(G)$ tends to infinity with n . We refer to [33] for a survey on clique graphs, and to [1, 2, 4–8, 10, 16–26, 34] for recent work on them. Iterated clique graphs have been applied to Loop Quantum Gravity [29–31] and to the study of the fixed point property in posets [9].

If G is a graph, its *Whitney complex* is the simplicial complex $\Delta(G)$ whose simplices are the complete subgraphs of G . A simplicial complex Δ will then be called a *Whitney complex* if there is a graph G such that $\Delta = \Delta(G)$ (such a G would have to be the 1-skeleton of Δ). A finite simplicial complex is not necessarily Whitney, but its first barycentric subdivision certainly is. We denote by $|\Delta(G)|$ the geometric realization of the complex $\Delta(G)$, and call it the *geometric realization* of the graph G . Whitney complexes are also called *clique complexes* in the literature, but in this term the use of *clique* is not consistent with ours, so we prefer naming these complexes after H. Whitney, who proved in [35] that any graph G such that $|\Delta(G)|$ is the two dimensional sphere is Hamiltonian. Rather than making explicit mention of $\Delta(G)$ or even $|\Delta(G)|$, we can apply topological concepts or constructions directly to G , for instance the Euler characteristic $\chi(G)$ of G is just $\chi(\Delta(G))$, and we say that G is a sphere, or a torus, if $|\Delta(G)|$ is so. We will also say that the graph G is a *Whitney triangulation* of the topological space X if G (or $|\Delta(G)|$, if you will) is homeomorphic to X .

Recall that a *compact surface* is a connected, compact and Hausdorff space \mathcal{S} in which any point has a neighbourhood homeomorphic to an open set in the closed upper half-plane. The *interior points* of \mathcal{S} have a neighbourhood homeomorphic to an open disk, and all others are the points of $\partial\mathcal{S}$, the *border* of \mathcal{S} . (From

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now on any further use of "neighbourhood" is graph theoretical.) If $\partial\mathcal{S} = \emptyset$ (i.e. \mathcal{S} is *borderless*) then \mathcal{S} is called *closed*, but in general $\partial\mathcal{S}$ has a finite number of connected components (the *holes* of \mathcal{S}), each homeomorphic to the circle \mathbb{S}^1 . If \mathcal{S} is not closed, attaching a closed disk \mathbb{D} to each hole H (by means of a homeomorphism $\partial\mathbb{D} \rightarrow H$) results in a closed surface, so each compact surface can be obtained from a closed surface by removing a finite number of open disks whose closures are mutually disjoint. As well as closed surfaces are completely characterized by their Euler characteristic and their orientability character, so are compact surfaces by their Euler characteristic, orientability character and number of holes, although we prefer to use the associated closed surface and the number of holes.

It is well known that all compact surfaces can be triangulated, so taking subdivisions if needed we have that they also admit Whitney triangulations. The Whitney triangulations of compact surfaces are characterized as those graphs in which the open neighbourhood of each vertex is either a cycle of length at least four (interior vertices) or a path of length at least one (border vertices). A graph is *locally cyclic* if the open neighbourhood of every vertex is a cycle. The Whitney triangulations of closed surfaces are thus the locally cyclic graphs (except the tetrahedron K_4) and in the regular case their clique behaviour is completely determined by their topology:

Theorem 1.1. [14] *If the graph G is regular and $|\Delta(G)|$ is a closed surface, then G is K -convergent if and only if its Euler characteristic $\chi(G)$ is negative.*

If we drop the regularity condition, this result no longer holds:

Theorem 1.2. [16, Thm. 4.3] *Every closed surface admits a K -divergent Whitney triangulation.*

And, on the other hand, we have:

Theorem 1.3. [15, Thm. 3.2] *Every closed surface with negative Euler characteristic admits a K -convergent Whitney triangulation.*

In fact, we conjecture that each Whitney triangulation of a closed surface with non-negative Euler characteristic is K -divergent. Therefore, in the non-regular case, it is only in the cases of the sphere, the projective plane, the torus and the Klein bottle that the topology of the closed surface could determine the clique behaviour of its Whitney triangulations.

In this work we consider the extension to compact surfaces of Theorems 1.2 and 1.3 and prove:

Theorem 1.4. *Every compact surface which is not homeomorphic to the disk, admits a K -divergent Whitney triangulation.*

Theorem 1.5. *If \mathcal{S} is a compact surface, which is not homeomorphic to the sphere, the projective plane, the torus nor the Klein bottle, then \mathcal{S} admits K -convergent Whitney triangulation.*

It is noteworthy that, even if the passage from closed surfaces to compact ones involves a new non-negative integral parameter (the number of holes), the unsolved cases in Theorems 1.3 and 1.5 remain the same, while only one additional unsolved case appears in Theorem 1.4 where none existed in Theorem 1.2. In a sense, our new two theorems for compact surfaces both strengthen and unify our previous conjectures that any Whitney triangulation of a closed surface with non-negative Euler characteristic is K -divergent [14, 15] and any Whitney triangulation of the disk is K -convergent [14]. Besides interlacing topology and clique graphs, an interesting feature of this work is the way it brings together various concepts and techniques (large local girth, clockwork graphs, retractions, coverings, rank divergence) from several other works. We will quickly review the needed language and results as they are needed.

Theorem 1.5 will be proved in section 2, whereas sections 3, 4 and 5 constitute the proof of Theorem 1.4.

2. CONVERGENT SURFACES

Recall that the *girth* of a graph G is the minimum length of its cycles (it is infinite if G does not have cycles). The *local girth* of G at a vertex v is the girth of the open neighbourhood of v . The *local girth* of G is the minimum of the local girths of G , and we say that G has *large local girth* if its local girth is at least 7. If G is a compact surface, then G has large local girth if and only if the minimum degree of its interior vertices is at least 7, as the local girth at the vertices in the border is infinite.

Theorem 2.1. [14, Thm. 8] *Any graph G with large local girth is K -convergent. In fact, $K^3(G) \cong K(G)$.*

In view of Theorems 1.3 and 2.1, in order to prove Theorem 1.5 it will be enough to give, for each compact surface with at least one hole, a Whitney triangulation with large local girth.

A sphere with one hole is just the disk, so even a triangle would do in this case. However, for the remaining cases it will be convenient to have larger triangulations of the disk. For any integers $q \geq 7$ and $r \geq 1$, our *r-layered disk* D_r is the disk of radius r of a vertex in the 1-skeleton of the triangulation of the hyperbolic plane by equilateral triangles with interior angle $2\pi/q$. Alternatively, just consider the disk triangulations D_1, D_2 and D_3 in Figure 1, and it should be clear what D_r is for each $r \geq 1$: we start with the q -wheel $D_1 = W_q$ and each new layer is added in such a way that the vertices in the border of the previous disk attain degree q .

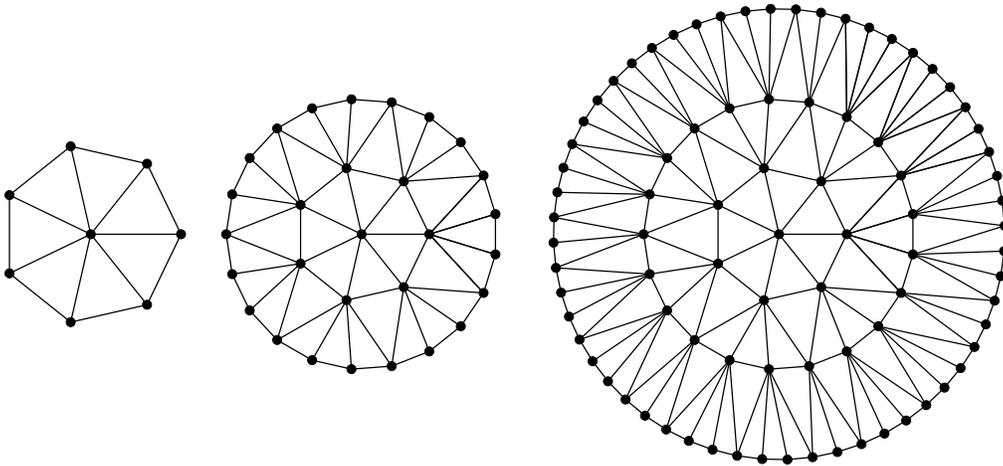


FIGURE 1. The triangulations D_1, D_2 and D_3 of the disk for $q = 7$.

Consider an edge $e = uv$ in D_r with r large enough. We need e to be *far from the border*: both u and v are interior vertices and have no common neighbours in the border of D_r . Then deleting e also removes two (interior) triangles of D_r , and we are left with a disk D' with a hole (a sphere with two holes), the new hole being bordered by a quadrilateral. If we need a third hole in our sphere, we remove another edge e' , but now taking care that e' is far from the border of D' . Continuing in this way we can, if r is large enough, obtain Whitney triangulations with large local girth of the sphere with any positive number of holes. This same considerations will also apply in the remaining cases, so in what follows it will be enough to give, for any closed surface \mathcal{S} other than the sphere, a Whitney triangulation with large local girth of the punctured surface \mathcal{S}° with just one hole.

If \mathcal{S} is the orientable surface of genus $g \geq 1$, then \mathcal{S}° is a sphere with g handles and a disk removed, i.e. a disk with g handles. In a disk D_r with sufficiently many layers, find $2g$ edges that are far from the border and also *very far* from each other in the following sense: for each pair of these edges, if we consider the quadrilateral holes produced by their deletion, the minimal distance in D_r from a vertex in one hole to one in the other is at least four. After removing these $2g$ edges, match the resulting holes by pairs and identify each of these quadrilaterals with its partner in any way of the four that preserve orientability. Taking this

quotient does not produce any new triangles because the chosen $2g$ edges were very far from each other, and the resulting graph triangulates the surface \mathcal{S}° . The interior vertices have still degree at least g , as those coming from the identification of 2 vertices have degree at least $2g - 2$.

If \mathcal{S} is the non-orientable surface of even Euler characteristic $\chi(\mathcal{S}) = -2k \leq 0$, then \mathcal{S}° is obtained by removing an open disk from the connected sum of $k + 1$ Klein bottles, so a Whitney triangulation can be constructed analogously to the previous case: start with a large D_r , remove $2(k + 1)$ edges which are far from the border and very far from each other, and match the quadrilateral holes by pairs. Now identify each of these quadrilaterals with its partner in a way that does not preserve orientability, and the resulting graph will triangulate the surface \mathcal{S}° and have large local girth. In fact, not all the gluings of matched holes have to be non-orientable, it is enough that at least one is.

If \mathcal{S} is the projective plane, \mathcal{S}° is a Möbius band, and this is obtained by identifying non-orientably two edges on the border of a large D_r , again taking care that this quotient does not produce any new triangles: the distance from a vertex in one of those edges to one in the other must be at least four.

The last case is that of the non-orientable surface \mathcal{S} of odd Euler characteristic $\chi(\mathcal{S}) = 1 - 2k < 0$. Here \mathcal{S}° is the connected sum of k Klein bottles and a projective plane with an open disk removed, so a combination of the previous two cases will clearly produce the sought-after triangulation.

Now that Theorem 1.5 is proved we shall devote our next three sections to the proof of Theorem 1.4.

3. DIVERGENT SURFACES: THE SPHERE

Except for the disk, we shall show the existence of a K -divergent Whitney triangulation for any compact surface \mathcal{S} . If \mathcal{S} is closed we use Theorem 1.2, so we can assume that \mathcal{S} has at least one hole.

This section is devoted to the sphere, so we shall assume that \mathcal{S} is a sphere with at least two holes (the sphere with one hole is the disk, which is not covered by Theorem 1.4). A sphere with two holes is the *cylinder* (or *annulus*) and we can use the triangulation in Figure 2.

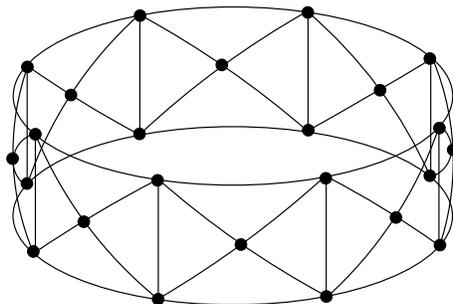
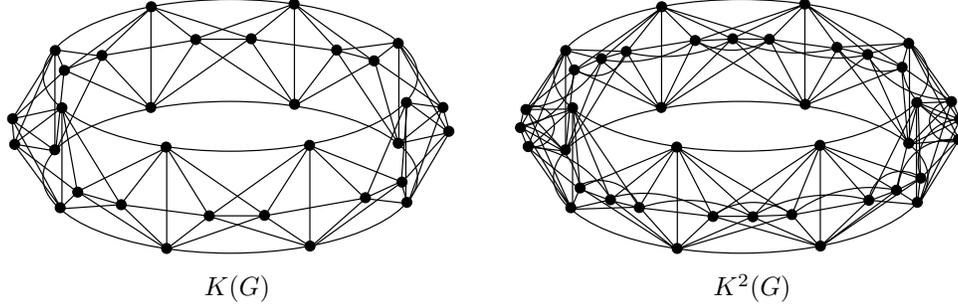


FIGURE 2. A K -divergent Whitney triangulation G of the cylinder.

This cylinder has eight *segments*, but any number $s \geq 4$ of segments will do. To construct the cylinder with $s \geq 5$ segments, start with a cylindrical ladder with s steps and then replace each quadrilateral with a 4-wheel, i.e. add a new common neighbour to the vertices of each quadrilateral. To construct the cylinder with 4 segments replace by 4-wheels the four walls of a cube, leaving alone the top and the bottom. All these cylinders are particular cases of *clockwork graphs*, and are known to be K -divergent since [11, 13]. Figure 3 shows the first two iterated clique graphs of the graph G in Figure 2. The cylindrical ladder is invariably present in all iterated clique graphs, but the wheel centers are $(n + 1)$ -plicated in $K^n(G)$, and they altogether induce the n -th power of a cycle of length $(n + 1)s$ for each $n \geq 0$. All of these are proved in detail in [11, 13].

For spheres with more than two holes we will use retractions. A *graph morphism* $\rho : G \rightarrow G'$ is a function $\rho : V(G) \rightarrow V(G')$ such that for any edge vv' of G the images $\rho(v)$ and $\rho(v')$ are either adjacent or equal. A *retraction* $\rho : G \rightarrow G'$ is a morphism that has a *right inverse*, that is a morphism $\sigma : G' \rightarrow G$ such that

FIGURE 3. The clique graph and the second clique graph of the graph G in Figure 2.

the composition $\rho \circ \sigma$ is the identity on G' . In this situation one says that G' is a *retract* of G , or that G *retracts* to G' . Notice that G' is isomorphic to the image of σ and that this is an induced subgraph of G , so an induced subgraph H is a retract of G if and only if there is a morphism $\pi : G \rightarrow H$ that restricts to the identity on H . It is quite clear that the composition of two retractions is again a retraction. Neumann-Lara's Retraction Theorem says that the clique graph operator behaves well with respect to retractions:

Theorem 3.1. [27] *For any retraction $\rho : G \rightarrow G'$ there is a retraction $\rho_K : K(G) \rightarrow K(G')$. In consequence, if G' is K -divergent, then so is G .*

A graph morphism $\pi : G \rightarrow G'$ is a *quotient map* if it is surjective in both vertices and edges, i.e. any $u' \in V(G')$ and $v'w' \in E(G')$ are of the form $u' = \pi(u)$ and $v'w' = \pi(v)\pi(w)$ for some $u, v, w \in V(G)$ with $vw \in E(G)$. In this case G' can be recovered from the (not necessarily connected) graph G by identifying any two vertices whenever they have the same image under π (this is an equivalence relation) and declaring adjacent any two distinct equivalence classes whenever they have adjacent representatives, hence the name “quotient map”, or also *natural projection* for π . Any retraction $\rho : G \rightarrow G'$ is a quotient map, as for any vertex or edge x' of G' we have $x' = \rho(\sigma(x'))$. The converse is not true in general.

Lemma 3.2. *Let $\rho : A \rightarrow G$ be a retraction, and let $\pi : A \rightarrow B$ be a quotient map such that $\pi(a) = \pi(a')$ implies $\rho(a) = \rho(a')$ for all $a, a' \in V(A)$. Then there exists a unique retraction $\rho' : B \rightarrow G$ such that $\rho' \circ \pi = \rho$, i.e. the following diagram commutes:*

$$\begin{array}{ccc}
 A & \xrightarrow{\pi} & B \\
 \rho \downarrow & & \swarrow \rho' \\
 & & G
 \end{array}$$

Proof: If $b \in V(B)$, any $a \in A$ with $\pi(a) = b$ will define the same $\rho'(b) = \rho(a)$ by hypothesis, and if $bb' \in E(B)$ we can take $aa' \in E(A)$ with $\pi(aa') = bb'$ because π is a quotient map. Thus, ρ' is a well defined graph morphism, and by construction makes the diagram commute. Let $\sigma : G \rightarrow A$ be a right inverse to ρ , and put $\sigma' = \pi \circ \sigma : G \rightarrow B$. Then ρ' is a retraction with right inverse σ' , because $\rho' \circ \sigma' = \rho' \circ \pi \circ \sigma = \rho \circ \sigma = 1_G$. Since π is vertex-surjective, it cancels on the right, so ρ' is unique. \square

The cylinders discussed before can be said to be single level cylinders, because piling up several of those (all with the same number of segments) we can construct many-storied cylinders as shown in Figure 4. Any of these new cylinders C retracts to its single level cylinder at ground level: keeping fixed this *ground cylinder* G , send any ladder vertex not in it to the bottommost vertex below it, and then it is clear where to send any wheel center. Since C retracts to G and G is K -divergent, C is K -divergent by Theorem 3.1.

Now we can construct K -divergent triangulations for the sphere with $k + 2$ holes, $k \geq 1$. We start with a large many-storied cylinder. It can have just three levels, but then a large number of segments will be

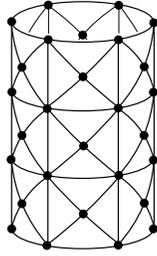
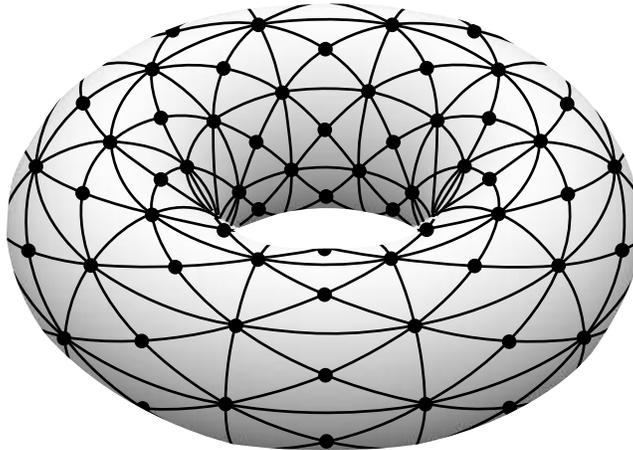


FIGURE 4. The cylinder with 6 segments and 3 stories.

needed, or it can have only four segments, but with many stories. If $k = 1$, we delete a step in the ladder above the ground cylinder G . Note that this edge is far from the border of the original cylinder (in the sense of the previous section), so after its deletion the resulting surface is a sphere with three holes. If $k > 1$, we take a new step edge not in G , far from the border of this new surface and delete it. Continuing this way we will end up with a Whitney triangulation of our many-holed sphere. As a graph, this triangulation is a subgraph of the original large cylinder and still contains the original ground-level cylinder, so the restriction of the above-discussed retraction shows, by Theorem 3.1, that our triangulation is K -divergent.

4. DIVERGENT SURFACES: THE ORIENTABLE CASE

Starting with a cylinder C with $s \geq 4$ segments and $t \geq 4$ stories as in Figure 4, identify each vertex in the upper border with that in the lower one lying directly below it, obtaining our *torus* $T(s, t)$, see Figure 5.

FIGURE 5. The torus $T(8, 10)$.

Using retractions, we show that these tori $T = T(s, t)$ are K -divergent: Let $\pi : C \rightarrow T$ be the natural projection, let us call G the ground cylinder of C , and let $\rho : C \rightarrow G$ be the retraction constructed in the previous section. Since π only identifies each vertex in the ceiling of the topmost story of C with the vertex in the floor of G lying below it, and ρ also identifies these vertices, we have a retraction $\rho' : T \rightarrow G$ by Lemma 3.2, and T is K -divergent by Theorem 3.1. We can also denote by G the image of G under the right inverse σ' of ρ' and call it the *ground cylinder* of the torus T , but notice that by the symmetry of T we have t different one level cylinders in T , and T clearly retracts to any of them.

It should now be clear that we can exhibit, in a similar way as in the previous sections, a K -divergent Whitney triangulation for the torus with any number of holes: taking $s + t$ large enough and deleting k edges of T far away from each other and not in the ground cylinder, the resulting graph would be a Whitney

triangulation of a torus with k holes and would still retract to the ground cylinder. There is, however, a neater way to do this that does not require such a large T .

For the punctured torus (one hole) we can proceed as just mentioned even starting with $T = T(4, 4)$, and obtain a torus T° whose only hole H is a quadrilateral. Now, if $k > 1$, the idea is to attach a many-holed pipe to T° . Take a cylinder C with four segments and $(k - 1)$ additional holes (a sphere with $2 + (k - 1)$ holes) as we did in the previous section, and identify the hole H of T° with the floor B of the ground cylinder G of C , which is also a quadrilateral, and call T' the resulting graph, which is then a Whitney triangulation of the torus with k holes. It is quite clear that C retracts to B : indeed, as the composition of two retractions is again a retraction, it is enough to check that G retracts to B . Using a retraction from C to B we can, in an obvious way, obtain one from T' to T° and, since this last is K -divergent, so is T' by Theorem 3.1.

The sphere and the torus having been disposed of, we need now to consider, for each $g \geq 2$, the orientable surface \mathcal{S} of genus g . Since the case of more than one hole can be dealt with by attaching a many-holed pipe, it is enough to construct a K -divergent Whitney triangulation of the punctured surface \mathcal{S}° .

Let T be one of our tori $T(s, t)$, call G its ground cylinder and fix two edges $e, f \in E(T)$ which are not in G and such that f is far from the border E of $T - e$, so $T - \{e, f\}$ is a torus with two holes with border $E \cup F$, where E and F are two disjoint quadrilaterals of T (F is the border of $T - f$). Now take g disjoint copies T_1, T_2, \dots, T_g of T . For any object X in T (vertex, edge, subgraph, etc.) we shall denote by X_i the corresponding object in T_i , for instance G_i will be the ground cylinder of T_i .

Put $T'_1 = T_1 - e_1$ and also define $T'_i = T_i - \{e_i, f_i\}$ for $2 \leq i \leq g$. Thus T'_1 is a torus with one hole whose border is the quadrilateral E_1 while T_i is a torus with the two holes E_i and F_i for $2 \leq i \leq g$. Let $A = \cup T'_i$ be the union of all the T_i and consider the following identifications in A . For each odd i with $1 \leq i < g$, identify E_i with E_{i+1} in the natural way (each vertex v_i of E_i with its twin v_{i+1} in E_{i+1}), and for each even i with $2 \leq i < g$, identify F_i with F_{i+1} also in the natural way. For $1 < i < g$, both holes of T_i were identified with other holes, and the same happened for the only hole E_1 of T_1 and one of the two holes of T_g , so only one of the holes remained: E_g or F_g , according to whether g is odd or even. Call B the resulting quotient graph, which is the connected sum of $g - 1$ tori and one punctured torus, so B is a Whitney triangulation of our punctured orientable surface \mathcal{S}° of genus g . We will call $\pi : A \rightarrow B$ the natural projection. Note that, if $x, y \in V(A)$, $x \neq y$ and $\pi(x) = \pi(y)$, we must have a vertex v of T such that $x = v_i$ and either $y = v_{i+1}$ or $y = v_{i-1}$ for some index i with $1 \leq i \leq g$.

Let us consider a retraction $\rho_0 : T \rightarrow G$. For any i with $1 \leq i \leq g$ we obtain, using the natural identification of T with T_i and restricting to T'_i , a retraction $\rho_i : T'_i \rightarrow G$. The union $\rho = \cup \rho_i : A \rightarrow G$ of all these retractions is also a retraction. Indeed, for any j , the right inverse $\sigma'_j : G \rightarrow T'_j$ of ρ_j , followed by the inclusion map $T'_j \rightarrow A$, is a right inverse of ρ . We observed above that π only identifies pairs of different vertices when they are of the form v_i and v_{i+1} , for some $v \in T$, but then $\rho(v_i) = \rho_0(v) = \rho(v_{i+1})$, and therefore π and ρ satisfy the hypotheses of Lemma 3.2, so there is a retraction $\rho' : B \rightarrow G$. Since G is K -divergent so is B by Theorem 3.1.

5. DIVERGENT SURFACES: THE NON-ORIENTABLE CASE

To construct K -divergent Whitney triangulations for non-orientable compact surfaces we will rely on covering techniques. The observant reader will have noticed that a morphism between two graphs $f : G \rightarrow H$ is exactly the same as a simplicial map $f : \Delta(G) \rightarrow \Delta(H)$ between their Whitney complexes. In a similar way, our covering maps for graphs coincide, at the level of the Whitney complexes, with the simplicial covering maps between them. We quickly review the needed material from [12].

A *covering map* $p : G \rightarrow H$ is a graph morphism that satisfies, in the obvious senses, the unique edge lifting property and the (necessarily unique) triangle lifting property. Equivalently, a covering map is just a *local isomorphism*, i.e. a graph morphism $p : G \rightarrow H$ such that the restriction to the closed neighbourhoods $p| : N_G[v] \rightarrow N_H[p(v)]$ is a graph isomorphism for each vertex $v \in V(G)$. In general, the simplicial covering maps do not coincide with the local isomorphisms (see [32]) but we are restricting to Whitney complexes. Here we only consider finite graphs, so our covering maps will all be finite (i.e. finite-to-one).

Theorem 5.1. [12, Prop. 2.2, Cor. 2.3] *If $p : G \rightarrow H$ is a covering map, there is a covering map $p_K : K(G) \rightarrow K(H)$ that has the same number of sheets as p . In particular, G is K -divergent if, and only if, H is K -divergent.*

A group Γ of automorphisms of the graph G is *admissible* if $d(v, \gamma(v)) > 3$ for each $v \in V(G)$ and $1 \neq \gamma \in \Gamma$. Admissible groups are the analogues of groups acting in a properly discontinuous way in topology. Indeed, denoting by G/Γ the quotient graph of the orbits under Γ of the vertices of G and by $p : G \rightarrow G/\Gamma$ the natural projection, we have:

Theorem 5.2. [12, Lem. 3.1] *If $\Gamma \leq \text{Aut}(G)$ is admissible, then $p : G \rightarrow G/\Gamma$ is a covering map.*

We shall apply this theorem in the simple case where Γ is a two-element group generated by an r -coaffine automorphism of G . Since r will be 4, we shall restrict to $r = 4$ in our much simplified review of the needed material from [16], and will omit most references to either r or 4 (our term *coaffine* is *4-coaffine* in [16]).

We say that $u, v \in V(G)$ are *far* (from each other) if $d(u, v) \geq 4$. A graph automorphism γ of G is *coaffine* (or a *coaffination* of G) if $\gamma(v)$ is far from v for each $v \in V(G)$. Notice that if this γ has order 2, then it generates a two-element admissible group. A *coaffine graph* is a graph G together with a fixed coaffination γ of G . For such a coaffine graph, a *coaffine subgraph* is a subgraph H such that $\gamma(V(H)) \subseteq V(H)$, i.e. γ can be restricted to a coaffination of H .

If G contains a K -divergent subgraph H , it by no means follows that G is K -divergent (for instance if G is obtained from H by adding a new vertex adjacent to all others, $K^2(G)$ is the one vertex graph). However, if H is a coaffine subgraph of G , there is a stronger kind of K -divergence, namely rank divergence, that propagates from H to G .

Let G be a coaffine graph whose coaffination γ is of order two, and let Γ be the two-element admissible group generated by γ . Let us say that $u, v \in V(G)$ are *close* if they are not far, i.e. $d(u, v) < 4$. Two Γ -orbits Γu and Γv are said to be *close* if any vertex in Γu is close to any vertex in Γv . Any two close Γ -orbits are different, since the vertices inside each of them are far from each other. The *rank* of G is the maximum possible cardinality of a set of mutually close Γ -orbits, and G is *rank divergent* if the rank of $K^n(G)$ is unbounded as n tends to infinity. As the rank of a coaffine graph is certainly lesser than its number of vertices, rank divergence implies K -divergence. The following two theorems are particular cases of results in [16].

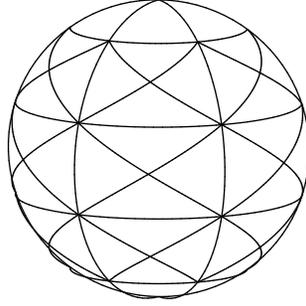
Theorem 5.3. [16, Thm. 2.6] *If H is a coaffine subgraph of G and H is rank divergent, then so is G and, in consequence, G is K -divergent.*

Let us call M the single level cylinder with 6 segments considered in Section 3, this has two segments less than the one in Figure 2. Also the single level cylinders with more than 6 segments would do for what follows, but to avoid technicalities we will stick to this M . Let $\gamma : M \rightarrow M$ be the symmetry consisting of a half-turn around the central axis of M followed by the top-down reflection. Direct inspection shows that γ is a coaffination of order two.

Theorem 5.4. [16, Thm. 3.1] *The cylinder M , with the coaffination γ , is rank divergent.*

Piling at least three copies of M , construct a many-storied cylinder C as that in Figure 4. We need an odd number of copies of M so as to have a middle one, the *middle cylinder* M of C . A half-turn plus the up-down reflection still define a coaffination of C , and the middle cylinder M is a coaffine subgraph of C . Now replace the hexagonal holes of C with 6-wheels, thus obtaining a Whitney triangulation S of the sphere which is coaffine with the only extension γ of the coaffination of C (interchange the two wheel centers, i.e. the *poles*). Now we can not retract S to M , but the middle cylinder is a coaffine subgraph of S , and S is K -divergent by Theorems 5.4 and 5.3. We can draw S as a sphere in Euclidean space in such a way that its coaffination γ is the antipodal map, see Figure 6.

Take an edge $e \in E(S)$ incident to the north pole, and call E the quadrilateral border of $S - e$. The antipodal image $e' = \gamma(e)$ of e is far from E , so $W^{\circ\circ} = S - \{e, e'\}$ is a sphere with two holes E, E' . It is coaffine and still has the middle cylinder M as a coaffine subgraph, so $W^{\circ\circ}$ is rank divergent.

FIGURE 6. The front hemisphere of S , three stories.

Using two disjoint copies S_0, S_1 of S , define $S'_i = S_i - \{e_i, e'_i\}$, $i = 1, 2$. In $S'_0 \cup S'_1$ identify, in the natural way of Section 4, E_0 with E_1 and also E'_0 with E'_1 . The resulting graph W is a Whitney surface triangulation: these identifications create no new triangles as the distance from a vertex in E to one in E' is at least two. A nice way [3] to see that W is a torus is to consider S'_1 as a bigger concentric sphere around S'_0 in such a way that twin vertices are aligned to the center, so the identifications can be achieved by just moving radially the vertices and edges of E_0 and E'_0 until they coincide with those of E_1 and E'_1 . The edges joining the moving quadrilaterals with their complements in S'_0 get dragged during the motion, but at the end they only touch S'_1 in the vertices of the target quadrilaterals. It is clear that the union $\gamma'_0 \cup \gamma'_1$ of the (restricted) coaffinations is a well defined coaffination of W . As W has (even two) rank divergent coaffine subgraphs (M_0 and M_1), it is K -divergent by Theorem 5.3. Fix now a ladder edge $f \in E(S)$ of the upper cylinder (i.e. an edge from the Arctic Circle to the Tropic of Cancer) such that the quadrilateral border F of $S - f$ is disjoint from E . If we remove f_1 and f'_1 from W , we end up with a coaffine graph $W^{\circ\circ}$ which is a rank divergent triangulation of the torus with two holes.

In general, for each $g \geq 0$, use $g + 1$ disjoint copies S_0, S_1, \dots, S_g of S and define $S'_0 = S_0 - \{e_0, e'_0\}$ and $S'_i = S_i - \{e_i, e'_i, f_i, f'_i\}$ for $i = 1, 2, \dots, g$. For even i with $0 \leq i < g$, identify E_i and E'_i with E_{i+1} and E'_{i+1} in the natural way. For odd i with $1 \leq i < g$, identify F_i and F'_i with F_{i+1} and F'_{i+1} also in the natural way. Thus we have a K -divergent coaffine Whitney triangulation $W^{\circ\circ}$ of the orientable surface of genus g with two holes, where the coaffination γ is of order two and reverses the orientation.

Now consider the admissible group $\Gamma \leq \text{Aut}(W^{\circ\circ})$ generated by γ , denote by $W' = W^{\circ\circ}/\Gamma$ the orbit graph under the action of Γ , and by $p : W^{\circ\circ} \rightarrow W'$ the natural projection. By Theorem 5.2, p is a covering map. Since covering maps are local isomorphisms and the open neighbourhoods of the vertices of $W^{\circ\circ}$ are either paths or cycles, so are the open neighbourhoods of the vertices of W' , and W' is a compact surface. Since Γ identifies the two holes of $W^{\circ\circ}$, then W' has just one hole. Since $W^{\circ\circ}$ is orientable and γ reverses the orientation, then W' is non-orientable. Since p is a covering map and $W^{\circ\circ}$ is K -divergent, then W' is K -divergent by Theorem 5.1.

Now consider any non-orientable compact surface \mathcal{S}' with just one hole. Denote by $f : \tilde{\mathcal{S}}' \rightarrow \mathcal{S}'$ the orientation cover of \mathcal{S}' , which is a two-to-one covering map where $\tilde{\mathcal{S}}'$ is connected, orientable and has two holes. Then $\tilde{\mathcal{S}}'$ admits a Whitney triangulation $W^{\circ\circ}$ as above. The natural projection for the quotient $W' = W^{\circ\circ}/\Gamma$ is also a two-to-one covering map $p : W^{\circ\circ} \rightarrow W'$, hence $2\chi(\mathcal{S}') = \chi(\tilde{\mathcal{S}}') = \chi(W^{\circ\circ}) = 2\chi(W')$. Being non-orientable compact surfaces with one hole and the same Euler characteristic, \mathcal{S}' and W' are homeomorphic. Now that we have a K -divergent triangulation for any non-orientable compact surface with one hole, the general case follows from Theorem 3.1 by attaching, as in Section 4, a retractable many-holed pipe.

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