Searching for square-complementary graphs: complexity of recognition and further nonexistence results

Ratko Darda

Institut de Mathématiques de Jussieu – Paris Rive Gauche (IMJ-PRG), Université Paris Diderot, France

Martin Milanič

University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia
University of Primorska, IAM, Muzejski trg 2, 6000 Koper, Slovenia

Miguel Pizaña

Universidad Autónoma Metropolitana, 09340 Mexico City, Mexico

Abstract

A graph is *square-complementary* (*squco*, for short) if its square and complement are isomorphic. We prove that there are no squco graphs with girth 6, that every bipartite graph is an induced subgraph of a bipartite squco graph, that the problem of recognizing squco graphs is graph isomorphism complete, and that no nontrivial squco graph is both bipartite and planar. These results resolve three of the open problems posed in Discrete Math. **327** (2014) 62–75.

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1 Introduction

Given a graph G, the *square* of G is the graph denoted by G^2 with the same vertex set as G in which two distinct vertices are adjacent if and only if they

Email addresses: `ratko.darda@imj-prg.fr` (Ratko Darda), `martin.milanic@upr.si` (Martin Milanič), `map@xanum.uam.mx` (Miguel Pizaña).
URL: `http://xamanek.izt.uam.mx/map` (Miguel Pizaña).

are at distance at most two in G. Squares of graphs and their properties are well-studied in literature (see, e.g., Section 10.6 in the monograph [7]). A graph G is said to be *square-complementary* (*squco* for short) if its square is isomorphic to its complement, that is, $G^2 \cong \overline{G}$, or, equivalently, $G \cong \overline{G^2}$. Examples of squco graphs are K_1, C_7 (the 7-cycle), and a cubic vertex-transitive bipartite squco graph on 12 vertices, known as the Franklin graph (see Fig. 1). We say that a graph is a *bipartite squco* graph whenever it is both bipartite and square-complementary.

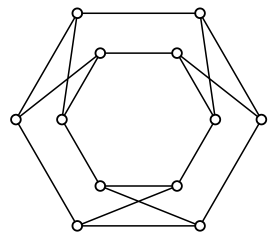

Fig. 1. The Franklin graph.

The terminology "square-complementary" ("squco") was suggested in [21], however the problem of characterizing squco graphs is much older; it was posed by Seymour Schuster at a conference in 1980 [25]. Since then, squco graphs were studied in the context of graph equations, which may in general involve a variety of operators including the line graph and complement, see, e.g., [3, 5, 9–11, 23]. The entire set of solutions of some of these equations was found (see for example [3] and references quoted therein). The set of solutions of the equation $G^2 \cong \overline{G}$ remains unknown, despite several attempts to describe it (see for example [5, 10, 21]). The problem of determining all squco graphs was also posed as Open Problem No. 36 in Prisner's book on graph dynamics [23]. Note that if we consider graphs up to isomorphism, then squco graphs are precisely the fixed points of the function ϕ defined on the class of all finite graphs by the rule $\phi(G) = \overline{G^2}$.

Every nontrivial squco graph has diameter 3 or 4 [10], but it is not known whether a squco graph of diameter 4 actually exists. In [21], several other questions regarding squco graphs were posed, and a summary of known necessary conditions for squco graphs was given. Among them (see Proposition 2.3), it was proved that the 7-cycle is the only squco graph of girth at least 7. This result leaves only five possible values for the girth g of a squco graph G, namely $g \in \{3, 4, 5, 6, 7\}$. The case $g = 7$ is completely characterized by Proposition 2.3. Baltić et al. [5] and Capobianco and Kim [10] asked whether there exists a squco graph of girth 3. An affirmative answer to this question was provided in [21] by a squco graph on 41 vertices with a triangle (namely, the circulant $C_{41}(\{4, 5, 8, 10\})$). As shown by the Franklin graph, there also exists a squco graph of girth 4. The questions regarding the existence of squco graphs of girth 5 or 6 were left as open questions in [21]. In Section 3, we answer one of them, namely Open Problem 3 in [21], by proving that there is no squco graph of girth 6. This leaves $g = 5$ as the only possible value of g for which the existence of a squco graph of girth g is unknown.

An important special case of the problem of characterizing squco graphs is the case of bipartite graphs. In this case, the squco property is closely related to the bipartite version of the property of being self-complementary. The *bipartite complement* of a bipartite graph G with bipartition $\{A, B\}$ is the graph $\overline{G}^{\mathrm{bip}}$ obtained from G by replacing $E(G)$ by $[A, B] \setminus E(G)$, where $[A, B]$ denotes the set of all pairs consisting of a vertex in A and a vertex in B. A bipartite graph G is said to be *bipartite self-complementary* if it is isomorphic to its bipartite complement. Bipartite self-complementary graphs are extensively studied in the literature, see, e.g., [1, 6, 15–19, 22, 24, 27]. Gangopadhyay and Hebbare proved in [19] that every connected bipartite self-complementary graph with at least two vertices has diameter at least 3 and at most 6. Note that every bipartite graph G with diameter 3 satisfies $\overline{G^2} = \overline{G}^{\mathrm{bip}}$. Thus, such a graph is squco if and only if it is bipartite self-complementary. As the next result shows, the connection between the two concepts is even stronger: except for the one-vertex graph, the bipartite squco graphs are exactly the bipartite self-complementary graphs with diameter 3.

Theorem 1.1 [21, Theorem 5.2] *For a nontrivial bipartite graph G, the following conditions are equivalent:*

(1) G is squco.

(2) G is bipartite self-complementary and of diameter 3*.*

Theorem 1.1 also connects bipartite squco graphs with the concept of self-antipodal graphs. Given a graph G, the *antipodal graph* $A(G)$ is the graph with vertex set $V(G)$ in which two vertices are adjacent if and only if the distance between them equals the diameter of G. The graph G is *self-antipodal* if $A(G)$ is isomorphic to G. Acharya and Acharya showed in [2] that a nontrivial bipartite graph G is self-antipodal if and only if it is a bipartite self-complementary graph with diameter 3.

Infinite families of bipartite self-complementary graphs with diameter 3 were constructed in [19, Remark 2] and in [21, Theorem 5.7]. A further construction will be given in Section 4. Using Theorem 1.1, each of these constructions leads to an infinite family of bipartite squco graphs.

In Section 4 we also prove that every bipartite graph is an induced subgraph of a bipartite squco graph. This implies that squco graphs can contain arbitrarily long induced paths and cycles, thus solving in particular Open Problem 8(2) in [21], asking whether squco graphs can contain arbitrarily long induced paths. Furthermore, we show that the problem of recognizing squco graphs is graph isomorphism complete, which solves Open Problem 10 in [21].

Our final result deals with the case of planar squco graphs. It is not difficult to see that every graph obtained from the 7-cycle by replacing one of its vertices with a nonempty independent set is a planar squco graph. Thus, there exist arbitrarily large planar squco graphs. This fact, together with the existence of arbitrarily large bipartite squco graphs motivate the following questions: Are there arbitrarily large squco graphs that are both bipartite and planar? If this is not the case, what is the maximum order of a bipartite planar squco graph? We answer these questions in Section 5, by proving that K_1 is the only squco graph that is both bipartite and planar.

2 Preliminaries

We use standard graph terminology [14]. We only consider finite, undirected, simple graphs. We briefly recall some useful definitions. Given two vertices u and v in a connected graph G, we denote by $d_G(u,v)$ the distance in G between u and v (that is, the number of edges in a shortest u,v-path). The *eccentricity* of a vertex u is the maximum distance from u to any vertex in the graph. The *radius* and the *diameter* of G are defined as the minimum and the maximum eccentricity of a vertex in G, respectively. For a positive integer i, we denote by $N_i(v,G)$ the set of all vertices u in G such that $d_G(u,v)=i$, and by $N_{\geq i}(v,G)$ the set of all vertices u in G such that $d_G(u,v)\geq i$. The *girth* of a graph G is defined as the shortest length of a cycle in G (∞ if G is acyclic). A *cut vertex* in a connected graph G is a vertex whose removal disconnects the graph. A graph G is *bipartite* if it has a *bipartition*, that is, a partition of $V(G)$ into two disjoint sets A and B such that every edge of G joins a vertex in A with a vertex in B. Note that a connected bipartite graph has a unique bipartition up to swapping the two parts.

A graph is *planar* if it can be drawn on the plane without crossings among its edges; it is *plane* if it is already equipped with a fixed planar embedding. The *faces* of a plane graph are the connected regions of the difference between the plane and the image of the embedding. Let G be a connected plane graph with n vertices, m edges, and f faces. Then, a consequence of Euler's formula $n-m+f=2$ is the inequality $m\leq 3n-6$, which strengthens to $m\leq 2n-4$ if G is bipartite.

For the reader's convenience, we transcribe here the known results on squco graphs that we shall need.

Proposition 2.1 [5] *Every squco graph is connected and has no cut vertices.*

Proposition 2.1 implies that the minimum degree of every nontrivial squco graph is at least two.

Proposition 2.2 [5, 10] *If G is a nontrivial squco graph, then we have* $\mathrm{rad}(G)=3$ *and* $3\leq \mathrm{diam}(G)\leq 4$. *Moreover, if G is regular, then* $\mathrm{diam}(G)=3$.

Proposition 2.3 [21, Proposition 3.6] *If G is a nontrivial squco graph with girth at least 7, then G is the 7-cycle.*

Proposition 2.4 [21, Proposition 4.2] *The only nontrivial squco graph with maximum degree at most 2 is the 7-cycle.*

A graph is *subcubic* if it has maximum degree at most 3.

Proposition 2.5 [21, Proposition 4.7] *The only subcubic squco graph on 12 vertices is the Franklin graph (see Fig. 1).*

Given a graph G with vertices labeled $v_1, v_2, \ldots, v_n$ and positive integers $k_1, k_2, \ldots, k_n$, we denote by $G[k_1, k_2, \ldots, k_n]$ the graph obtained from G by replacing each vertex v_i of G with a set U_i of k_i (new) vertices and joining vertices $u_i \in U_i$ and $u_j \in U_j$ with an edge if and only if v_i and v_j are adjacent in G.

Theorem 2.6 [21, Theorem 4.8] *Let G be a graph with at most 11 vertices. Then, G is squco if and only if G is one of the following eight graphs:*

$$K_1, \quad C_7, \quad C_7[2,1,1,1,1,1,1], \quad C_7[3,1,1,1,1,1,1], \quad C_7[4,1,1,1,1,1,1],$$
$$C_7[1,2,1,2,2,1,1], \quad C_7[5,1,1,1,1,1,1], \quad C_7[2,1,1,2,1,2,2].$$

By Theorem 1.1, every bipartite squco graph has diameter at most 3. This implies the following.

Lemma 2.7 [21, Lemma 5.1] *Let G be a bipartite squco graph with bipartition $\{A, B\}$. Then, every two vertices in A have a common neighbor (in B).*

3 The girth of a squco graph is not 6

Theorem 3.1 *There is no squco graph of girth 6.*

Proof. Suppose for a contradiction that G is a squco graph of girth 6. First, we observe that if x is a vertex of G, then there are no edges in any of the sets $N_i(x, G)$ for $i = 1, 2$ and that no two distinct vertices in $N_1(x, G)$ have a common neighbor in $N_2(x, G)$. Let $k = \Delta(G)$ be the maximum degree of G and let w be a vertex of degree k. By Proposition 2.4, the only squco graphs with maximum degree at most 2 are K_1 and C_7; hence we have $k \geq 3$.

We consider two cases.

Case 1. w has a neighbor of degree at least three.

Let v be a neighbor of w of degree at least three, and let p and q be two neighbors of v other than w. If one of them, say p, is of degree at least 3, then p has at least two neighbors in $N_2(v, G)$ and thus

$$\Delta(\overline{G^2}) \geq |N_1(q, \overline{G^2})| \geq |N_1(w, G) \setminus \{v\}| + |N_1(p, G) \setminus \{v\}| = (k-1) + 2 = k+1\,,$$

(see Fig. 2), contrary to the fact that $\overline{G^2} \cong G$.

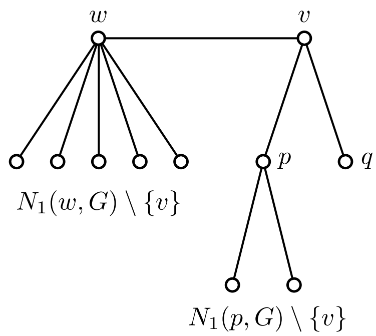

Fig. 2. Neighborhood of w and v in G when p has degree at least 3.

Since Proposition 2.1 excludes the possibility of having degree 1 vertices, both p and q are of degree 2. Thus, there exist unique vertices a and b contained in $N_1(p,G)\setminus\{v\}$ and in $N_1(q,G)\setminus\{v\}$, respectively. Since G has girth 6, vertices a and b are distinct, nonadjacent to each other, and nonadjacent to w. Clearly, they both belong to $N_2(v,G)$. The set $N_3(v,G)$ is nonempty, because the radius of G is 3 by Proposition 2.2. Vertices a and b must be adjacent to all vertices in $N_3(v,G)$, otherwise, if, say, vertex a is nonadjacent to a vertex $x \in N_3(v,G)$, then

$$\Delta(\overline{G^2}) \geq |N_1(p,\overline{G^2})| \geq |N_1(w,G) \setminus \{v\}| + |\{b,x\}| \geq k+1\,,$$

contrary to the fact that $\overline{G^2} \cong G$. To avoid a 4-cycle in G, we conclude that $|N_3(v,G)| = 1$. But now, the degree of v in $\overline{G^2}$ is 1, which implies that $\overline{G^2}$ has a cut vertex, contrary to the fact that G is squco and Proposition 2.1.

Case 2. All neighbors of w are of degree at most two.

In this case, all neighbors of w are of degree exactly two. In particular, $|N_2(w,G)| = |N_1(w,G)| = k \geq 3$. Now we will show that every vertex x from $N_2(w,G)$ is of degree at least $|N_3(w,G)|$. Let $x \in N_2(w,G)$. Since G is of girth more than 4, vertex x has a unique neighbor y in $N_1(w,G)$. Vertex x has at least $|N_3(w,G)|-1$ neighbors in $N_3(w,G)$, since otherwise $|N_1(y,\overline{G^2})| \geq k+1$. This implies that any two vertices from $N_2(w,G)$ (the size of $N_2(w,G)$ is at least 3) have at least $|N_3(w,G)| - 2$ common neighbors in $N_3(w,G)$. This bounds $|N_3(w,G)| \leq 3$, otherwise we would have a 4-cycle.

Also we have that $N_4(w,G) = \emptyset$ since, otherwise, for any $z \in N_4(w,G)$ we would have $|N_1(z,\overline{G^2})| \geq k+1 > \Delta(G) = \Delta(\overline{G^2})$, which is a contradiction. It follows that

$$|V(G)| = |\{w\} \cup N_1(w,G) \cup N_2(w,G) \cup N_3(w,G)| = 1 + 2k + |N_3(w,G)|\,.$$

Suppose $|N_3(w,G)| = 3$. Then, each vertex in $N_2(w,G)$ is adjacent to at least two vertices in $N_3(w,G)$. Since $|N_2(w,G)| = k \geq 3$ and no two vertices in $N_2(w,G)$ have two common neighbors in $N_3(w,G)$, we infer that each vertex in $N_2(w,G)$ is adjacent to exactly two vertices in $N_3(w,G)$. If $|N_2(w,G)| \geq 4$, then there exist two distinct vertices in $N_2(w,G)$ with the same neighborhood in $N_3(w,G)$, leading to a 4-cycle in G, a contradiction. We thus have $|N_1(w,G)| = |N_2(w,G)| = k = 3$ and $|N_{\geq 4}(w,G)| = 0$. This implies that our graph has exactly ten vertices. All squco graphs with at most 11 vertices are listed in Theorem 2.6 and none of them has girth 6. Hence this is a contradiction with G having girth 6.

Suppose $|N_3(w,G)| = 2$. If $k \leq 4$, then our graph has no more than 11 vertices, which is not possible. Hence $k \geq 5$. There must be at least $2k-1$ vertices of degree two in G (all k vertices in $N_1(w,G)$; at most one of the k vertices in $N_2(w,G)$ has both vertices from $N_3(w,G)$ for neighbors, otherwise we have a 4-cycle as before). In $\overline{G^2}$ at most $k+3$ of them are of degree two, because every vertex in $N_1(w,G)$ will be connected to all but one vertex in $N_2(w,G)$ in $\overline{G^2}$, which is a contradiction, because $k \geq 5$.

The last possibility is that $|N_3(w,G)| = 1$, but then w would be of degree 1 in $\overline{G^2}$, again a contradiction. This completes the proof. $\square$

4 A bipartite construction for squco graphs

Given two vertex-disjoint bipartite graphs G and G' with respective bipartitions $\{A,B\}$ and $\{A',B'\}$, construct a bipartite graph $H = H(G,G')$ with vertex set $V(G) \cup V(G') \cup C \cup D$, where $C = \{c_1,c_2\}$ and $D = \{d_1,d_2\}$ are sets of two new vertices each, and edge set $E(G) \cup E(G') \cup \{c_1d_1, c_2d_2\} \cup [A,D] \cup [A',B] \cup [C,B']$ (recall that $[A,D] = \{ad : a \in A, d \in D\}$ and so on). See Fig. 3 for an illustration.

Note that the Franklin graph (see Fig. 1) is the result of this construction when each of G and G' is isomorphic to the graph consisting of two disjoint copies of K_2.

We say that a bipartite graph G is *nice* if neither of the two parts of G is empty and every vertex has a neighbor both in G and in $\overline{G}^{\text{bip}}$.

Lemma 4.1 *Given a nice bipartite graph G, the bipartite graph $H(G,\overline{G}^{bip})$ is squco.*

Proof. Let G' be a disjoint copy of $\overline{G}^{\text{bip}}$. Let $\{A,B\}$ be a bipartition of G, and let $\{A',B'\}$ be the corresponding bipartition of G', in the sense that A' is a disjoint copy of A and B' is a disjoint copy of B.

Since $G' \cong \overline{G}^{\text{bip}}$, the graph $H = H(G,G')$ is bipartite self-complementary: the required isomorphism, viewed as a permutation of $V(H)$, leaves the subsets C and D fixed and exchanges A with A' and B with B'.

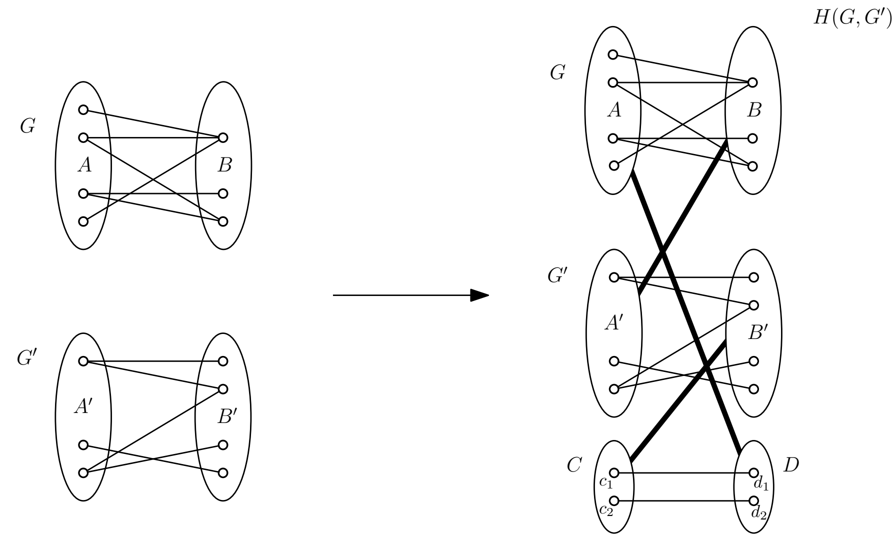

Fig. 3. The construction of $H(G, G')$. A thick edge between two sets of vertices denotes the presence of all possible edges.

Since $A, A', B, B' \neq \emptyset$ and none of these sets contain isolated vertices in G or G', it is easy to check that $H(G, G')$ has diameter 3. It follows by Theorem 1.1 that $H = H(G, G')$ is a bipartite squco graph. $\square$

Observe that every bipartite graph is an induced subgraph of some nice bipartite graph. A simple construction establishing this is shown in Fig. 4.

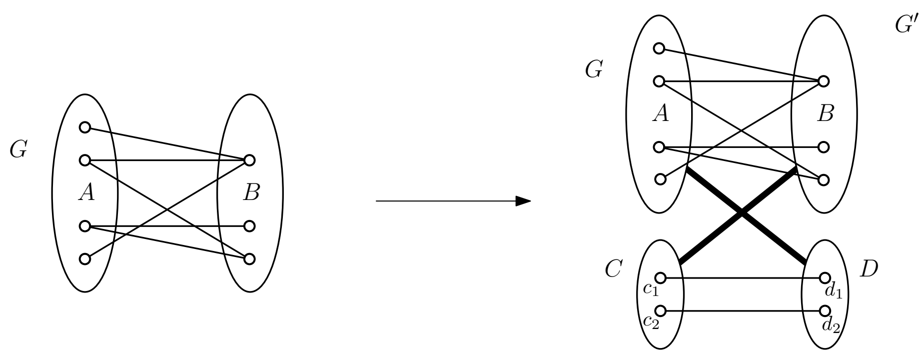

Fig. 4. A general transformation showing that every bipartite graph G is an induced subgraph of some nice bipartite graph G'. A thick edge between two sets of vertices denotes the presence of all possible edges.

Therefore, we have the following result as an immediate corollary of Lemma 4.1.

Theorem 4.2 *Every bipartite graph is an induced subgraph of a bipartite squco graph. In particular, there are squco graphs containing arbitrarily long induced paths and cycles.*

The bipartite construction mapping G and G' to $H(G, G')$ is also instrumental in the proof of the following result.

Theorem 4.3 *The problem of recognizing squco graphs is graph isomorphism complete.*

Proof. Given a graph G, consider $I(G)$, the vertex-edge bipartite incidence graph of G, that is, the left part of $I(G)$ is $A = V(G)$, the right part is $B = E(G)$, and $x \in A = V(G)$ is adjacent in $I(G)$ to $e \in B = E(G)$, if and only if x is incident to e in G. Clearly $G_1 \cong G_2$ if and only if $I(G_1) \cong I(G_2)$. Without loss of generality we can consider only graphs G satisfying

(1) $m = |E(G)| > |V(G)| = n > 1$, and

(2) $I(G)$ and $\overline{I(G)}^{\text{bip}}$ do not have isolated vertices.

This is so because $G_1 \cong G_2$ if and only if $G_1 * K_2 \cong G_2 * K_2$, and because $G * K_2$ always satisfies conditions (1) and (2) (we denote by $G * K_2$ the result of adding two universal vertices to G). Also, it suffices to consider only pairs of graphs G_1, G_2 having $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| = |E(G_2)|$, since otherwise the fact that G_1 and G_2 are not isomorphic can be detected in linear time.

Now let G_1, G_2 be two such graphs, and $I_1 := I(G_1)$ and $I_2 := \overline{I(G_2)}^{\text{bip}}$. In the rest of the proof, we show that $G_1 \cong G_2$ if and only if $H(I_1, I_2)$ is squco.

If $G_1 \cong G_2$, then $H(I_1, I_2)$ is squco because of Lemma 4.1.

Suppose now that $H(I_1, I_2)$ is squco. We shall show that the required isomorphism, seen as a permutation of the vertex set of $H(I_1, I_2)$, exchanges $V(I_1)$ and $V(I_2)$ in the expected way: $A = V(G_1)$ with $A' = V(G_2)$ and $B = E(G_1)$ with $B' = E(G_2)$. This will imply that $I(G_1) \cong I(G_2)$ and thus $G_1 \cong G_2$.

First, observe that the left part of $H(I_1, I_2)$ has $2n + 2$ vertices and the right part contains $2m + 2$ vertices. Since we assumed $m > n$, it follows that any isomorphism must preserve each of the two parts. Next, observe that the degrees of the vertices in the right part determine the subset to which they belong: vertices in B have degree $n + 2$, vertices in B' have degree n, and vertices in subset D have degree $n + 1$. Furthermore, also the subsets A, A', and C are uniquely determined as the subsets of vertices on the left part that have 2, 0, and 1 neighbor in D, respectively.

It follows that the required isomorphism between $H(I_1, I_2)$ and $\overline{H(I_1, I_2)^2} = \overline{H(I_1, I_2)}^{\text{bip}}$ must exchange $V(I_1)$ and $V(I_2)$ in the expected way and the result follows. $\square$

5 There are no nontrivial planar bipartite squco graphs

First we reduce the problem to a finite number of cases. To this end, the following property of bipartite squco graphs will be useful.

Lemma 5.1 *Every nontrivial bipartite squco graph has minimum degree at least three, and each of its parts has at least six vertices.*

Proof. Let G be a nontrivial bipartite squco graph with bipartition $\{A, B\}$. By Proposition 2.1, G is connected and of minimum degree at least 2. Suppose that there is a vertex $x_1 \in V(G)$ of degree 2, say $N_G(x_1) = \{y_1, y_2\}$. By symmetry, we may assume that $x_1 \in A$ (and then $y_1, y_2 \in B$). The fact that G is bipartite and squco implies by Theorem 1.1 that $G \cong \overline{G}^{\text{bip}}$. Thus, by Lemma 2.7, there is a vertex $x_2 \in A$ adjacent to both y_1, y_2 in $\overline{G}^{\text{bip}}$. But then x_1 and x_2 cannot have a common neighbor in B (in G), contrary to Lemma 2.7. It follows that every vertex in G has degree at least 3. Then, every vertex must also have degree at least 3 in $\overline{G}^{\text{bip}}$. Hence $|N_G(x) \cup N_{\overline{G}^{\text{bip}}}(x)| \geq 6$ for all $x \in A \cup B$, and each part must have at least 6 vertices. $\square$

Theorem 5.2 *If a nontrivial, planar bipartite squco graph G exists, it has a bipartition $\{A, B\}$ with $6 = |A| \leq |B| \leq 8$.*

Proof. Let G be a planar bipartite squco graph on n vertices and m edges, where $n > 1$, with bipartition $\{A, B\}$ such that $|A| \leq |B|$. Let $a = |A|$ and $b = |B|$. By Proposition 2.1 and Lemma 5.1, G is connected and of minimum degree at least 3, and $a \geq 6$. The fact that G is bipartite self-complementary implies that $|E(\overline{G}^{\text{bip}})| = |E(G)|$ and consequently $m = ab/2$. By Euler's formula for planar bipartite graphs, we have that $m \leq 2n - 4$, and hence, we get that $ab \leq 4n - 8 = 4(a + b) - 8$. This yields

$$(a - 4)(b - 4) \leq 8\,. \tag{1}$$

Since $6 \leq a \leq b$, we thus have $(a - 4)^2 \leq 8$, which implies $a \leq 6$. Thus, $a = 6$ and by (1), we have $6 \leq b \leq 8$, as claimed. $\square$

The previous theorem reduces the problem of the existence of nontrivial planar bipartite squco graphs to a finite number of cases manageable by a computer. To finally prove that there are none, we can apply two possible approaches.

One approach is a traditional human-readable proof, a case by case analysis, which we did in a previous preprint version of this paper available in [13]. The full proof is four pages long.

The other approach is a computer-assisted proof, which we also did, twice (in our opinion, computer-assisted proofs benefit from redundancy). First we produced an exhaustive DFS (backtracking) algorithm with automorphism reductions (pruning) using GAP with YAGS [12, 20]. This way we generated up to isomorphism, all bipartite graphs with minimum degree at least 3, whose bipartite complements also have minimum degree at least 3, and with parts of orders given by 6+6, 6+7, and 6+8 vertices respectively. Checking these graphs for the square-complementary property, we found a total of $1 + 3 + 11 = 15$

bipartite squco graphs. We then checked that none of them is planar (all of them contain a subdivision of $K_{3,3}$). The code is available upon request.

Second, we also used plantri [8] to generate all the relevant planar bipartite graphs, namely those that are 2-connected, with minimum degree at least 3, and having orders and sizes given by: 12 and 18, 13 and 21, and 14 and 24, respectively; a total of $1 + 2 + 12 = 15$ graphs. We then checked that none of them is squco, again using GAP and YAGS.

Whatever your preferred approach, we can now claim the following theorem.

Theorem 5.3 *The only planar bipartite squco graph is* K_1.

6 Conclusion

The results of the present paper provide further insight on the solution set of the graph equation $G^2 \cong \overline{G}$. We showed that, while no solutions can be found among graphs of girth six or among nontrivial planar bipartite graphs, recognizing square-complementary graphs is in general as difficult as the Graph Isomorphism problem. This is a notorious problem that is not known to be either in P or NP-complete, and for which Babai recently announced a quasipolynomial-time algorithm [4].

While our work answers three of the open problems on squco graphs posed in [21], several problems on squco graphs mentioned therein remain open. This includes existence of squco graphs within the classes of graphs of girth five, graphs of diameter four, and nontrivial chordal graphs. Furthermore, the proof of Theorem 4.3 shows that the problem of recognizing squco graphs is graph isomorphism complete within the class of bipartite graphs. This motivates the study of the problem of recognizing squco graphs within subclasses of bipartite graphs for which the graph isomorphism problem is GI-complete. This is the case, for example, for the class of chordal bipartite graphs [26].

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