DISCRETE MORSE THEORY
AND THE HOMOTOPY TYPE OF CLIQUE GRAPHS

F. LARRIÓN, M.A. PIZAÑA, AND R. VILLARROEL-FLORES

Abstract. We attach topological concepts to a simple graph by means of the simplicial complex of its complete subgraphs. Using Forman's discrete Morse theory we show that the strong product of two graphs is homotopic to the topological product of the spaces of their complexes. As a consequence, we enlarge the class of clique divergent graphs known to be homotopy equivalent to all its iterated clique graphs.

1. Introduction

Our graphs are finite and simple. As in Harary's book [12], to which we refer for graph theory, a *clique* of a graph G is a maximal complete subgraph of G, or just its set of vertices, as we identify induced subgraphs with their vertex sets. The *clique graph* $K(G)$ is the intersection graph of the cliques of G. The *iterated clique graphs* $K^n(G)$ are defined recursively by $K^0(G) = G$, $K^{n+1}(G) = K\big(K^n(G)\big)$. If the sequence $G, K(G), K^2(G), \dots$ has only a finite number of non-isomorphic graphs (equivalently, there are $0 \leq m < n$ such that $K^m(G) \cong K^n(G)$), we say that G is *K-convergent*, otherwise G is *K-divergent*. We refer to [23] for a survey on clique graphs, and to [1, 4, 6, 19] for recent work on them. Iterated clique graphs have been applied to Loop Quantum Gravity [22].

If G is a graph, its *Whitney complex* is the simplicial complex $\Delta(G)$ whose simplices are the complete subgraphs of G. A simplicial complex Δ will then be called a *Whitney complex* if there is a graph G such that $\Delta = \Delta(G)$. (Such a G would have to be the 1-skeleton of Δ.) We denote by $|G|$ the geometric realization of the complex $\Delta(G)$, and call it the *geometric realization* of the graph G. We will say that the graphs G_1, G_2 are *homotopy equivalent*, and denote it by $G_1 \simeq G_2$, if their geometric realizations $|G_1|$ and $|G_2|$ are so. Whitney complexes are also called *clique complexes* in the literature, but in this term the use of *clique* is not consistent with ours, so we prefer naming these complexes after H. Whitney, who proved in [25] that any graph G such that $|G|$ is the two dimensional sphere is Hamiltonian. Note that every order complex $\Delta(P)$ (where P is a poset) is a Whitney complex $\Delta(G)$, where G is the comparability graph of P.

Several studies have considered the relation between the topology of the Whitney complex of a graph and the dynamical behavior of the graph under the clique operator, as well as the effect of the clique operator on the homotopy type of a graph. More specifically, the first general condition ensuring that a graph is homotopy equivalent to its clique graph ($G \simeq K(G)$) was given by Prisner in [21] (see our § 5). A stronger result was found by Larrión, Neumann-Lara and Pizaña in [14], which was further strengthened and generalized by the present authors in [18]. Moreover, two infinite families of K-divergent graphs that satisfy $G \simeq K^n(G)$ for all $n \geq 0$ were shown in [17].

In this note we apply Forman's discrete Morse theory [11] to the analysis of the homotopy type of Whitney complexes. In §2 we review quickly the needed facts from this theory, and use it to prove a result of Welker which generalizes previous conditions under which a vertex or an edge can be removed from a graph without affecting its homotopy type. In §3 we illustrate the use of this result by giving a family of graphs G such that we can determine explicitly the homotopy type of $K(G)$ and $K^2(G)$. Our chief application of discrete Morse theory (Theorem 4.2) can be of independent interest, it shows that, given any two graphs G_1, G_2, the following holds:

$$|G_1 \boxtimes G_2| \simeq |G_1| \times |G_2|, \tag{1}$$

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where we are using Nešetřil's notation for the boxtimes product (also called strong product). In fact, the equivalence (1) is derived from a *simple homotopy*, that is, one obtained by collapsing free faces. Since the clique operator distributes over the boxtimes product [20], Theorem 4.2 can be used, as we shall do in §5, to extend the class of known K-divergent graphs such that all their iterated clique graphs have the same homotopy type. We also prove that given any graph G, there is another graph G' which is K-divergent and has the same homotopy type as G.

2. Elements of discrete Morse theory

Let Δ be a (finite) simplicial complex, which we identify with its set of simplices, or faces. We follow Jonsson's presentation [13, Chap. 4] of Forman's discrete Morse theory and all its conventions, in particular, we admit that $\varnothing \in \Delta$ whenever the set of vertices of Δ is not empty. The reader can also find the fundamentals on homotopy type of simplicial complexes in [13]. A *matching* $\mathcal{M}$ on Δ is a family of pairs $\{\sigma, \tau\} \subseteq \Delta$ such that no face of Δ is in more than one pair. The faces that are in a pair of $\mathcal{M}$ are said to be *matched*, the rest are called *critical*. We say that $\mathcal{M}$ is an *element matching* if for every pair of $\mathcal{M}$ there are $\sigma \in \Delta$ and a vertex v such that said pair is equal to $\{\sigma - v, \sigma + v\}$. We will consider only element matchings. For any fixed vertex v we always have an element matching $\mathcal{M}(v)$ on Δ given by:

$$\mathcal{M}(v) = \{\, \{\sigma - v, \sigma + v\} \mid \sigma + v \in \Delta \,\}\,. \tag{2}$$

If $\mathcal{M}$ is an element matching, the digraph $D(\Delta, \mathcal{M})$ has vertex set Δ, and an arrow $\sigma \to \tau$ if either:

(1) $\{\sigma, \tau\} \in \mathcal{M}$, and $\tau = \sigma + v$ for some $v \notin \sigma$, or
(2) $\{\sigma, \tau\} \notin \mathcal{M}$ and $\sigma = \tau + v$ for some $v \notin \tau$.

We say that $\mathcal{M}$ is *acyclic* if $D(\Delta, \mathcal{M})$ has no directed cycle. For $\sigma, \tau \in \Delta$ we write $\sigma \rightsquigarrow \tau$ if there is a directed path in $D(\Delta, \mathcal{M})$ from σ to τ. For $\mathcal{V}, \mathcal{W} \subseteq \Delta$ we write $\mathcal{V} \rightsquigarrow \mathcal{W}$ if there are $\sigma \in \mathcal{V}$, $\tau \in \mathcal{W}$ such that $\sigma \rightsquigarrow \tau$, otherwise we will write $\mathcal{V} \not\rightsquigarrow \mathcal{W}$. The following is a special case of Lemma 4.1 of [13]:

Lemma 2.1. *Let the vertex v of Δ be such that $\sigma + v \in \Delta$ for all $\sigma \in \Delta$, i.e. Δ is a* cone *with* apex *v. Then the element matching $\mathcal{M}(v)$ of equation* (2) *is in fact an acyclic matching on Δ with no critical simplices.*

Lemma 2.2. (Lemma 4.2 from [13]) *Let Δ be a simplicial complex, Q a poset, and $F \colon \Delta \to Q$ a poset map, where Δ is ordered by inclusion. For each $q \in Q$, let $\mathcal{M}_q$ be an acyclic matching on Δ such that all elements of the pairs of $\mathcal{M}_q$ are elements of $F^{-1}(q)$, and let $\mathcal{M} = \cup_{q \in Q} \mathcal{M}_q$. Then $\mathcal{M}$ is an acyclic matching on Δ.*

The following is a reformulation of a theorem due to Forman ([11]).

Theorem 2.3. (Theorem 4.4 from [13]) *Let $\mathcal{M}$ be an acyclic matching on a simplicial complex Δ. Assume that the subcomplex Δ_0 contains all the critical faces, and that $\Delta_0 \not\rightsquigarrow \Delta \setminus \Delta_0$. Then Δ collapses to Δ_0.*

In particular, in Theorem 2.3, $|\Delta|$ and $|\Delta_0|$ are homotopy equivalent. On the other hand, if Δ collapses to its subcomplex Δ_0, there is an acyclic matching $\mathcal{M}$ on Δ satisfying the hypotheses of Theorem 2.3. The complex Δ is called *collapsible* if it collapses to the one-vertex complex, or, equivalently, if it admits an acyclic matching with no critical faces. The *link* of σ in Δ is the complex $\mathrm{lk}_\Delta(\sigma) = \{\, \tau \in \Delta \mid \sigma \cup \tau \in \Delta, \sigma \cap \tau = \varnothing \,\}$. We will use the notation $[\sigma, \infty) = \{\, \rho \in \Delta \mid \sigma \subseteq \rho \,\}$, and say that the face σ is *removable* if Δ collapses to $\Delta \setminus [\sigma, \infty)$. Notice that $[\sigma, \infty) = \{\, \sigma \cup \tau \mid \tau \in \mathrm{lk}_\Delta(\sigma) \,\}$.

If the face σ of Δ is such that $\mathrm{lk}_\Delta(\sigma)$ is collapsible, let $\mathcal{M}$ be an acyclic matching on $\mathrm{lk}_\Delta(\sigma)$ with no critical faces. Now take $\mathcal{M}' = \{\, \{\sigma \cup \tau, \sigma \cup \rho\} \mid \{\tau, \rho\} \in \mathcal{M} \,\}$, which is an acyclic matching in Δ whose critical faces are precisely those of the subcomplex $\Delta_0 = \Delta \setminus [\sigma, \infty)$, so $\Delta_0 \not\rightsquigarrow \Delta \setminus \Delta_0$ and we have, by Theorem 2.3, the following result of Welker:

Proposition 2.4. (Lemma 2.7 from [24]) *Any face with collapsible link is removable.*

In the case of a Whitney complex $\Delta(G)$ the removals of vertices and edges of the graph G are specially useful, because they lead again to Whitney complexes. The link in $\Delta(G)$ of a vertex $v \in G$ is just (the Whitney complex of) its open neighborhood $N(v)$, the set of vertices adjacent to v, and the link of an edge $e = \{u, v\}$ is the common neighborhood $N(e) = N(u) \cap N(v)$. Some particular cases of Proposition 2.4 have been discovered for Whitney complexes. The removability of a vertex v such that $N(v)$ is a cone goes back to Prisner [21, Prop. 3.2] and was rediscovered in the context of independence complexes, see [9, Lemma 2.4]. That any edge e with $N(e)$ complete is removable was used in [17, Prop. 2.3]. A variant of the removability of vertices and edges with collapsible links (s-dismantlability) appears in [5, Prop.1.7, Lemma 1.6].

3. A family of examples

If the space X is homeomorphic to $|\Delta(G)|$ we say that the graph G is a *Whitney triangulation* of X. A graph is a Whitney triangulation of some compact surface if, and only if, the open neighborhood of each vertex is either a cycle with at least four vertices or a path with at least two, with paths occurring only when the surface is not closed [15]. We shall illustrate the usefulness of Proposition 2.4 by determining the homotopy type of the second clique graph for a family of Whitney triangulations of the two-dimensional sphere $\mathbb{S}^2$.

We start by classifying vertices of $K^2(G)$ into two types: the stars and the neckties. In any graph G, for $v \in G$ we denote by v^* the set $\{ Q \in K(G) \mid v \in Q \}$, which is called a *star* with *center* v and is a complete subgraph (sometimes even a clique) of $K(G)$. Any clique of $K(G)$ which is not a star is called a *necktie*.

The *girth* of a graph G is the length of a shortest cycle in G. The *local girth* of a graph G, denoted by $\lg(G)$, is the minimum of the girths of the open neighborhoods of vertices of G. For example, from the first paragraph of this section we obtain that $\lg(G) \geq 4$ if G is a Whitney triangulation of a compact surface. More generally, if $\lg(G) \geq 4$, we can define a particular case of neckties. A triangle T in a graph G is called *inner* if for any of its three edges there is a triangle T' such that $T \cap T'$ is precisely that edge. Given an inner triangle T in G, the *necktie centered at* T is the set of all cliques of G that share at least an edge with T. It is shown in [15] that if $\lg(G) \geq 4$, they are indeed neckties, and that if $\lg(G) \geq 5$, all neckties are centered at some inner triangle of G.

Lemma 3.1. *Let G be a Whitney triangulation of a closed surface, different from the octahedron (where the octahedron is the complement of three disjoint edges). Then:*

1. *the clique graph $K(G)$ is obtained from the dual graph of G, adding all edges between vertices on the same face,*
2. *for all vertices $v \in G$, v^* is a clique in $K(G)$,*
3. *every clique of $K(G)$ is either a star, or a centered necktie,*
4. *the subgraph of $K^2(G)$ induced by the stars is isomorphic to G,*
5. *the subgraph of $K^2(G)$ induced by the neckties is obtained from the dual of G, adding edges between vertices at distance two and on the same face.*

Proof: The cliques of G are its triangles, so the vertices of $K(G)$ correspond bijectively to the vertices of the dual graph of G. Two vertices in $K(G)$ are adjacent whenever they share an edge or a vertex. But two triangles share an edge whenever their corresponding vertices of the dual are adjacent, and two triangles share exactly a vertex whenever their corresponding vertices in the dual are in the same face. This proves (1).

Now, if v^* were not a clique for some $v \in G$, then there would be a triangle T intersecting all the triangles (at least 4, the surface is closed) that contain v, but with $v \notin T$. However, this would imply that there is a triangle in $N_G(v)$, contradicting that $|G|$ is a compact surface, and proving (2).

For (3), we just observe that the proof of Proposition 10 in [15] still works under the hypotheses that $\lg(G) \geq 4$ and G does not have an induced octahedron.

For (4), we have that v, w are adjacent in G if and only if $\{v, w\} \subseteq Q$ for some clique Q, this is equivalent to $Q \in v^* \cap w^*$, and this means that v^*, w^* (which are vertices of $K^2(G)$ by (2)) are adjacent in $K^2(G)$ because they are distinct (in our case, they only share the two triangles containing the edge $\{v, w\}$).

Now, for (5), we have in the paragraph before Theorem 11 from [15], that under our hypothesis two neckties $Q_T, Q_{T'}$ (centered respectively on the triangles T, T') are adjacent in $K^2(G)$ if and only if T, T' share an edge, or they share exactly one vertex and are joined by a *crossbar* (that is, an edge joining a vertex $a \in T$ to some vertex $b \in T'$, where neither a nor b is the point in $T \cap T'$). Hence the adjacencies in $K^2(G)$ determine exactly the adjacencies in the dual of G mentioned in (5), and this finishes the proof. □

If $n \geq 5$, let G be the suspension of the n-gon C_n. Then G is obtained from C_n adding two new vertices which we call the *poles* and are adjacent to all the *equatorial* vertices of C_n but not to each other. Then G is a Whitney triangulation of $\mathbb{S}^2$. We view G as inscribed in the unit sphere with the poles in the z-axis and the equatorial vertices in the xy-plane.

The cliques of G are its triangles, and by Lemma 3.1.1, the clique graph $K(G)$ is the n-gonal prism $\mathcal{P}_1$, but with the faces made complete. We view $K(G)$ as inscribed in a smaller sphere, centered also at the origin, in such a way that the n-gonal faces of $\mathcal{P}_1$ are horizontal, so the n rectangular *walls* are vertical. By the results of [14] and [18] mentioned in the introduction, the only Whitney triangulation of a compact surface which is not homotopic to its clique graph is the octahedron, so $K(G) \simeq G$.

From Lemma 3.1.4, the stars induce a subgraph of $K^2(G)$ which is isomorphic to G and they can be depicted as above in the unit sphere. From Lemma 3.1.3, the neckties correspond to the triangles of G, and induce a subgraph of $K^2(G)$ which is isomorphic to an n-gonal prism with the rectangular faces made complete and the n-gonal faces squared (that is, add the chords between vertices at distance 2), and can be depicted in the smaller sphere as above in such a way that, for any equatorial vertex $v \in G$, the star v^* projects radially into the center of the wall of $\mathcal{P}_1$ defined by the neckties centered on each of the four triangles in v^* (this is *the wall of* v).

The connections between the two subgraphs are as follows. For each pole $v \in G$ the star v^* intersects all the neckties, whereas, if $v \in C_n$ is an equatorial vertex of G, the star v^* only intersects the eight neckties in the union of the wall of v with the two neighboring walls. In fact, a star intersects a necktie if, and only if, the center of the star lies in the union of the necktie (that is, the set of all vertices in some clique of the necktie)

Proposition 3.2. *Let G be the suspension of C_n for $n \geq 5$. Then $K(G) \simeq G \simeq \mathbb{S}^2$, $K^2(G) \simeq \mathbb{S}^3$ if $n = 5$, $K^2(G) \simeq \mathbb{S}^3 \vee \mathbb{S}^3 \vee \mathbb{S}^3$ if $n = 6$, and $K^2(G) \simeq \mathbb{S}^2$ if $n \geq 7$.*

Proof: Let us denote by $\mathcal{P}_2$, the subgraph of $K^2(G)$ induced by the neckties. We observe that $K^2(G)$ is the suspension of its subgraph H induced by the neckties and the equatorial stars. We will prove first that $H \simeq \mathcal{P}_2$. See Figure 1 for a partial depiction of H in the case that $n = 5$.

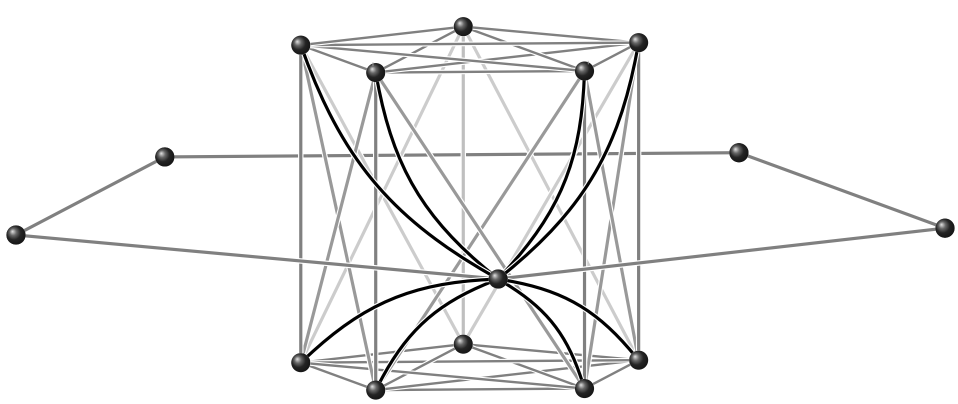

FIGURE 1. The graph H for $n = 5$. Only the edges from one of the equatorial stars to $\mathcal{P}_2$ were depicted.

Consider an edge $e = \{u^*, v^*\}$ in H between equatorial stars. Then $N_H(e)$ consists of the 6 vertices of $\mathcal{P}_2$ in the union of the walls of u and v, and this is a cone with apex in the intersection of those walls. By

Proposition 2.4 we can remove from H the edges between equatorial stars, obtaining a graph which is homotopic to H and which we still call H.

Now the open neighborhood $N = N_H(v^*)$ of any equatorial star v^* consists of the eight vertices of $\mathcal{P}_2$ in the union of the wall of v with the two neighboring walls. By Proposition 2.4, we will be able to remove the equatorial stars, thus ending our proof that $H \simeq \mathcal{P}_2$, if we show that N is collapsible.

Take a vertex x in N, but not in the wall of v. Then $N_N(x)$ is the cone over a path of length three if $n = 5$ and a diamond (that is, a cone over a path of length two) if $n \geq 6$, hence we can remove x from N by Proposition 2.4. The resulting subgraph N' is a cone and therefore collapsible. It follows that $H \simeq \mathcal{P}_2$, as we claimed. In Figures 2, 3 and 4 we show the graph $\mathcal{P}_2$ for several values of n.

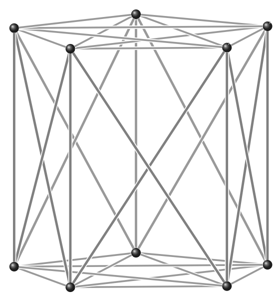

FIGURE 2. $n = 5$

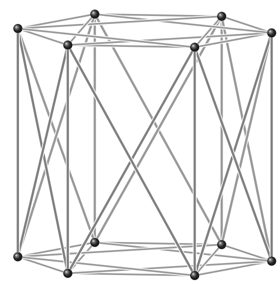

FIGURE 3. $n = 6$

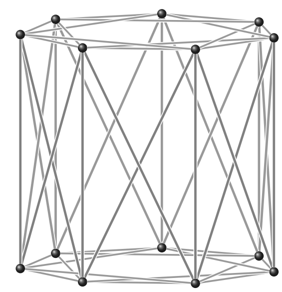

FIGURE 4. $n = 7$

Now, for $n = 5$, we have that $\mathcal{P}_2$ is isomorphic to $K(G)$, which we know is homotopic to $\mathbb{S}^2$.

For $n = 6$, consider a (graph theoretic) perfect matching in $\mathcal{P}_2$, formed with one diagonal edge from each wall. By Proposition 2.4 we can successively remove each edge of the matching, since its link is a path of length three, hence contractible. More precisely, if we number the vertices at the top of $\mathcal{P}$ with the numbers 1 to 6 and those at the bottom with 7 to 12, with 1 above 7 and both increasing in the same direction, then we remove the edges $2-7$, $3-8$, etc. Call T the the resulting graph. Then the 2-dimensional complex $\Delta(T)$ has a subcomplex Δ_0, defined as $\Delta_0 = \Delta(T) \setminus \{\{1,3,5\},\{2,4,6\},\{7,9,11\}\}$, and which is depicted in Figure 5. The complex Δ_0 is collapsible. Applying Lemma 10.2 from [3] by collapsing Δ_0 in Δ, we obtain that $\Delta(T) \simeq \mathbb{S}^2 \vee \mathbb{S}^2 \vee \mathbb{S}^2$, which finishes the proof of the claim for $n = 6$.

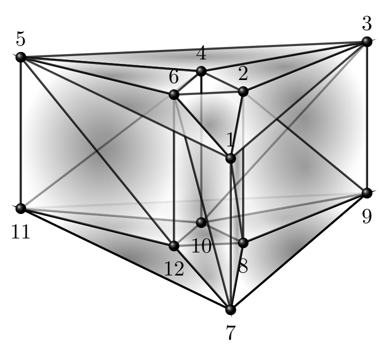

FIGURE 5. Collapsible subcomplex Δ_0

Finally, for $n \geq 7$ clearly we can remove all the chords in both the top and the bottom of $\mathcal{P}_2$ to get a graph which is homotopic to $\mathbb{S}^1$.

Since $K^2(G)$ is the suspension of H and H is homotopy equivalent to $\mathcal{P}_2$, it follows that $K^2(G)$ is homotopy equivalent to the suspension of $\mathcal{P}_2$, and the theorem follows. □

4. The boxtimes product

We begin with some definitions and remarks for simplicial complexes in general. If Δ and Δ' are simplicial complexes, their *categorical product* is the complex $\Delta \otimes \Delta'$ with vertex set $V(\Delta) \times V(\Delta')$ and $\sigma \in \Delta \otimes \Delta'$ whenever $\pi(\sigma) \in \Delta$ and $\pi'(\sigma) \in \Delta'$, where π and π' are the projections. As observed in [7, §3], one problem is that the categorical product "is not well behaved, in the sense that it does not commute with geometric realization". The usual way to solve this problem is to order the simplexes. In each of Δ and Δ', fix a partial order of the set of vertices such that each simplex is totally ordered, and give $V(\Delta) \times V(\Delta')$ the product order. Then the *ordered product* of Δ and Δ' is the simplicial complex $\Delta \oslash \Delta'$ whose simplexes are the totally ordered subsets σ of $V(\Delta) \times V(\Delta')$ such that $\pi(\sigma) \in \Delta$ and $\pi'(\sigma) \in \Delta'$. For instance, $\Delta(K_2) \otimes \Delta(K_2) \cong \Delta(K_4)$ and $\Delta(K_2) \oslash \Delta(K_2) \cong \Delta(D)$, where D is the diamond. The fact that $|\Delta \oslash \Delta'|$ is homeomorphic to the topological product $|\Delta| \times |\Delta'|$ is well known, see for instance [8, p. 68] or [7, Prop.3.7].

We now turn to the corresponding products for graphs. Recall that the *boxtimes product* $G_1 \boxtimes G_2$ of two graphs G_1, G_2, has $V(G_1 \boxtimes G_2) = V(G_1) \times V(G_2)$. If $(x, y) \neq (x', y')$ then (x, y) is adjacent to (x', y') in $G_1 \boxtimes G_2$ if and only if x, x' are equal or adjacent in G_1 and also y, y' are equal or adjacent in G_2. The boxtimes product is also called *strong product* in the literature. Now fix a partial order in each of $V(G_1)$, $V(G_2)$ in such a way that adjacent vertices are always comparable (we could, for example, just take total orders). With respect to these choices of partial orders, we define an *ordered product* $G_1 \text{œ} G_2$, that has the same vertex set $V(G_1) \times V(G_2)$ as $G_1 \boxtimes G_2$, and in which (x, y) is adjacent to (x', y') if they are adjacent in the boxtimes product and they are comparable as elements in the product poset $V(G_1) \times V(G_2)$. The proof of the following is straightforward:

Lemma 4.1. *Let G_1, G_2 be graphs and fix partial orders as above. Then:*

$$\Delta(G_1 \boxtimes G_2) = \Delta(G_1) \otimes \Delta(G_2),$$
$$\Delta(G_1 \text{œ} G_2) = \Delta(G_1) \oslash \Delta(G_2).$$

We can prove now another application of discrete Morse theory:

Theorem 4.2. *For any graphs G_1, G_2, the complexes $\Delta(G_1 \boxtimes G_2)$ and $\Delta(G_1 \text{œ} G_2)$ are simple-homotopic, and so the space $|G_1 \boxtimes G_2|$ is homotopic to the product space $|G_1| \times |G_2|$.*

Proof: Let G_1, G_2 be two finite graphs. We will use the notation $A = V(G_1)$, $B = V(G_2)$, and fix a partial order in each of A, B in such a way that adjacent vertices are comparable. Using Lemma 4.1, we have by the result of [8, p. 68] that $|G_1 \text{œ} G_2| \simeq |G_1| \times |G_2|$; hence it is enough to show that $|G_1 \boxtimes G_2| \simeq |G_1 \text{œ} G_2|$. For simplicity's sake, we will denote by ab any ordered pair $(a, b) \in A \times B$.

We intend to apply Lemma 2.2 in order to get an acyclic matching on $\Delta(G_1 \boxtimes G_2)$. For this, we define a poset Q with underlying set:

$$Q = \{\, (ab, a'b') \in (A \times B) \times (A \times B) \mid a > a', b < b' \,\}, \tag{3}$$

and order relation given by:

$$(ab, a'b') > (cd, c'd') \text{ if } \begin{cases} a > c, \text{or} \\ a = c, \text{ and } b > d, \text{ or} \\ a = c, b = d, \text{ and } a' > c' \text{ or} \\ a = c, b = d, a' = c' \text{ and } b' > d'. \end{cases} \tag{4}$$

We observe that if σ is a complete subgraph of $G_1 \boxtimes G_2$, then the set of ordered pairs of elements of σ which are in Q is totally ordered under (4). This is because if $(ab, a'b'), (cd, c'd') \in \sigma \times \sigma$, then $\{a, a', c, c'\}$ is a complete subgraph in G_1 and $\{b, b', d, d'\}$ is a complete subgraph in G_2. On the other hand, it is immediate that $\sigma \in \Delta(G_1 \boxtimes G_2) \setminus \Delta(G_1 \text{œ} G_2)$ if and only if there are $ab, a'b' \in \sigma$ such that $(ab, a'b') \in Q$.

We define a map $F\colon \Delta(G_1 \boxtimes G_2) \setminus \Delta(G_1 \text{œ} G_2) \to Q$, such that:

$$F(\sigma) = \max \{\, (ab, a'b') \in Q \mid \{ab, a'b'\} \subseteq \sigma \,\}$$

We have that for $\sigma, \sigma' \in \Delta(G_1 \boxtimes G_2) \setminus \Delta(G_1 \text{ œ } G_2)$, the inclusion $\sigma \subseteq \sigma'$ implies that $F(\sigma) \leq F(\sigma')$, so F is a poset map. We now aim to prove that if $\sigma \in F^{-1}\big((ab, a'b')\big)$, then $\sigma + a'b \in F^{-1}\big((ab, a'b')\big)$ and $\sigma - a'b \in F^{-1}\big((ab, a'b')\big)$.

Let p_A (p_B) be the natural projection $A \times B \to A$ ($A \times B \to B$). First, since $a' \in p_A(\sigma)$ and $b \in p_B(\sigma)$ we have $\sigma - a'b \subseteq \sigma + a'b \subseteq p_A(\sigma) \times p_B(\sigma)$, but since $p_A(\sigma), p_B(\sigma)$ are complete subgraphs, so is $p_A(\sigma) \times p_B(\sigma)$. Hence $\sigma + a'b$ and $\sigma - a'b$ are both complete subgraphs of $G_1 \boxtimes G_2$. Now $a'b \notin \{ab, a'b'\}$ implies $F(\sigma - a'b) = (ab, a'b')$.

Next, we will prove that any possible pair of elements in Q that consists of $a'b$ and some $xy \in \sigma$ is less or equal than $(ab, a'b')$.

Suppose then that there is $xy \in \sigma$ such that $(xy, a'b) \in Q$ (so that $x > a'$, $y < b$). It cannot happen that $x > a$, since then $(xy, ab) \in Q$ is greater than the maximum $(ab, a'b')$ among pairs of elements of σ. It follows that $x \leq a$, but then, since by hypothesis $y < b$, $(xy, a'b) < (ab, a'b')$. Then, suppose that $xy \in \sigma$ and $(a'b, xy) \in Q$. Since $a > a'$, it follows that $(ab, a'b') > (a'b, xy)$. Hence $F(\sigma + a'b) = (ab, a'b')$.

Now by Lemma 2.1 the element matching $\mathcal{M}(a'b)$ on $F^{-1}((ab, a'b'))$ is an acyclic matching without critical simplices. By Lemma 2.2, we have defined an acyclic matching $\mathcal{M} = \bigcup_{(ab,a'b') \in Q} \mathcal{M}(a'b)$ on $\Delta(G_1 \boxtimes G_2) \setminus \Delta(G_1 \text{ œ } G_2)$ without critical simplices. From Theorem 2.3, we get that $\Delta(G_1 \boxtimes G_2)$ collapses to $\Delta(G_1 \text{ œ } G_2)$. Hence we have $|G_1 \boxtimes G_2| \simeq |G_1 \text{ œ } G_2| \simeq |G_1| \times |G_2|$. □

5. Homotopy clique permanent graphs

Let us say that the graph G is *homotopy clique invariant* (or just *homotopy K-invariant*) if it is homotopic to its clique graph: $K(G) \simeq G$. Of course this happens whenever $K(G) \cong G$, but a graph needs not to be clique invariant to be homotopy clique invariant. Not every graph is homotopy K-invariant: The n-octahedron $\mathcal{O}_n$ (i.e. the complement of a 1-factor in K_{2n}) is homeomorphic to the sphere $\mathbb{S}^{n-1}$, and Neumann-Lara [10] proved that $K(\mathcal{O}_n) = \mathcal{O}_{2^{n-1}}$ for $n \geq 3$. As mentioned before, the first general condition ensuring that a graph is homotopy clique invariant was given by Prisner in [21]: Any clique-Helly graph is homotopy K-invariant. Here, a graph is *clique-Helly* if any collection of pairwise intersecting cliques has a nonempty intersection, as for instance triangleless graphs or cones. Prisner's result was generalized and strengthened in [14] and [18], and the sufficient conditions found there are satisfied by any graph which is free of induced octahedra and whose cliques are at most triangles, so all these graphs are homotopy clique invariant. In particular, except for the 3-octahedron, any Whitney triangulation of a compact surface (with or without border) is homotopy K-invariant.

Call the graph G *homotopy clique permanent* (or *homotopy K-permanent*) if it is homotopic to all its iterated clique graphs: $K^n(G) \simeq G$ for all $n \geq 0$.

Not every homotopy K-invariant graph is homotopy K-permanent, as the first two graphs of Proposition 3.2 explicitly show. Indeed, we propose the following Conjecture:

Conjecture 5.1. *Every suspension of an n-cycle for $n \neq 3$ is not homotopy K-permanent.*

If a family $\mathcal{F}$ of graphs is closed under the clique operator ($K(G) \in \mathcal{F}$ for all $G \in \mathcal{F}$) and each $G \in \mathcal{F}$ is homotopy clique invariant, it is clear that each $G \in \mathcal{F}$ is homotopy clique permanent. For instance, Escalante proved in [10] that the clique graph of a clique-Helly graph is again clique-Helly, so by the above-mentioned result of Prisner we see that all clique-Helly graphs are homotopy K-permanent.

Another example is the class of dismantlable graphs: A vertex $x \in G$ is called *dominated* if there is a vertex y, $y \neq x$, such that $N[x] \subseteq N[y]$ (in other words, $N(x)$ is a cone, y is the apex). The class of *dismantlable* graphs can be defined recursively: the one-vertex graph is dismantlable, and a graph G with at least two vertices is dismantlable if it has a dominated vertex x with $G - x$ dismantlable. Every dismantlable graph G is contractible by Prisner [21]. We also have that $K(G)$ is again dismantlable by Bandelt and Prisner [2]. Hence all dismantlable graphs are homotopy K-permanent.

There is a common feature in the previous two families of homotopy K-permanent graphs. If G is either clique-Helly or dismantlable, we certainly have that $K^n(G) \simeq G$ for all n, but the family of the $K^n(G)$ is finite up to isomorphism, as G is K-convergent. Indeed, clique-Helly graphs are K-convergent by Escalante [10], and dismantlable graphs by Bandelt and Prisner [2].

However, the present authors showed in [17] that it is in fact possible for a K-divergent graph G to satisfy $K^n(G) \simeq G$ for all n, giving two infinite families of K-divergent, homotopy K-permanent graphs. All members in one family (clockwork graphs) have the homotopy type of the circle $\mathbb{S}^1$ and all graphs in the other family (locally C_6 triangulations of the torus) have the homotopy type of the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Now, with the help of Theorem 4.2, we can in particular obtain infinitely many non homotopy-equivalent, clique divergent graphs G having homotopy type different from $\mathbb{S}^1$ and $\mathbb{S}^1 \times \mathbb{S}^1$ and with the property that $K^n(G) \simeq G$ for all n:

Theorem 5.2. *Let $G_1, G_2 \ldots, G_n$ be graphs, and $G = G_1 \boxtimes G_2 \cdots \boxtimes G_n$. Then:*

(1) *If all G_i are homotopy clique invariant, then so is G.*
(2) *If all G_i are homotopy clique permanent, then so is G.*
(3) *If at least one G_i is clique divergent, then so is G.*

Proof: The proof is immediate from Theorem 4.2 and the fact, proven by Neumann-Lara in [20], that $K(G_1 \boxtimes G_2)$ is isomorphic to $K(G_1) \boxtimes K(G_2)$. Item (3) had already been noted in [20]. □

We conclude with still another application of Theorem 4.2 and the previous observations.

Theorem 5.3. *For any graph G there is a K-divergent graph G' such that $G \simeq G'$.*

Proof: The existence of a contractible and K-divergent graph B is shown explicitly in [16]. The graph $G' = G \boxtimes B$ then satisfies our claim. □

Remarks 5.4.

(1) Since for every simplicial complex Δ there is a graph G with $\Delta(G) \simeq \Delta$ (it suffices to take the 1-skeleton of the barycentric subdivision of Δ), it follows that every "interesting" homotopy type is realized by a K-divergent graph. As opposed to this, we do not know if there is any K-convergent graph homotopic to $\mathbb{S}^2$, and we conjecture that there is none.
(2) We could obtain homotopy K-invariant and divergent graphs of homotopy types different than those of $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, taking strong product of $\mathbb{T}^n$ with a clique-Helly graph, for example, a graph consisting of several 4-cycle subgraphs sharing a vertex, which clearly has the homotopy type of a wedge of 1-spheres.

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