Clique Divergent Clockwork Graphs and Partial Orders

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Abstract

S. Hazan and V. Neumann-Lara proved in 1996 that every finite partially ordered set whose comparability graph is clique null has the fixed point property and they asked whether there is a finite poset with the fixed point property whose comparability graph is clique divergent. In this work we answer that question by exhibiting such a finite poset. This is achieved by developing further the theory of clockwork graphs. We also show that there are polynomial time algorithms that recognize clockwork graphs and decide whether they are clique divergent.

Key words: Iterated clique graphs, fixed point property, clique divergence.

1 Introduction

Our graphs are finite, simple and connected. A *clique* of a graph G is a maximal complete subgraph of G. We shall often identify induced subgraphs (hence cliques) with their vertex sets. The clique graph $K(G)$ of G is the intersection graph of its cliques: the vertices are the cliques and two of them are adjacent if and only if they share at least one vertex. The iterated clique graphs are defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. Iterated clique graphs were introduced by Hedetniemi and Slater in [6]. We refer to [18], [9] and [22] for the literature on iterated clique graphs.

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We say that G is *K-divergent* (*clique divergent*) if $lim_{n\to\infty}|K^n(G)| = \infty$, and that G is *K-bounded* if $K^n(G) \cong K^m(G)$ for some $n \neq m$. The graph G is K-divergent if and only if G is not K-bounded. A special case of a K-bounded graph is a *K-null* graph: $K^n(G) \cong K_1$ (the trivial graph) for some n. The dynamical behaviour of G under the iterated application of the clique operator K is called the *K-behaviour* of G. By determining the K-behaviour of a graph G we mean determining whether G is K-null, K-bounded but not K-null, or K-divergent. It is an open problem to determine whether the K-behaviour is computable or not. In §3 we shall prove that the K-behaviour of a clockwork graph can be determined in polynomial time and also that these graphs can be recognized in polynomial time.

The useful family of clockwork graphs was introduced in [11] and has already been applied: The presence of a K-divergent clockwork graph inside the clique graph of the icosahedron is an important part of the proof of its K-divergence in [16] and a specific clockwork graph was used in [3] to show that the period of a K-periodic graph G is not invariant under removal of dominated vertices. In this paper we will give a new application of these graphs. In order to achieve this, we will develop further their theory in §3 after we review the basic terminology and results about clockwork graphs in §2.

Our new application of clockwork graphs will be to a problem in (finite) partially ordered sets. Such a poset P is said to have the *fixed point property* (FPP) if and only if every endomorphism of P has a fixed point. In 1928, Knaster and Tarski [7] proved that the lattice of all subsets of a set has the FPP. In 1955, Tarski [23] and Davis [1] proved that a lattice is complete if and only if it has the FPP. In 1976, I. Rival proved the FPP for any finite poset satisfying a condition called *dismantlability* [19]. The connection of this subject to iterated clique graphs was established by S. Hazan and V. Neumann-Lara in [5]. This connection came via the comparability graph, which has as vertices the points of P and in which two points $x, y \in P$ are adjacent if and only if $x \leq y$ or $x \geq y$. The authors of [5] gave a weaker sufficient condition than that in Rival's result: the poset P has the FPP whenever the comparability graph of P is K-null. A question remained open in [5]: Is it possible to find a poset with the FPP whose comparability graph is not K-null? In §4 we will apply our results to study Schröder's list [21] of posets with the FPP, and we will show that there are posets with this property which are K-divergent, thus solving in the affirmative that question.

Finally, the last section introduces some problems related to this work. An extended abstract of this paper is [12].

We thank the referees for their careful revisions. Their comments and questions led us to include Theorems 3.4 and 3.5 at revision time.

2 Preliminaries

Clockwork graphs were introduced in [11]. For the reader's convenience, we recall the basic terminology and results.

Let G, H be graphs. A *morphism* $\alpha : G \to H$ is a vertex-function $\alpha : V(G) \to V(H)$ such that the images under α of adjacent vertices of G either coincide or are adjacent in H. Let $s \geq 3$ and let $\mathcal{C}_s$ be the cyclic graph with s vertices labeled from 0 to $s-1$. Let G be a graph and assume that there is a surjective morphism $\pi : G \to \mathcal{C}_s$ such that $\pi^{-1}(i)$ induces a complete subgraph of G for $i = 0, \ldots, s-1$. In case $s = 3$ assume further that the image under π of any triangle of G is a vertex or an edge of $\mathcal{C}_s$. Put $G_i = \pi^{-1}(i)$. Then we say that (G, π) is a *cyclically segmented graph* with *cyclic segmentation* $(G_0, G_1, \ldots, G_{s-1})$. Since π is determined by the *segments* G_i we won't mention π if the segments are known. In particular, we will say that G is a (cyclically segmented) graph with cyclic segmentation $(G_0, G_1, \ldots, G_{s-1})$. Also, we consider the indices as being taken modulo s, and say that G_i and G_{i+1} are *consecutive* segments. Equivalently, a cyclically segmented graph G is a graph having an ordered partition $(G_0, G_1, \ldots, G_{s-1})$ into complete subgraphs such that every edge and every triangle of G is contained in the union of two consecutive segments. We denote by $N_G(v)$ or just $N(v)$ the set of all neighbours of the vertex v.

Let C be a graph with cyclic segmentation $(C_0, C_1, \ldots, C_s)$. Assume that there is a strict linear order $<$ on each segment C_i. We say that C is a *core graph* if:

C1: $x, y \in C_i$ and $x < y$ imply $N(y) \cap C_{i-1} \subseteq N(x) \cap C_{i-1}$.
C2: $x, y \in C_i$ and $x < y$ imply $N(x) \cap C_{i+1} \subseteq N(y) \cap C_{i+1}$.

This is a more symmetric reformulation of the original definition in [11]. Note that a given core graph C admits several core graph structures, even with the same set of segments $\{C_0, C_1, \ldots, C_{s-1}\}$. For instance, the *rotated* structures $(C_j, C_{j+1}, \ldots C_{s-1}, C_0, \ldots C_{j-1})$, and the *reversed* structure $(C_{s-1}, \ldots, C_1, C_0)$ with the reversed order in each C_i also satisfy **C1** and **C2**. Another example: If we define the equivalence relation $x \sim y$ in each C_j by $N(y) \cap C_{i-1} = N(x) \cap C_{i-1}$ and $N(x) \cap C_{i+1} = N(y) \cap C_{i+1}$ then arbitrarily redefining the strict linear order within each equivalence class we obtain a *permuted* structure which also satisfies **C1** and **C2**. Two core structures of a graph will be called *kindred* if one can be obtained from the other by repeated application of rotations, permutations and reversals. From now on, when we say that a graph C is a core graph we usually assume that a specific core structure admitted by C is at hand.

In a core graph C we say that C_i is a *good segment* if for every vertex $u \in C_i$ there is a vertex $v \in C_{i+1}$ such that u and v are not adjacent. A vertex $v \in C_i$

is *covered by* $u \in C_i$ if $u < v$ and $N(u) \cap C_{i+1} = N(v) \cap C_{i+1}$. This covering relation was called *strong* in [11].

Let B be a graph with cyclic segmentation $(B_0, B_1, \ldots, B_{s-1})$. We say that B is a *crown graph* if:

B1: Each segment B_i has at least 2 vertices.
B2: The edges of B connecting B_i and B_{i+1} constitute a perfect matching for each i.

We say that two rotated and/or reversed crown structures are *kindred*.

If $(B_0, \ldots, B_{s-1})$ and $(C_0, \ldots, C_{s-1})$ are cyclic segmentations for the graphs B and C respectively, their *segmented sum* is the graph G with $V(G) = V(B) \cup V(C)$ and $E(G) = E(B) \cup E(C) \cup \{\{b, c\} : b \in B_i \cup B_{i+1}, c \in C_i, 0 \leq i < s\}$. In this case, we write $G = B \oplus C$. Note that G is a cyclically segmented graph, with segments $G_i = B_i \cup C_i$. We will always assume that a segmented sum comes with this natural cyclic segmentation.

Finally, A *clockwork graph* G is the segmented sum $G = B \oplus C$ of a crown graph B and a core graph C. When we say that G is a clockwork graph we usually assume that a specific decomposition $G = B \oplus C$ is given.

Figure 1 shows a drawing of a simple clockwork graph G with three segments $G_0 = \{1, 2, 7, 8\}$, $G_1 = \{3, 4, 9, 10\}$ and $G_2 = \{5, 6, 11, 12\}$ indicated as dashed ovals. The core and crown subgraphs C and B are induced by the vertex sets $\{1, \ldots, 6\}$ and $\{7, \ldots, 12\}$ respectively. For clarity, we do not draw edges joining a vertex in the crown with a vertex in the core. This clockwork graph has two good segments (namely C_0 and C_1) and no covered vertices.

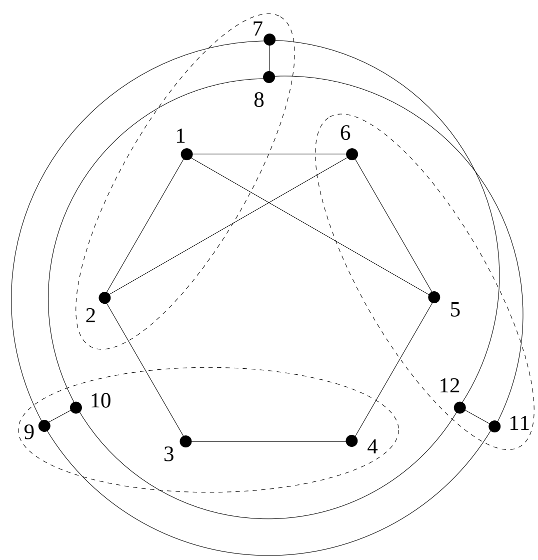

Fig. 1. A clockwork graph.

For more complex clockwork graphs, there are too many edges in the core for the drawing to be clear, thus we just draw arrows joining every vertex in the core graph with its most distant counter-clockwise neighbour in the core as in Figure 2. Thus, for example, vertex 4 is adjacent to every vertex in $\{1, 2, 3, 5, 6, 7, 8, 9, 10, 18, 19, 20, 21, 22, 23, 24\}$. This clockwork graph has three good segments and one covered vertex, namely vertex 13.

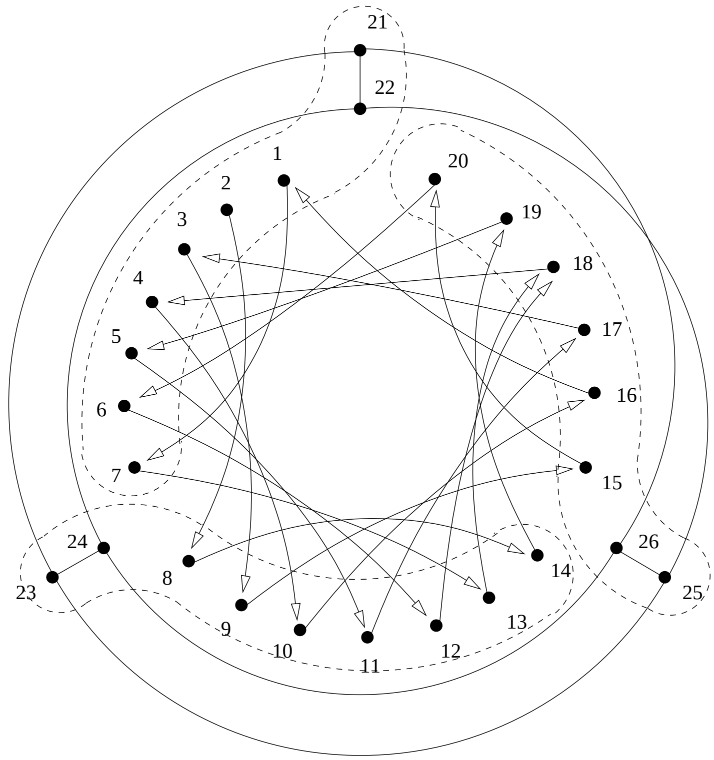

Fig. 2. Another clockwork graph.

Larrión and Neumann-Lara proved in [11] that the clique graph of any clockwork graph $G = B \oplus C$ is a clockwork graph $K(G) = \bar{B} \oplus \bar{C}$ where (up to isomorphism) $\bar{B}$ can be taken to be identical to B, and $\bar{C}$ can be obtained from C applying two simple operations:

tick: Let C' be the subgraph of C induced by the vertices which are not covered, with the cyclic segmentation given by $C'_i = C_i \cap V(C')$.

tock: Add a new vertex to each segment C'_i whenever C_i is a good segment in C. Make this new vertex greater than the old vertices in C'_i and adjacent to every old vertex in $C'_i \cup C'_{i+1}$ thus obtaining the core graph $\bar{C}$.

We will always assume that $K(G)$ has been constructed in this way; in particular a vertex in $K(G) \cap G$ is an *old vertex* of $K(G)$, while a vertex in $K(G) - G$ is a *new vertex* of $K(G)$.

We shall also use the following two theorems from [11]:

Theorem 2.1 *The clockwork graphs G and $K(G)$ have the same number of good segments. In fact, $\bar{C}_i$ is good if and only if C_{i+1} is good.* $\square$

Theorem 2.2 *If G is a clockwork graph without covered vertices, then $K(G)$ does not have covered vertices.* $\square$

It follows from Theorems 2.1 and 2.2, and from the explicit construction of $K(G)$ given above, that a clockwork graph with no covered vertices is clique divergent if and only if it has at least one good segment. In particular, the graph in Figure 1 is clique divergent.

In any graph, the vertex u is said to be *dominated* by the vertex v if $v \in N(u)$ and $N(u) - v \subseteq N(v)$. Notice that $u \neq v$ in this case (G has no loops). We will use a result from Frías, Neumann-Lara and Pizaña [3]:

Theorem 2.3 *Let G be a graph and let u be a dominated vertex of G. Then, G and $G - u$ have the same K-behaviour. That is, both are K-null, both are K-bounded but not K-null, or both are K-divergent.* $\square$

3 Further Results on Clockwork Graphs

Theorem 3.1 *If $G = B \oplus C$ is a clockwork graph, then $K(G)$ has at most as many covered vertices as G does.*

PROOF. We will first assign to each covered vertex v of $K(G)$ a covered vertex x_v of G, and then we will prove that this assignment is injective.

Let $K(G) = \bar{B} \oplus \bar{C}$ as described in the preceding section. Let $u, v \in \bar{C}_i$ and assume that v is covered by u in $K(G)$. As $u < v$ it follows that u is an old vertex of $K(G)$. On the other hand, v can be either new or old.

Assume that v is old. Since $v \in K(G)$, v can not be covered in G. Thus $N_G(v) \cap C_{i+1}$ is non-empty (otherwise, u would cover v in G). Using the order relation in C_{i+1}, set $x_v = \max(N_G(v) \cap C_{i+1})$. By condition **C1**, $N_G(v) \cap C_{i+1} = \{y \in C_{i+1} : y \leq x_v\}$. If we had $x_v \in N_G(u)$, again by **C1** we would get $N_G(u) \cap C_{i+1} \supseteq \{y \in C_{i+1} : y \leq x_v\} = N_G(v) \cap C_{i+1}$. But then **C2** would imply that u covers v in G. Therefore $x_v \notin N_G(u)$. Now x_v is covered in G (i.e. $x_v \notin K(G)$) for otherwise $x_v \in (N_{K(G)}(v) - N_{K(G)}(u)) \cap \bar{C}_{i+1}$ and v would not be covered by u in $K(G)$.

Now assume that v is new. Then C_i must be a good segment. Take $x_v = \max C_{i+1}$. As before, x_v is a covered vertex in G or else $N_G(u) = C_{i+1}$ contradicting the fact that C_i is a good segment.

Finally assume $x_v = x_{v'}$ for some pair of different vertices $v, v' \in \bar{C}_i$. We assume that $v' < v$. Thus v' is old, and v may be either new or old.

If v is new, $x_v = \max C_{i+1} = x_{v'}$ and then v' is adjacent by **C1** to every vertex in C_{i+1} contradicting that C_i is good.

If v is old, $x_v = \max(N_G(v) \cap C_{i+1}) = x_{v'}$. But then $N_G(v) \cap C_{i+1} = N_G(v') \cap C_{i+1}$ and v is covered by v' in G, contradicting that $v \in K(G)$. □

It follows from Theorems 2.1 and 3.1 that any clockwork graph with more good segments than covered vertices is clique divergent. In particular, it follows that the clockwork graph represented in Figure 2 is clique divergent.

Theorem 3.2 *Every covered vertex of a clockwork graph is dominated.*

PROOF. Assume that v is covered by u. There is some segment C_i with $u, v \in C_i$, so u and v are adjacent. Both $N(u)$ and $N(v)$ are contained in $C_{i-1} \cup B_i \cup C_i \cup B_{i+1} \cup C_{i+1}$, but by construction both u and v are adjacent to all other vertices in $B_i \cup C_i \cup B_{i+1}$. By definition of covered vertices we have $N(v) \cap C_{i+1} = N(u) \cap C_{i+1}$. As $u < v$, condition **C1** implies $N(v) \cap C_{i-1} \subseteq N(u) \cap C_{i-1}$. Thus $N(v) - u \subseteq N(u)$ and v is dominated by u. □

Larrión and Neumann-Lara proved implicitly in [11] that if G is a clockwork graph and u is a covered vertex, then $G - u$ is also a clockwork graph. In view of the previous theorem, the following is a generalization of this result.

Theorem 3.3 *If G is a clockwork graph and $u \in G$ is a dominated vertex, then $G - u$ is also a clockwork graph.*

PROOF. Let $G = B \oplus C$ have crown subgraph B, core subgraph C, and cyclic segmentation (G_i). Recall that $G_i = B_i \cup C_i$ for $i = 0, ..., s-1$.

Observe first that $u \in C_i$ and $C_i - u$ non-empty imply that $C' = C - u$ is a core graph: indeed, the restricted morphism $\pi' : C' \to \mathcal{C}_s$ is still surjective and the remaining properties of a core graph are inherited from C. It follows that $G - u = B \oplus C'$ is a clockwork graph. Thus it will suffice to show that, if u is dominated by v in G, then both u and v lie in the same segment C_i of C (since $u \neq v$, $C_i - u$ is non-empty).

Let us assume that $u \in G_i = B_i \cup C_i$. Since u and v are adjacent and (G_j) is a cyclic segmentation, $v \in G_{i-1} \cup G_i \cup G_{i+1}$.

If $u \in B_i$, then u is adjacent to every other vertex in B_i (at least one by **B1**) and to some vertices $x \in B_{i-1}$ and $y \in B_{i+1}$. Then $v \in B_{i-1} \cup B_i \cup B_{i+1}$ because $N(u) - v \subseteq N(v)$ and no vertex in C is adjacent to vertices in three segments of B. If $v \in B_i$, then $\{x, u\}, \{x, v\} \in E(B)$ contradicts **B2**. If $v \in B_{i-1} \cup B_{i+1}$, either $\{u, v, x\}$ or $\{u, v, y\}$ is a triangle with vertices in three segments of B, contradicting that (B_j) is a cyclic segmentation. Therefore $u \in C_i$. Now v must be adjacent to every vertex in $B_i \cup B_{i+1}$, but this is possible only if v is also in C_i. □

The following theorem will lead to a polynomial time recognition algorithm.

Theorem 3.4 *Let $G = B \oplus C$ be a clockwork graph with crown subgraph B and core subgraph C. Then a vertex $v \in G$ is in C if and only $N(v)$ contains an induced 4-cycle.*

PROOF. Assume that v is in the segment C_i of C. By **B1** and **B2** we can take two different vertices $b_1, b_2 \in B_i$ and their unique neighbours b_1' and b_2' in B_{i+1}. Then (b_1, b_2, b_2', b_1') is an induced square in $N(v)$.

Now assume that v is in the segment B_i of B, and that S is an induced square in $N(v)$. Let x, y be the unique neighbours of v in B_{i-1} and B_{i+1} respectively. Then $N(v) = \{x\} \cup C_{i-1} \cup (B_i - v) \cup C_i \cup \{y\}$. Since B is cyclically segmented, x and y are not adjacent (otherwise $s = 3$ and $\pi((x, v, y))$ is a triangle in $\mathcal{C}_s$). Since $N(x) \cap N(v) = C_{i-1}$ is complete, $x \notin S$. Similarly $y \notin S$. Thus $S \subseteq X := C_{i-1} \cup (B_i - v) \cup C_i$. Any vertex in $B_i - v$ is universal in X, so $S \subseteq C_{i-1} \cup C_i$. But these two segments are complete, so S has exactly two vertices in each of them, say $a < b$ in C_{i-1} and $c < d$ in C_i. Now d must be adjacent to some $w \in \{a, b\}$ but, by **C1**, c is also adjacent to w and (c, d, w) is a triangle in S. Contradiction. □

Now we shall address the recognition problem. Given an input graph G it is required to decide if G is a clockwork graph, and we will do so by determining, if it exists, some specific clockwork structure (decomposition $G = B \oplus C$ with the associated structures for B and C). In fact, if the input graph G already had a specific clockwork structure, it will be clear that our algorithm produces the same decomposition $G = B \oplus C$, with kindred structures on B and C.

Theorem 3.5 *Clockwork graphs can be recognized, and some clockwork structure determined for them, in polynomial time.*

PROOF. Let G be a graph. We shall sketch an algorithm whose steps are easily seen to take polynomial time. The algorithm should stop and answer NO as soon as one of the checks below fails: Indeed, all those checks must succeed if G is a clockwork graph. The used constructions can always be performed provided that the previous tests succeeded.

Compute $C = \{v \in G : N(v) \text{ has an induced 4-cycle}\}$ and $B = G - C$. Check that B and C are both non-empty and that B is connected.

Now construct a partition $B = (B_0, B_1, \ldots, B_{s-1})$ using the equivalence relation $x \sim y$ iff $N(x) \cap C = N(y) \cap C$. Check that $s \geq 3$, that $|B_i| \geq 2$ and B_i is complete for all i, and that for each i and $v \in B_i$ we have $N(v) \cap B - B_i \cong \bar{K}_2$. Thus, the two neighbours in $B - B_i$ of each vertex $v \in B_i$ lie in two other different B_j's. Now check that these two B_j's are the same for all $v \in B_i$.

Renumber $(B_0, B_1, \ldots, B_{s-1})$ in such a way that the vertices in each B_i have neighbours in B_{i-1} and B_{i+1} (indices always modulo s).

For each $i = 0, \ldots, s-1$, compute $C_i = C \cap N(B_i) \cap N(B_{i+1})$, where $N(X) = \bigcap_{x \in X} N(x)$. Check that the C_i's are non-empty, are complete and form a partition of C. Check that $N(v) \cap C \subseteq C_{i-1} \cup C_i \cup C_{i+1}$ for each i and each vertex $v \in C_i$. Now check that no triangle in C has vertices in three different C_j's.

Finally we have to define a linear order on each C_i and verify that C satisfies **C1** and **C2**: First define, on each C_i, the preorders $\lesssim$ and $\precsim$ by the formulae: $x \lesssim y$ iff $N(y) \cap C_{i-1} \subseteq N(x) \cap C_{i-1}$ and $x \precsim y$ iff $N(x) \cap C_{i+1} \subseteq N(y) \cap C_{i+1}$. Now check that any pair of vertices $x, y \in C_i$ are comparable under both preorders $\lesssim$ and $\precsim$. Check also that $\lesssim$ and $\precsim$ agree, in the sense that:

$$x \lesssim y \text{ and } y \not\lesssim x \Rightarrow x \precsim y,$$

$$x \precsim y \text{ and } y \not\precsim x \Rightarrow x \lesssim y.$$

Now, taking on each C_i any strict linear order $<$ that is compatible with both $\lesssim$ and $\precsim$ (in the sense that $x < y \Rightarrow x \lesssim y \;\&\; x \precsim y$) we have that C satisfies **C1** and **C2**. Therefore, $G = B \oplus C$ is a clockwork graph with crown subgraph B and core subgraph C with segmentations $B = (B_0, B_1, \ldots, B_{s-1})$ and $C = (C_0, C_1, \ldots, C_{s-1})$ respectively. $\square$

Theorem 3.6 *Clique divergence for clockwork graphs is decidable in polynomial time.*

PROOF. As before, each step is easily seen to take polynomial time. Let G be a clockwork graph. Remove dominated vertices from G until there is no such vertex in the resulting graph G'. By Theorems 2.3 and 3.3 we know that G' is a clockwork graph with the same K-behaviour as G. By Theorem 3.5 we can find a clockwork structure for G' in polynomial time. By Theorem 3.2 G' does not have covered vertices. Then using Theorems 2.1 and 2.2 and the tick and tock operations we know that $|K^n(G')| = |G'| + ng$ where g is the number of good segments of G', so G' is K-divergent if and only if $g > 0$. □

Note that since the clique graph of a clockwork graph is again a clockwork graph and they always have at least 9 vertices, they are never clique null. In particular, the previous theorem tells us that we can compute K-behaviour for clockwork graphs in polynomial time.

4 Partial Orders

We shall say that a poset is clique null (clique divergent, etc.) when its comparability graph is so. In a similar way, we shall often use the same symbol for both the poset and its comparability graph.

Among other results, Hazan and Neumann-Lara [5] studied Rutkowky's list [20] and Schröder's list [21] of posets with the fixed point property. They found by computer that Rutkowsky's posets $P1$, $P3$, $P4$, $P5$, $P7$, $P8$ and $P9$ and Schröder's P_1^{443}, are clique null and they said that the remaining cases (three from Rutkowsky and seven from Schröder) seem to be divergent. Recently, increased computer performance and better implemented algorithms have shown that Schröder's P_2^{443} is also clique null.

We shall prove below that Schröder's P_4^{443} and P_5^{443} are clique divergent. The remaining cases still seem to be divergent, not only because of the computer evidence (which failed to be good enough for P_2^{443}) but also because the remaining cases are related to existing conjectures as we shall see in the next section.

For the reader's convenience, we have included a Hasse diagram of P_4^{443} in Figure 3. The comparability graph is drawn in Figure 4. The Hasse diagram of P_5^{443} can be obtained from that of P_4^{443} adding an edge in Figure 3 from vertex 6 to vertex 11. The comparability graph of P_5^{443} can thus be obtained adding two new edges to Figure 4 joining vertex 11 with 6 and 1 respectively.

Before proving that these posets are clique divergent, we will prove that they are not clique null. The arguments illustrate powerful topological techniques first studied by Prisner. We recall that he proved in [17] that the first modulo 2 Betti number $\hat{\beta}_1$ of (the simplicial complex of) a graph G is invariant under the clique operator, that is $\hat{\beta}_1(G) = \hat{\beta}_1(K(G))$. Here the i-th modulo 2 Betti number of a graph G is defined by $\hat{\beta}_i(G) = \dim H_i(G^\uparrow, \mathbb{Z}_2)$, where $G^\uparrow$ is the simplicial complex whose simplexes are the complete subgraphs of G. In this paper Betti number always means modulo 2 Betti number. For more details

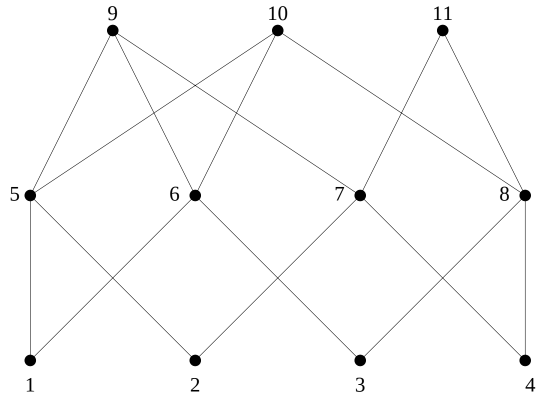

Fig. 3. Hasse diagram of P_4^{443}.

on these topics we refer to [17] and the literature cited therein. A clear and self-contained introduction to these topics can be found in [14].

From the above result it readily follows that P_4^{443} and P_5^{443} are not clique null: Indeed the geometric realization of $(P_4^{443})^\uparrow$ is the Möbius band as depicted in Figure 4. The two additional edges in the comparability graph of P_5^{443} do not alter the homotopy type as the triangles $\{11, 3, 6\}$ and $\{11, 6, 1\}$ can be retracted into the path $(11, 3, 6, 1)$. Thus $\hat{\beta}_1(P_4^{443}) = \hat{\beta}_1(P_5^{443}) = 1 \neq \hat{\beta}_1(K_1) = 0$.

Alternatively, we can prove this from the fact [10] that the triangular fundamental group of a graph is invariant under the clique operator: The triangular fundamental groups of $(P_4^{443})^\uparrow$ and $(P_5^{443})^\uparrow$ are isomorphic to the fundamental group of the Möbius band, which is non-trivial. By the way, since $\hat{\beta}_1(G)$ is the dimension of the modulo 2 reduction of the abelianized triangular fundamental group of G, even the above-mentioned result of Prisner's follows from the invariance of the triangular fundamental group.

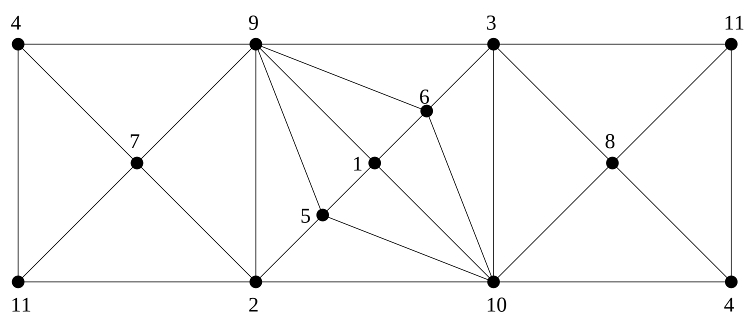

Fig. 4. Comparability graph of P_4^{443} as a triangulation of the Möbius band.

We prove now that P_4^{443} and P_5^{443} are indeed clique divergent:

A computer verification (we used GAP [4]) shows that the ninth iterated clique graph of the comparability graph of P_4^{443} is the clockwork graph shown in Figure 2, which we already know to be clique divergent. Alternatively, calculate the fourth iterated clique graph and then remove all dominated vertices to obtain the clockwork graph G_0 in Figure 1 which is also clique divergent; now use Theorem 2.3.

It is not surprising that P_4^{443} led to a clockwork graph, because clockwork graphs were designed to deal with it. In fact, they were designed to deal with the ninth iterated clique graph of P_4^{443} in order to skip the whimsical behaviour of its first 8 clique graphs shown in Table 1. What is surprising is that the ninth iterated clique graph of the comparability graph of P_5^{443} is also a clockwork graph with three good segments and one covered vertex and thus it is also clique divergent. Another proof: taking the fourth iterated clique graph of P_5^{443} and then removing dominated vertices we get again a clique divergent clockwork graph G_1: indeed (another surprise) $G_1 \cong K(G_0)$!

Thus we have the following:

Theorem 4.1 *The comparability graphs of P_4^{443} and P_5^{443} are clique divergent. In particular, there are posets with the fixed point property whose comparability graphs are clique divergent.* □

5 Problems

Table 1 summarizes what we know about the K-behaviour of Rutkowsky's and Schröder's posets. We have added a column describing the first Betti numbers. All these graphs have $\hat{\beta}_i = 0$ for $i \geq 3$. There is also a column specifying the order sequence of the iterated clique graphs $|K^0(G)|$, $|K^1(G)|$, etc. A "$\geq n$" in the table means that we aborted the calculation after 2 days with n cliques already computed.

The question arises: Is there a strong connection between Betti numbers and K-behaviour? This is not so in general, as any cycle of length at least 4 is K-invariant and has the same Betti numbers as P_4^{443}. However, we do not know an example of a non K-null graph with the same Betti numbers as $P1$ (i.e. as the disk), or one of a non K-divergent graph with the same Betti numbers as $P2$ (i.e. as the sphere). Moreover, Prisner [17] has conjectured that a planar graph is K-null if and only if its Betti numbers are those of the disk. We have two similar, long-standing conjectures (A triangulation is Whitney if every triangle of the graph is a face of the triangulation [13]):

Conjecture 5.1 *Except the tetrahedron, every Whitney triangulation of the sphere is K-divergent.*

Conjecture 5.2 *Every Whitney triangulation of the disk is K-null.*

Since the comparability graph of $P2$ is a Whitney triangulation of the sphere,

Graph	K-behaviour	$\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$	Order Sequence
$P1$	K-null	1, 0, 0	9, 14, 19, 30, 26, 1
$P2$	Unknown	1, 0, 1	10, 16, 26, 56, 310, 154752
$P3$	K-null	1, 0, 0	10, 18, 22, 37, 27, 1
$P4$	K-null	1, 0, 0	10, 18, 22, 44, 72, 1
$P5$	K-null	1, 0, 0	10, 18, 22, 36, 29, 1
$P6$	Unknown	1, 0, 1	10, 18, 27, 70, 467, $\geq$185637
$P7$	K-null	1, 0, 0	10, 16, 22, 32, 24, 1
$P8$	K-null	1, 0, 0	10, 16, 23, 40, 79, 185, 1
$P9$	K-null	1, 0, 0	10, 16, 22, 32, 9, 1
$P10$	Unknown	1, 0, 1	10, 16, 26, 66, 454, $\geq$243000
P_1^{443}	K-null	1, 0, 0	11, 20, 26, 52, 204, 617, 5, 1
P_2^{443}	K-null	1, 0, 0	11, 18, 25, 44, 137, 1376, 9, 1
P_3^{443}	Unknown	1, 0, 1	11, 21, 29, 71, 458, $\geq$133850
P_4^{443}	K-divergent	1, 1, 0	11, 16, 21, 26, 30, 28, 26, 26, 26, 26, 28, 30, 32, 34, 36, ...
P_5^{443}	K-divergent	1, 1, 0	11, 18, 23, 34, 47, 47, 27, 28, 29, 30, 32, 34, 36, 38, 40, ...
P_1^{353}	Unknown	1, 0, 1	11, 20, 31, 80, 444, $\geq$106652
P_2^{353}	Unknown	1, 0, 1	11, 20, 30, 72, 396, $\geq$189713
P^{3323}	Unknown	1, 0, 1	11, 20, 53, 172, 164790

Table 1
Rutkowsky's and Schröder's posets

this graph lies within the scope of Prisner's conjecture as well as within that of Conjecture 5.1. Our Conjecture 5.2 is weaker than Prisner's conjecture.

The comparability graphs of the posets in the list whose behaviour is unknown have the same Betti numbers as the sphere.

Problem 5.3 *Is it true that every graph which has the same Betti numbers as the sphere is K-divergent?*

There is an evident gap in Table 1:

Problem 5.4 *Is there a poset with the fixed point property such that its comparability graph is K-bounded but not K-null?*

Finally we recall that we obtained that it is computationally easy to determine the K-behaviour of a given clockwork graph. That the K-behaviour of a clique-Helly graph is also computable in polynomial time was implicitly established by Escalante [2], and Neumann-Lara also established this for complements of cycles [15]. The forthcoming papers [8] and [13] will show similar results for cographs and regular locally cyclic graphs respectively. But in general, this task is much more difficult; in fact, we do not know if it is possible at all:

Problem 5.5 *Is there an algorithm that can determine the clique behaviour of any given graph?*

It is easy to see that there is an algorithm for deciding K-divergence (or K-boundedness) if and only if there is also an algorithm for determining K-behaviour. On the other hand, an algorithm for deciding K-nullity (if such exists) would not be enough to determine K-behaviour.

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