A Hierarchy of Self-Clique Graphs

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Abstract

The clique graph $K(G)$ of G is the intersection graph of all its (maximal) cliques. A connected graph G is self-clique whenever $G \cong K(G)$. Self-clique graphs have been studied in several papers. Here we propose a hierarchy of self-clique graphs: Type 3 $\subsetneq$ Type 2 $\subsetneq$ Type 1 $\subsetneq$ Type 0. We give characterizations for classes of Type 3, 2 and 1 (including Helly self-clique graphs) and several new constructions of families of self-clique graphs. It is shown that all (but one) previously published examples of self-clique graphs are of Type 2. Our methods provide a unified approach and generalizations of those examples. As further applications, we give a characterization of the self-clique graphs such that at most 3 cliques have more than two vertices (they are all of Type 2) and a description of the diamond-free graphs of Type 2.

Key words: Self-clique graphs, Clique Helly graphs, Vertex-clique bipartite graph

1 Introduction

We are mainly interested in finite, non-empty, simple graphs G. For auxiliary purposes we shall also consider graphs (always denoted by H) which are al-

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lowed to have loops (at most one at each vertex). We will call *possibly loopy* this kind of graphs. A *graph* is thus a simple graph, i.e. a possibly loopy graph with no loops.

A *clique* of a graph G is a maximal complete subgraph of G, which we usually identify with its vertex set. The *clique graph* $K(G)$ of G has the cliques of G as vertices and two cliques $Q \neq Q'$ are adjacent in $K(G)$ whenever $Q \cap Q' \neq \varnothing$. By definition, G is *self-clique* if G is connected and $K(G) \cong G$. The study of self-clique graphs was initiated by Escalante in [5], and has been continued in [14,1,4,3]. In [10], Hedman asked if such graphs can be characterized. We say that any self-clique graph is of *Type 0*: this universe is our main object of study in this work.

The Helly property (see §2.2) has played an important role ever since the rise of the study of clique graphs [8,16]. A graph G is *clique-Helly* if the family of cliques of G satisfies the Helly property. Since we will focus mainly on clique-Helly graphs, we will often call them simply *Helly graphs.* Escalante [5] already made the distinction between Helly self-clique graphs and non-Helly self-clique graphs. A Helly self-clique graph will be said to be of *Type 1.* The *vertex-clique bipartite graph* of a graph G is the graph $BK(G)$ whose vertices are $V(G) \cup V(K(G))$ and where $\{v, Q\}$ is an edge iff $v \in Q$. We will prove in §4 that a connected graph G is a Helly self-clique graph if and only if $BK(G)$ admits a part-switching automorphism.

The case in which $BK(G)$ admits a part-switching *involution* (automorphism of order 2) is particularly interesting. A connected graph G is *involutive* or a *Type 2 self-clique graph* iff $BK(G)$ has a part-switching involution. In §5 we will show that G is involutive iff $G \cong H^{[2]}$ for some possibly loopy, good, non-bipartite and connected graph H. Here, H is *good* iff its family of neighbourhoods is an antichain and satisfies the Helly property (for the *strict square* $H^{[2]}$ see §2). It will be seen in §§ 6, 7 and 8 that, apart from the few non-Helly examples in Escalante's paper [5], all previously published examples of self-clique graphs are involutive. It will also transpire that our methods allow for stronger versions and further developments.

In [3] Bondy, Durán, Lin and Szwarcfiter gave a large family of self-clique graphs, all of which turn out to be involutive. Motivated by this, we defined our third type of self-clique graphs. A *disk* (of radius 1) of a graph R is the closed neighbourhood of a vertex of R. We say that a connected graph G is *clique-disk* or a *Type 3 self-clique graph* if G does not have twins, there exists a graph R with $G = R^2$ and the cliques of G are precisely the disks of R. It was proved implicitly in [3] that any clique-disk graph is self-clique, and also a sufficient condition for a graph to be clique-disk was implicitly given. We say that H is *totally loopy* if it has one loop at each vertex. In §6 we will prove that a connected graph G is clique-disk if and only if there is a totally loopy good graph H such that $G \cong H^{[2]}$ or, equivalently, $BK(G)$ admits a part-switching involution σ such that v and $\sigma(v)$ are adjacent for any vertex v. In particular, any clique-disk graph is involutive.

As a consequence of our characterizations, it follows that these classes of self-clique graphs are properly contained one in another, which we shall prove in the Hierarchy Theorem 6.4.

In § 7 we introduce a fairly general method (the vertex-clique construction) for constructing possibly loopy good graphs, and hence involutive graphs. This yields an unified approach to all the families of self-clique graphs constructed or studied by Escalante [5] (except the non-Helly family), Lim and Peng [14], Balakrishnan and Paulraja [1] and Chia [4], which at first looked very different from each other and were studied in those papers by ad-hoc methods. Besides this unification, we get that all those graphs are indeed involutive (not only self-clique) and, furthermore, we get significant extensions of those families of involutive graphs. For instance, in [4], Chia characterized all self-clique graphs with at most one clique that is not an edge. In § 8 we will give a characterization of the self-clique graphs G with at most 3 cliques that are not edges and show that they are all involutive. In § 9, all the diamond-free involutive graphs will be completely described.

The authors of [3] have recently informed us that they also arrived independently to our Theorem 4.4 in a matricial language. They will publish their new results in a forthcoming paper. We used Gap [6] for many computer verifications. We learned recently that the paper [2] by Balconi contains material of related interest.

2 Preliminaries

2.1 Basic Definitions

All our graphs are finite, non-empty, without multiple edges and, unless otherwise stated, they are also loopless. We will need to consider also *possibly loopy graphs* (always denoted by H in this work) which are allowed to have at most one loop at each vertex. Each loop is considered to be an edge, and it adds only 1 to the degree of the vertex it lies at. We denote by $\delta(H)$ and $\Delta(H)$ the minimum and the maximum degrees of the possibly loopy graph H. By $d(u, v)$ or $d_H(u, v)$ we denote the distance between the vertices u and v of H, and $g(G)$ denotes the *girth* of a graph G: the minimum length of a cycle in G. The *neighbourhood* of a vertex $v \in V(H)$ is the set $N(v)$ of all neighbours of v (notice: $v \in N(v)$ iff v has a loop) and the *closed neigh-*

bourhood of v is $N[v] = N(v) \cup \{v\}$ and coincides with the disk of radius 1: $N[v] = \{x : d_H(x, v) \leq 1\}$.

If $X \subseteq V(G)$, we denote by $G[X]$ the subgraph induced by X.

If H is a graph and e is a non-negative integer, the e-th *power* of H is the loopless graph H^e with $V(H^e) = V(H)$ and $\{u, v\} \in E(H^e)$ if and only if $0 \neq d_H(u, v) \leq e$. In particular, the *square* of H is H^2. The *strict square* of H is the (necessarily loopless) graph $H^{[2]}$ with $V(H^{[2]}) = V(H)$ and $\{u, v\} \in E(H^{[2]})$ if and only if u and v can be joined by exactly two distinct edges $\{u, x\}$, $\{x, v\}$ of H.

Regarding graph products, the notations and names used in the literature are far from universal. For the two products that will be used we have adopted Nešetřil's symbols (which look like the product of K_2 by K_2) and the TEX name for them.

The *times product* $H \times H'$ has vertex set $V(H \times H') = V(H) \times V(H')$ and $\{(x, x'), (y, y')\} \in E(H \times H')$ if and only if $\{x, y\} \in E(H)$ and $\{x', y'\} \in E(H')$. Notice that (x, x') has a loop only if both x and x' have loops.

The *boxtimes product* $G \boxtimes G'$ also has $V(G \boxtimes G') = V(G) \times V(G')$. If $(x, x') \neq (y, y')$, then there is an edge $\{(x, x'), (y, y')\} \in E(G \boxtimes G')$ if and only if x and y are equal or adjacent in G and also x' and y' are equal or adjacent in G'.

(There is also a *box product* $G \square G'$ which will not be used here.)

A *complete subgraph* (or just *a complete*) of H is a subgraph S such that any two distinct vertices of S are neighbours, so loops have no bearing in this concept. By a *clique* of H we mean a (vertex-)**maximal** complete subgraph of H, and we identify it with its vertex set. The *clique graph* $K(G)$ is the intersection graph of all the cliques of G, so $V(K(G))$ is the set of all cliques Q of G, and $\{Q, Q'\} \in E(K(G))$ if and only if $Q \neq Q'$ and $Q \cap Q' \neq \varnothing$. The *iterated clique graphs* are recursively defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. We say that G is *K-periodic* (or just *periodic*) if there is a positive integer p with $K^p(G) \cong G$ (the smallest such p is the *period* of G). More generally, G is *eventually K-periodic* (of period p) if there is a non-negative integer n such that $K^n(G)$ is periodic (of period p).

A connected graph G is said to be *self-clique* if $K(G) \cong G$. As already indicated by the usage in [5], the difference between "self-clique" and "periodic of period 1" is connectedness.

A family $\mathcal{F}$ of subsets of a set $X \neq \varnothing$ has the *Helly property* (or *is Helly*) if $\cap\mathcal{S} \neq \varnothing$ for any $\mathcal{S} \subseteq \mathcal{F}$ such that $S, S' \in \mathcal{S} \Rightarrow S \cap S' \neq \varnothing$.

For example, the Helly property holds for stars: If v is a vertex of a graph G, the *star* of v is the set $v^* = \{Q \in K(G) : v \in Q\}$ which is complete in $K(G)$. Notice that $u^* \cap v^* \neq \varnothing$ iff u and v are either equal or neighbours. The family $\mathcal{F} = \{v^* : v \in G\}$ is Helly [8]: If $\mathcal{S} = \{v^* : v \in C\}$ is a pairwise intersecting family, C must be complete in G, so any clique Q with $C \subseteq Q$ belongs to $\cap\mathcal{S}$. Another example is any family $\mathcal{F} \subseteq E(G)$ where no three edges in $\mathcal{F}$ form a triangle. There is a restatement of the Helly property due to Roberts and Spencer [16]:

Theorem 2.1 *Let $\mathcal{F}$ be a family of subsets of X. Put $\widehat{A} = \{S \in \mathcal{F} : |S \cap A| \geq 2\}$ for $A \subseteq X$. Then $\mathcal{F}$ is Helly if and only if $\cap\widehat{A} \neq \varnothing$ for any $A = \{x, y, z\} \subseteq X$.* $\square$

A graph G is said to be *clique-Helly* (or just *Helly*, for short) if the family of all cliques of G has the Helly property. For instance, graphs without triangles are Helly: the non-trivial cliques are edges. If T is a triangle of a graph G, the *extended triangle* of T is $\widehat{T} = G[\{x \in V(G) : |N(x) \cap V(T)| \geq 2\}]$. The following characterization is due to Szwarcfiter [17]:

Theorem 2.2 *The graph G is clique-Helly if and only if each extended triangle of G has a universal vertex.* $\square$

The boxtimes product of graphs behaves well with respect to the clique operator [15] and the Helly property:

Proposition 2.3 *For any two graphs G_1 and G_2 we have:*

(1) $K(G_1 \boxtimes G_2) \cong K(G_1) \boxtimes K(G_2)$ and
(2) $G_1 \boxtimes G_2$ is clique-Helly if and only if G_1 and G_2 are so.

PROOF. Let $G = G_1 \boxtimes G_2$ and let $\pi_i : G \to G_i$ be the projections $(i = 1, 2)$. A subset $U \subseteq V(G)$ is complete if and only if both $\pi_1(U) \subseteq V(G_1)$ and $\pi_2(U) \subseteq V(G_2)$ are complete. Therefore, the cliques of G are the Cartesian products $Q_1 \times Q_2$ with $Q_i \in K(G_i)$ for $i = 1, 2$. Now (1) and (2) follow easily. $\square$

Let G be a clique-Helly graph. Any clique $\mathcal{Q}$ of $K(G)$, being a mutually intersecting family of cliques of G, must be contained in a star v^* but then, since stars are complete, $\mathcal{Q} = v^*$. Thus all the cliques of $K(G)$ are stars. Since the Helly property holds for stars, we have **Escalante's Satz 1** *[5]: If G is a clique-Helly graph, then so is $K(G)$.*

If v, w are vertices of a graph G, we say that w *dominates* v iff $N[v] \subseteq N[w]$. Equivalently, w dominates v iff $v^* \subseteq w^*$. Let G be clique-Helly and $v \in G$. Then v^* is a clique of $K(G)$ iff v dominates any $w \in G$ which dominates v. Thus, if $E \subseteq V(G)$ is such that any vertex of $V(G) \setminus E$ (but none of E) is dominated by another vertex in E, the set of cliques of $K(G)$ is precisely $\{v^* : v \in E\}$ and $|V(K(G))| = |E|$. Then, since $u^* \cap v^* \neq \varnothing$ iff u and v are adjacent or equal, we have **Escalante's Satz 2** *[5]: Let G be a clique-Helly graph. Let E be a minimal induced subgraph of G satisfying: for any vertex $v \in G$ there exists a vertex $w \in E$ such that w dominates v. Then we have $K^2(G) \cong E$.*

It follows that any clique-Helly graph is eventually K-periodic of period one or two. We say that a vertex v is *dominated* (without specifying by whom) only if v is dominated by a different vertex $w \neq v$. We will use the following reformulation of Escalante's Satz 3:

Theorem 2.4 [5] *Let G be a clique-Helly graph. The following are equivalent:*

(1) G is K-periodic.
(2) $K^2(G) \cong G$.
(3) G does not have dominated vertices.
(4) $\nu(v) = v^*$ defines an isomorphism $\nu : G \to K^2(G)$. $\square$

2.3 The Vertex-Clique Bipartite Graph

Let B be a bipartite graph. Then $V(B) = X \cup Y$, where X, Y are some disjoint and nonempty independent sets; $\{X, Y\}$ is a *bipartition* of B. As in [19], the notation (X, Y) for a bipartition of B means that we distinguish the vertices of X and Y as *left vertices* and *right vertices* respectively; $B = (X, Y)$ also denotes this. Given a bipartition (X, Y) of B, the *dual bipartition* is (Y, X). The relation of this with the theory of hypergraphs is explained by Wallis and Wu in [19].

The *vertex-clique bipartite graph* of a graph G is the graph $BK(G)$ with vertex set $V(G) \cup V(K(G))$ and edge set $\{\{v, Q\} : v \in G,\ Q \in K(G) \text{ and } v \in Q\}$. We will always consider $BK(G)$ as endowed with its *standard bipartition*: $X = V(G)$ and $Y = V(K(G))$. The neighbourhoods in $BK(G)$ are as follows: $N(v) = v^* \subseteq Y$ for $v \in X$ and $N(Q) = Q \subseteq X$ for $Q \in Y$. The following is immediate:

Lemma 2.5 *Let $B = BK(G)$ and let $B = (X, Y)$ be the standard bipartition. Then $G = B^2[X]$ and $K(G) = B^2[Y]$.* $\square$

A bipartition (X, Y) of B is said to be *right N-Sperner* (resp. *left N-Sperner*) if $N(z) \subseteq N(z') \Rightarrow z = z'$ for all $z, z' \in Y$ (resp. $z, z' \in X$). In a similar way, (X, Y) is *left N-Helly* (resp. *right N-Helly*) if the family $\{N(x) : x \in X\}$ (resp. $\{N(y) : y \in Y\}$) is Helly. If it is clear which ordered bipartition is involved, we can just talk about B itself being left (or right) N-Helly or N-Sperner.

Theorem 2.6 *Let B be a bipartite graph. Then there exists a graph G such that $B \cong BK(G)$ if and only if B has no isolated vertices and has a bipartition (X, Y) which is left N-Helly and right N-Sperner. For any such bipartition, the graph $G = B^2[X]$ satisfies $B \cong BK(G)$ and $K(G) \cong B^2[Y]$.*

PROOF. If $B = BK(G)$, take the standard bipartition. Cliques are maximal, so B is right N-Sperner. The Helly property holds for stars, so B is left N-Helly. An isolated vertex in B would be a vertex of G not contained in any clique or a void clique of G.

If $B = (X, Y)$ is left N-Helly and right N-Sperner, put $G = B^2[X]$. If $y \in Y$, the set $N(y)$ is complete in G. We claim that any complete of G is contained in one of these sets, hence they will be the cliques of G by the right N-Sperner condition. If $C \subseteq X$ is complete in G, the sets $N(v)$ for $v \in C$ are non-void and meet by pairs; since B is left N-Helly, there is a $y \in \cap\{N(v) : v \in C\}$, but then $C \subseteq N(y)$. $\square$

2.4 Periodic Helly Graphs

A graph G will be called *N-Sperner* whenever $N(v) \subseteq N(w) \Rightarrow v = w$ for all $v, w \in G$. Note that the only N-Sperner simple graph having isolated vertices is the trivial one. Similarly, G is called N-Helly if the family $\{N(v) : v \in G\}$ is Helly. We will say that G is *good* if G is both N-Sperner and N-Helly. Notice that for a bipartite graph, B is good iff some (every) bipartition (X, Y) of B and its dual (Y, X) are right N-Sperner and left N-Helly.

Theorem 2.7 *Let B be a bipartite graph. Then B is good if and only if there exists a periodic clique-Helly graph G with $B \cong BK(G)$. If this is the case, for any bipartition (X, Y) of B the graph $G = B^2[X]$ is periodic, clique-Helly, and satisfies $B \cong BK(G)$ and $K(G) \cong B^2[Y]$.*

PROOF. If $B = BK(G)$ and G is periodic and Helly, use the standard bipartition. By 2.6, B is left N-Helly and right N-Sperner. Since G is clique-

Helly, B is right N-Helly. By 2.4, G does not have dominated vertices, so B is left N-Sperner.

Assume now that B is good. Fix a bipartition $B = (X, Y)$ and put $G = B^2[X]$. Since (X, Y) is left N-Helly and right N-Sperner, $B \cong BK(G)$ and $K(G) \cong B^2[Y]$ by 2.6. Since (X, Y) is also right N-Helly and left N-Sperner, G is clique-Helly and has no dominated vertices. By 2.4, G is K-periodic. $\square$

Each connected periodic Helly graph yields a self-clique Helly graph:

Theorem 2.8 *Let G be a connected periodic Helly graph, and let $G' = G \boxtimes K(G)$. Then G' is a self-clique Helly graph.*

PROOF. By [5], Satz 1, we have that $K(G)$ is also Helly, and 2.4 yields $K^2(G) \cong G$. Using 2.3, G' is Helly and $K(G') \cong K(G) \boxtimes K^2(G) \cong G'$. $\square$

The same technique also yields non-Helly self-clique graphs. Recall that Escalante proved ([5], Satz 10) that for any period $p \geq 1$ there exists a periodic non-Helly graph with period p. Furthermore, using clockwork graphs (see [12], [13]) one can easily construct several infinite families of such graphs. Then we have:

Theorem 2.9 *Let G be a connected periodic non-Helly graph with period $p \geq 2$, and let $G' = G \boxtimes K(G) \boxtimes \cdots \boxtimes K^{p-1}(G)$. Then G' is a non-Helly self-clique graph.*

PROOF. Immediate from 2.3. $\square$

It also follows that $G \boxtimes G'$ is a non-Helly self-clique graph whenever G and G' are self-clique and at least one of them is non-Helly.

3 Conditions for Goodness

When we say that a graph H is possibly loopy we mean that it may or may not be the case that H has some loops, at most one at each vertex. The notion of goodness, hitherto used only for simple graphs, can also be applied to possibly loopy graphs H: we still say that H is *good* iff it is N-Helly and N-Sperner. One needs only to remember that, if $v \in H$, then $v \in N(v)$ iff there is a loop at v. The existence of a good H (or a good B) will be an important element in the characterizations of several families of graphs to be studied in this work.

The N-Sperner and N-Helly conditions are polynomially verifiable, but often it is very useful to have some necessary or sufficient conditions for a possibly loopy graph to be N-Sperner, N-Helly, or good. This section introduces several conditions of that kind which will be used in the following sections.

Proposition 3.1 *Let H be a connected possibly loopy graph with more than 2 vertices. Then, if H is N-Sperner, $\delta(H) \geq 2$.*

PROOF. Let $v \in H$. Assume that $|N(v)| = 1$. If $N(v) = \{v\}$, then v would be isolated. But then $N(v) \subseteq N(w)$ for some $w \in H$ with $d(v, w) = 2$, contradicting that H is N-Sperner. $\square$

If we add a loop to each vertex of K_2 we obtain a *dumbbell.*

Proposition 3.2 *Let H be a possibly loopy graph with $\delta(H) \geq 2$. Assume that*

(1) H has neither triangles nor squares.
(2) Each vertex in a dumbbell of H has degree greater than 2.

Then H is N-Sperner.

PROOF. Suppose that $u, v \in H$ are distinct and $N(u) \subseteq N(v)$. Assume first that $u \in N(v)$, so there is a loop at v. Let $w \in N(u)$, $w \neq v$. If $w \neq u$, H has a triangle. If $w = u$, then u is in a dumbbell and there is a third vertex $w' \in N(u)$, so H has a triangle. Assume now that $u \notin N(v)$. Take two distinct $w, w' \in N(u)$; since these vertices are in $N(v)$, no one of them is u; since also neither of them is v, there is a square in H. $\square$

Proposition 3.3 *Let the possibly loopy graph H be N-Helly. Then:*

(1) Any clique of H with more than 2 *vertices has a loop.*
(2) If H has no triangles, any chordless hexagon of H is embedded in an induced subgraph of H which is a cube with a diagonal (and perhaps some loops).

PROOF. (1): If $Q \in K(H)$ and $|Q| > 2$, the neighbourhoods of the vertices of Q must intersect by pairs. Since H is N-Helly, there exists $v \in H$ which is a neighbour of all the vertices of Q. Then $v \in Q$ by the maximality of Q. But then $v \in N(v)$, so there is a loop at v.

(2): Let $\mathcal{H} = (x, u, y, v, z, w)$ be a chordless hexagon in H. Since H is N-Helly, there exists $a \in H$ with $x, y, z \in N(a)$. Since $\mathcal{H}$ is chordless, $a \notin \mathcal{H}$ and $\mathcal{H} + a$ has only 9 edges because H has no triangles. Using again that H is N-Helly, there exists $b \in H$ with $a, u, v, w \in N(b)$. Again b is a new vertex, and $\mathcal{H}+a+b$ induces a cube with the only diagonal $\{a, b\}$. □

Theorem 3.4 *Let H be a possibly loopy graph. Then H is N-Helly if and only if* Gilmore's condition *holds: For any three different vertices with pairwise intersecting neighbourhoods there exists a vertex whose neighbourhood contains the union of the pairwise intersections of the neighbourhoods of those vertices.*

PROOF. This follows from 2.1: Put $\mathcal{F} = \{N(v) : v \in H\}$ and fix $A = \{v_1, v_2, v_3\}$. Put $I_{ij} = N(v_i) \cap N(v_j)$ for $i \neq j$, and $I = I_{12} \cup I_{13} \cup I_{23}$. Assume that $|A| = 3$ and all $I_{ij} \neq \emptyset$, for otherwise $\cap\widehat{A} \neq \emptyset$. Since $|N(v) \cap A| \geq 2 \Leftrightarrow v \in I$, if $w \in H$: $w \in \cap\widehat{A} \Leftrightarrow w \in N(v)$ $\forall v$ s.t. $|N(v) \cap A| \geq 2 \Leftrightarrow v \in N(w)$ $\forall v \in I \Leftrightarrow I \subseteq N(w)$. □

Consider the diagrams in Fig. 1. We shall say that a possibly loopy graph H is *compatible* with these diagrams if every subgraph of H which is isomorphic to one of these (solid line) graphs induces at least one of the dashed edges. Notice that compatibility with a set of diagrams is just shorthand for a set of forbidden induced subgraphs: replace each diagram with all graphs with the same vertex set having all the solid edges (perhaps others) and having none of the dashed edges. For instance, call the solid graph of diagram (a) a *barbell*, then compatibility with (a) just means that there are not induced barbells.

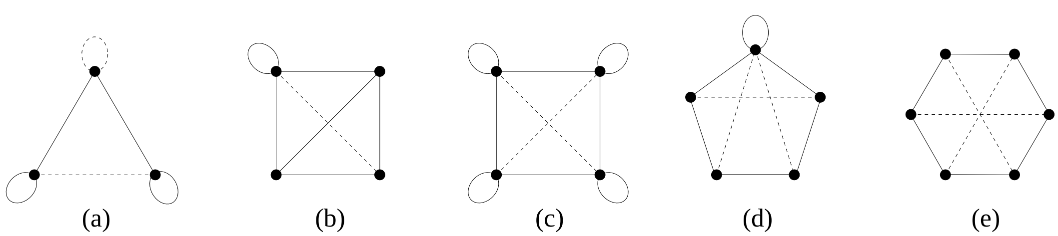


Fig. 1. A set of diagrams.

Theorem 3.5 *Let H be a possibly loopy graph such that every clique with at least three vertices has a loop. If H is compatible with the diagrams in Fig. 1, then H is N-Helly.*

PROOF. We use the notation $N(u, v) = N(u) \cap N(v)$. We will use 3.4. Fix 3 distinct vertices x, y, z of H having pairwise intersecting neighbourhoods.

We assume that none of these vertices is adjacent to every vertex in $I = N(x,y) \cup N(x,z) \cup N(y,z)$. In each case, we will get either a contradiction or another vertex w which is adjacent to every vertex in I.

Since none of x, y, z is adjacent to all of I, there are three (different) vertices x', y' and z' such that $x' \in N(y,z) - N(x)$, $y' \in N(x,z) - N(y)$ and $z' \in N(x,y) - N(z)$. If $\{x,y,z\} \cap \{x',y',z'\} = \varnothing$, we get an hexagon $\{x,z',y,x',z,y'\}$ and then H is not compatible with (e): Contradiction.

Suppose $\{x,y,z\} \cap \{x',y',z'\} \neq \varnothing$. In each of the following cases, we assume that only the indicated equalities hold, so vertices which are not declared equal are held to be distinct. By symmetry, we only have the following cases:

Case $x = x'$, $y = y'$ and $z = z'$: Since $\{x,y,z\}$ is complete, it is contained in some clique which has a loop. Either this loop is at (say) x and we get a contradiction to $\{x,x'\} \notin E(H)$, or it is at another vertex w which is adjacent to x, y and z. In the last case w is adjacent to all of I since H is compatible with (b).

Case $x = x'$, $y = z'$ and $z = y'$: Both y and z would have loops and compatibility with (a) would imply that either $\{x,x'\} \in E(H)$ or $\{y,y'\} \in E(H)$. Contradiction.

Case $x = y'$, $y = z'$ and $z = x'$: As $\{z',x\} \in E(H)$ we would get $\{y,y'\} \in E(H)$.

Case $x = x'$ and $y = y'$: Since $\{x,y,z\}$ is complete, there is a loop either at z (contradicting compatibility with (b)) or at another vertex w which is adjacent to x, y, z. Then w is adjacent to all of I since H is compatible with (b).

Case $x = x'$ and $y = z'$: Since x has no loop and $\{x,y',z\}$ is complete, there is a loop at y' or z or at a different vertex w which is adjacent to x, y' and z. In the first two cases, compatibility with (a) would imply $\{y,y'\} \in E(H)$ or $\{z,z'\} \in E(H)$. In the last case, compatibility with (a) yields $\{y,w\} \in E(H)$, and then compatibility with (b) would imply $\{y,y'\} \in E(H)$: A contradiction.

Case $x = y'$ and $y = x'$: Since $\{z,z'\} \notin E(H)$ and $\{x,x'\} \notin E(H)$, compatibility with (a) implies a loop at z and a loop at z'. Then H would not be compatible with (c).

Cases $x = y'$ and $y = z'$; $x = y'$ and $z = x'$: We would have, respectively, $\{y,y'\} \in E(H)$, $\{x,x'\} \in E(H)$.

Case $x = y'$: Then H would not be compatible with (d).

Case $x = x'$: Since $\{x, y, z'\}$ and $\{x, y', z\}$ are complete and $x = x'$ has no loop, it follows that there is a loop at y' or z or at a different vertex w which is adjacent to $\{x, y', z\}$ and also that there is a loop at y or z' or at a different vertex w' which is adjacent to $\{x, y, z'\}$. Notice that it may be the case that $w = w'$.

If y' has a loop, compatibility with (a) and (b) imply $\{y, y'\} \in E(H)$. On the other hand, if z has a loop then $\{z, z'\} \in E(H)$ by compatibility with (a) and (b) too. The symmetric cases in which there is a loop at z' or y give, respectively, $\{z, z'\} \in E(H)$ and $\{y, y'\} \in E(H)$ by the same reasons.

Assume that the loops are at w and w'. If $w \neq w'$ we have $\{w, w'\} \in E(H)$ since H is compatible with (a), and then $\{w, y\} \in E(H)$ by compatibility with (b). If $w = w'$ then w is adjacent to y because w' is so. Therefore, in any case, w is adjacent to x, y and z.

Thus, w is adjacent to all of $N(x, y) \cup N(x, z)$ by compatibility with (b). Take $u \in N(y, z)$, $u \notin \{x, y, z\}$. We have an hexagon $\{x, z', y, u, z, y'\}$ and $\{x, u\} \in E(H)$ by compatibility with (e). Then $\{w, u\} \in E(H)$ as G is compatible with (b). Now w is adjacent to all of I. $\square$

Corollary 3.6 *Let H be possibly loopy with $\delta(H) \geq 2$. Assume that:*

(1) There are no induced barbells in H.
(2) There are no triangles, squares or hexagons in H.
(3) No pentagon of H has a loop.
(4) Each vertex in a dumbbell of H has degree greater than 2.

Then H is good.

PROOF. By 3.5 and conditions (1), (2) and (3), H is N-Helly. By 3.2 and conditions (2) and (4), H is N-Sperner. $\square$

Corollary 3.7 *Any loopless graph G with $g(G) \geq 7$ and $\delta(G) \geq 2$ is good.*

PROOF. Immediate from 3.6 $\square$

Proposition 3.8 *Let $H = H_1 \times H_2$ for some non-trivial possibly loopy graphs H_1 and H_2. Then H is good if and only if both H_1 and H_2 are good.*

PROOF. This is due to $N((u, v)) = N(u) \times N(v) \neq \varnothing$ for all $(u, v) \in H$. $\square$

4 Type 1: Helly Self-Clique Graphs

A bipartition (X, Y) of a bipartite graph B is said to be *self-dual* if there exists an automorphism σ of B which transforms (X, Y) into its dual (Y, X) i.e. $\sigma(X) = Y$ and $\sigma(Y) = X$. Such a σ is called a *self-duality* or a *part-switching automorphism* of $B = (X, Y)$. Whenever B is connected, the bipartition $\{X, Y\}$ of B is unique, so one can speak about B itself being *self-dual* or not.

Lemma 4.1 *Let G be a connected graph such that $B = BK(G)$ is self-dual. Then $G \cong B^2[Z]$ for any bipartition (Z, T) of B.*

PROOF. Obviously B is connected iff G is so. Then the result follows from 2.5 and the uniqueness of $\{Z, T\}$ $(= \{X, Y\})$ because any self-duality of B is also an automorphism of B^2. Of course, $G \cong B^2[T]$ too. $\square$

Lemma 4.2 *Let G be a graph, and let $B = BK(G)$. Then the standard bipartition of B is self-dual if and only if there exist two isomorphisms $f : G \to K(G)$ and $g : K(G) \to G$ such that for all $v \in G$, $Q \in K(G)$ we have $v \in Q \Rightarrow g(Q) \in f(v)$.*

PROOF. If $\sigma : B \to B$ is a self-duality, consider it as an automorphism of B^2. The restrictions of σ to $B^2[X]$ and $B^2[Y]$ are isomorphisms $f : G \to K(G)$ and $g : K(G) \to G$. Thus $v \in Q \Rightarrow g(Q) \in f(v)$ because $v \in Q \Rightarrow \sigma(Q) \in \sigma(v)$.

Given f and g as in the statement, define $\sigma : B \to B$ by the rules $\sigma(v) = f(v)$ and $\sigma(Q) = g(Q)$ for any $v \in G$ and $Q \in K(G)$. Then σ is bijective as a vertex-map, and it is an automorphism of B because $v \in Q \Rightarrow \sigma(Q) \in \sigma(v)$. $\square$

Theorem 4.3 *If $B = (X, Y)$ is connected and bipartite, the following statements are equivalent:*

(1) B is good and self-dual.
(2) B is good and $B^2[X] \cong B^2[Y]$.
(3) $B \cong BK(G)$ for some Helly self-clique graph G.

Under these conditions, the graph G in (3) *is determined by $G \cong B^2[X]$.*

PROOF. That (1)$\Rightarrow$(2) is immediate. The last claim follows from 4.1.

Let B be good as in (2). By 2.7 we know that the graph $G = B^2[X]$ is Helly and satisfies $B \cong BK(G)$ and $K(G) \cong B^2[Y]$. Since $B^2[X] \cong B^2[Y]$, G is self-clique.

If $B = BK(G)$ with G as in (3), B is good by 2.7. Let $f : G \to K(G)$ be any isomorphism. Then $f_K : K(G) \to K^2(G)$ given by $f_K(Q) = f(Q) = \{f(x) : x \in Q\}$ is also an isomorphism. Let $g = \nu^{-1} \circ f_K : K(G) \to G$, where $\nu : G \to K^2(G)$ is the star isomorphism of 2.4(4). Then g is an isomorphism and $f(Q) = f_K(Q) = g(Q)^*$ for each $Q \in K(G)$. Since $v \in Q \Leftrightarrow f(v) \in f(Q) \Leftrightarrow f(v) \in g(Q)^* \Leftrightarrow g(Q) \in f(v)$, B is self-dual by 4.2. $\Box$

We can prove now our characterization of Helly self-clique graphs:

Theorem 4.4 *If G is a connected graph, the following are equivalent:*

(1) G is Helly and self-clique.
(2) $BK(G)$ is self-dual.

PROOF. That (1) implies (2) is immediate from 4.3. By 2.6, $BK(G)$ is left N-Helly and right N-Sperner, so $BK(G)$ self-dual implies $BK(G)$ self-dual and good. Then, by 2.5 and 4.3, $G = BK(G)^2[V(G)]$ is Helly and self-clique. $\Box$

5 Type 2: Involutive Graphs

A connected graph G will be said to be *involutive* if $BK(G)$ has a part-switching *involution*, i.e. a part-switching automorphism $\pi \in \mathrm{Aut}(BK(G))$ with $\pi^2 = \mathrm{id}$. Therefore, if G is involutive, then $BK(G)$ is self-dual and by 4.4 we have that G is a Helly self-clique graph.

If H is possibly loopy and $V(K_2) = \{0, 1\}$, the graph $B = K_2 \times H$ is clearly bipartite, and $B = (\{0\} \times H, \{1\} \times H)$ admits a part-switching involution. We will use the following result due to George, Porter and Wallis:

Theorem 5.1 [7] *Let $B = (X, Y)$ be a bipartite graph. Then B has a part-switching involution if and only if there exists some possibly loopy graph H such that $B \cong K_2 \times H$. Indeed, given the involution $\sigma : B \to B$, H can be taken as the quotient graph $H = B/\sigma$ which is constructed by identifying each vertex $x \in B$ with $\sigma(x) \in B$. A loop appears in H whenever x is adjacent to $\sigma(x)$.* $\Box$

Lemma 5.2 *Let H be possibly loopy, and let $B = K_2 \times H$. Then*

(1) B is connected if and only if H is connected and non-bipartite.
(2) If B is connected and $B = (X, Y)$, then $B^2[X] \cong H^{[2]}$.

PROOF. (1): Take two vertices $v = (k, h)$ and $v' = (k', h')$ in B. There is a path from v to v' in B iff either $k = k'$ and there is a walk of even length from h to h' in H, or $k \neq k'$ and there is a walk of odd length from h to h' in H.

(2): That $B^2[X] \cong H^{[2]}$ follows from $\{X, Y\} = \{\{0\} \times H, \{1\} \times H\}$. $\square$

Theorem 5.3 *Let B be a bipartite graph. The following are equivalent:*

(1) $B \cong BK(G)$ for some involutive graph G.
(2) B is connected, good and has a part-switching involution.
(3) B is connected and $B \cong K_2 \times H$ for some good possibly loopy graph H.
(4) $B \cong K_2 \times H$ with H possibly loopy, good, connected and non-bipartite.

PROOF. (1) and (2) are equivalent by 2.7. Since K_2 is good, (2) and (3) are equivalent by 5.1 and 3.8. That (3) and (4) are equivalent follows from 5.2(1). $\square$

Theorem 5.4 *A graph G is involutive if and only if $G \cong H^{[2]}$ for some possibly loopy, good, connected, non-bipartite graph H. In this case, $BK(G) \cong K_2 \times H$.*

PROOF. Let $B = BK(G)$. If G is involutive, we know by 5.3 that $B \cong K_2 \times H$ with H as in the statement. Using 2.5 and 5.2(2), we get $G = B^2[V(G)] \cong H^{[2]}$. Conversely, if $G \cong H^{[2]}$ with H as in the statement, we have by 5.2(1) that the bipartite graph $B_1 = K_2 \times H$ is connected. By 5.3, $B_1 \cong BK(G_1)$ for some involutive graph G_1. If $B_1 = (Z, T)$, $G_1 \cong B_1^2[Z] \cong H^{[2]} \cong G$ by 4.1 and 5.2(2), so G is involutive. $\square$

Some of the oldest and easiest examples of self-clique graphs are the small powers of cycles; they were used in [11], but a squared octagon appeared already in [5]. Let $n \geq 4$ and take the cycle C_n with vertices $\{0, 1, ..., n-1\}$ and edges $\{i, i+1\}$ (modulo n sum). Let s be an integer with $0 < s < \frac{n}{3}$. Then $G = C_n^s$ is an involutive graph. Indeed, the cliques of G are the sets $Q_i = \{i, i+1, ...i+s\}$. The pairing $i \leftrightarrow Q_{-i}$ gives a part-switching involution of $BK(G)$, so G is involutive.

Another family of examples is provided by our products in 2.8:

Proposition 5.5 *Let G be a connected periodic Helly graph, and consider the self-clique graph $G' = G \boxtimes K(G)$. Then G' is involutive.*

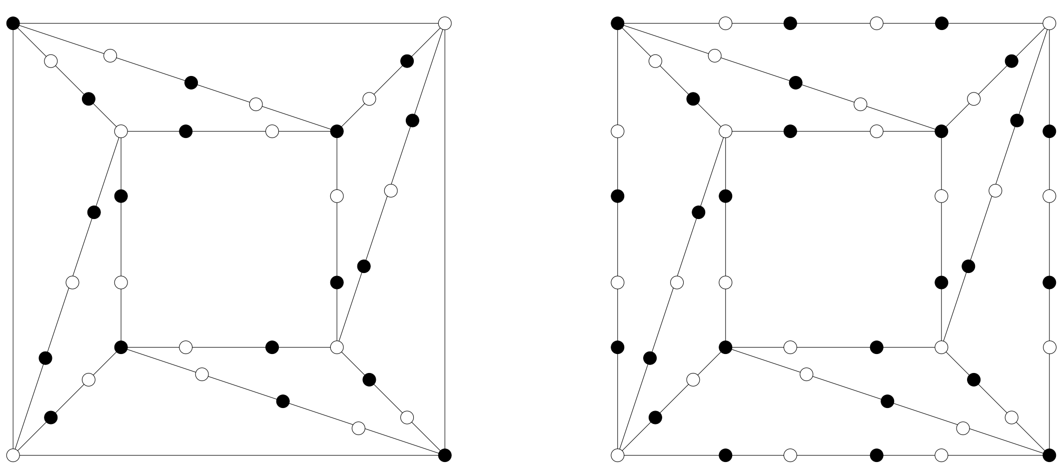
PROOF. By 2.4 and the proof of 2.3, the vertices of $K(G')$ are all the $Q \times v^*$ with $Q \in K(G)$ and $v^* \in K^2(G)$ (i.e. $v \in G$). Let $B' = BK(G')$. The vertex $(v, Q) \in G'$ is adjacent in B' to the vertex $R \times w^* \in K(G')$ iff $v \in R$ and $Q \in w^*$, iff $w \in Q$ and $R \in v^*$. Therefore, (v, Q) and $R \times w^*$ are neighbours in B' iff (w, R) and $Q \times v^*$ are so. This says that the pairing $(v, Q) \leftrightarrow Q \times v^*$ is a part-switching involution of B', so G' is involutive. □

Proposition 5.6 *Let G be a connected, non-bipartite, loopless graph. Assume that $g(G) \geq 7$ and $\delta(G) \geq 2$. Then the strict square $G^{[2]}$ is involutive.*

PROOF. By 3.7, G is good. By 5.4, G' is involutive. □

Proposition 5.7 *Not every Helly self-clique graph is involutive.*

PROOF. Consider the bipartite graphs B_1 and B_2 in Fig. 2. Since they are loopless, triangleless and free of induced hexagons, it follows from 3.5 that they are N-Helly. Since they are also N-Sperner, they are good. A quarter-turn is a part-switching automorphism for both of them. Therefore, $B_1 = BK(G_1)$ and $B_2 = BK(G_2)$ for some Helly self-clique graphs G_1 and G_2 by 4.3. However, none of these B_i has a part-switching involution: Indeed, the automorphism group in both cases is $\mathbb{Z}_4$ (look at the outer vertices) and the only involution does not switch the parts. □



B_1 B_2

Fig. 2. Two bipartite graphs.

Proposition 5.8 *Let G be a Helly self-clique graph. Assume that G does not have automorphisms of order two. Then G is involutive.*

PROOF. Let $B = BK(G) = (X, Y)$. By 4.4 there is a self-duality $\sigma : B \to B$. If the order of σ is $o(\sigma) = 2^m n$ with n odd, then $m \geq 1$. Now $\tau = \sigma^n$ is also part-switching and $o(\tau) = 2^m$. By 4.1, the restrictions of τ^2 to X and Y yield automorphisms of G of orders dividing $o(\tau^2) = 2^{m-1}$, so both restrictions are identity maps. Therefore, τ^2 is the identity in B and τ is a part-switching involution. $\square$

Proposition 5.9 *Let $G = H^{[2]}$ be an involutive graph, with H as in 5.4. Then the cliques of G are precisely the sets $N_H(v)$, $v \in G$, and we have that $N_H(v) = N_H(w)$ if and only if $v = w$; in symbols: $V(K(G)) = \dot{\bigcup}_{v \in G} \{N_H(v)\}$.*

PROOF. Identify $B = BK(G)$ with $K_2 \times H$. Thus $N_B(1, v) = \{0\} \times N_H(v)$. $\square$

6 Type 3: Clique-Disk Graphs

A connected graph G is *clique-disk* if there is a graph R such that $G = R^2$ and $V(K(G)) = \dot{\bigcup}_{v \in G} \{N_R[v]\}$. In other words, G has a (square) root, G does not have twins, and the cliques of G are the disks of radius 1 of R. If R is a graph, construct the totally loopy graph R° by attaching a loop to every vertex of R. Notice that, if G is clique-disk with root R, then $H = R^\circ$ satisfies

$$G = H^{[2]} \text{ and } V(K(G)) = \dot{\bigcup}_{v \in G} \{N_H(v)\}.$$

Theorem 6.1 *A connected graph G is clique-disk if and only if there is a totally loopy good graph H such that $G \cong H^{[2]}$. In particular, any clique-disk graph is involutive.*

PROOF. If G is clique-disk with root R, let $H = R^\circ$. We know that $G = R^2 = H^{[2]}$. Using that $N_H(v) = N_R[v]$ for all $v \in H$, we show that H is good: Let $S \subseteq V(H)$ be such that the sets $N_R[v]$ for $v \in S$ intersect by pairs. Then S is complete in G, and therefore there is a clique $N_R[w]$ with $S \subseteq N_R[w]$. Then $w \in N_R[v]$ for all $v \in S$ and H is N-Helly. Since no $N_R[u]$ is contained in $N_R[v]$ for $v \neq u$, H is N-Sperner too.

Assume now that $G = H^{[2]}$ with H as in the statement. Since H is not bipartite, G is involutive by 5.4. Let R be the underlying loopless graph of H. Therefore $G = H^{[2]} = R^2$ and, since $N_H(v) = N_R[v]$ for all $v \in H$, G is clique-disk by 5.9. $\square$

An immediate consequence is the following:

Theorem 6.2 *A connected graph G is clique-disk if and only if $B = BK(G)$ has a part-switching involution that maps every vertex x of B to a neighbour of x.* $\square$

The hypotheses in the following result were proved to be sufficient for self-cliqueness by Bondy, Durán, Lin and Szwarcfiter [3]. In fact, the first part of the proof of Theorem 1 in [3] shows, in our language, that G^{2k} is clique-disk:

Theorem 6.3 [3] *Let G be connected satisfying $\delta(G) \geq 2$ and $g(G) \geq 6k+1$ for some $k \geq 1$. Then G^{2k} is clique-disk, hence involutive.* $\square$

Let us just mention another proof : If $R = G^k$, it can be shown using 3.4 that R° is good, so 6.1 yields that $(R^\circ)^{[2]} = R^2 = G^{2k}$ is clique-disk.

Not every involutive graph is clique-disk. Indeed most involutive graphs considered in this paper are not clique-disk. The simplest example is C_4: the square is not a square!

Not every clique-disk graph comes from 6.3: For $n, m \geq 7$, $G = (C_n \boxtimes C_m)^2$ is clique-disk, but not of the form $G = P^{2k}$ with $g(P) \geq 6k+1$. Another example: Let us remove the vertex 1 from Escalante's graph R_8 (see Fig. 3) and call $R = R_8 - 1$. Then $g(R) = 3$, and it can be checked that R^2 is clique-disk, but R^2 is not of the form G^{2k} with $g(G) > 6k+1$.

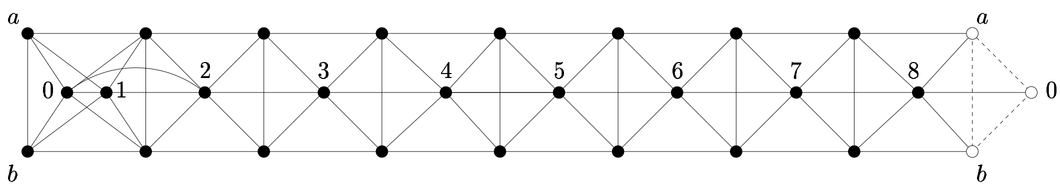


Fig. 3. Escalante's graph R_8 (identify vertices with same labels).

There are also square graphs which are self-clique but not clique-disk, an example is $(R_8)^2$ which is non-Helly and therefore it is not clique-disk.

We have already the elements for the **Hierarchy Theorem:**

Theorem 6.4 *The following proper inclusions hold:*

$$\textit{Type 3} \subsetneq \textit{Type 2} \subsetneq \textit{Type 1} \subsetneq \textit{Type 0}$$

PROOF. That Type 3 $\subseteq$ Type 2 follows from 6.2. By 4.4, Type 2 $\subseteq$ Type 1. It is clear from the definitions that Type 1 $\subseteq$ Type 0.

We have just remarked that the first inclusion is proper, and the second one is proper by 5.7. That the third inclusion is proper is due to Escalante [5], who gave the graphs R_k of Fig. 3 (see also our 2.9). $\square$

7 The Vertex-Clique Construction

Simple as it is, the following method for constructing possibly loopy good graphs will be useful to present in a unified way most of the previously known families of Helly self-clique graphs, thus showing them to be involutive. Furthermore, this unified approach easily yields expansions of those families and stronger versions of the results.

7.1 The construction

The *vertex-clique construction* is a method for obtaining good graphs H starting with the vertex-clique bipartite graph of a Helly graph P.

By attaching a *p-leg* to a graph we shall mean attaching a path of length $p \geq 0$ to one of the vertices and putting a loop at the free end of the new path (attaching a 0-leg is just attaching a loop). In a similar way, we can attach a *q-handle* to a graph by just joining two distinct vertices by a new path of length $q \geq 3$.

Let us begin with any Helly graph P (periodic or not) and take $B = BK(P)$ with its standard bipartition $B = (X, Y)$. Then we know that B is already left N-Helly, right N-Sperner and right N-Helly, so B is "almost good": only the dominated vertices of P prevent B from being left N-Sperner. It is easy to obtain from B a good graph H by attaching some legs and handles in such a way that each vertex $x \in X$ which is dominated in P gets a leg pending from it or a handle joining it to some other vertex. Of course, nothing forbids us to do this to other vertices of B (non-dominated vertices of X or vertices in Y). Since we also want H to be non-bipartite we must use at least one leg, or an odd length handle joining vertices in the same part of B, or an even length handle joining vertices in distinct parts. In principle, even more than one leg or handle can be attached at the same vertex, but this requires extra care: for instance, attaching a 0-leg and a 1-leg to a vertex results in a non-N-Sperner graph.

If the good graph H was constructed by our above vertex-clique construction, then $G = H^{[2]}$ is involutive and we have $B \subseteq H$, so P and $K(P)$ are subgraphs of G, and these are indeed induced in G.

An easy case works always fine: attach only p-legs and q-handles with $p \geq 1$ and $q \geq 6$, at most one at each vertex. We may also attach some 0-legs and shorter handles, but then we should be careful not to destroy N-Hellyness (for instance, by creating a 6-cycle or a barbell). The following result will be helpful here:

Proposition 7.1 *Let H be possibly loopy and N-Helly. Let H' be obtained from H by attaching either a p-leg with $p \geq 1$ or a q-handle with $q \geq 3$ in such a way that no new subgraph as the solid graphs in Fig. 1 is created. Then H' is N-Helly.*

PROOF. It is straightforward to check Gilmore's condition 3.4 for any 3 vertices x, y, z in H'. The four easy cases to consider are $|\{x, y, z\} \cap V(H)| = 0, 1, 2, 3$. $\square$

Attaching legs and handles to B can also be interpreted as operations on $B^{[2]}$. For instance, let us assume that every vertex of P gets at most one leg or handle (but not both): Starting with $H = B$ means starting with $G = P \cup K(P) = H^{[2]}$. Attaching a p-leg to a vertex $x \in P \subseteq H$ means (when we take the strict square of the resulting graph H') attaching a path of length p to the vertex $x \in P \subseteq G$ and then joining the free end of this path to all the cliques in $x^* \subseteq V(K(P)) \subseteq G$. If the p-leg is attached to a vertex $Q \in K(P) \subseteq H$, this is the same as attaching a path of length p to the vertex $Q \in K(P) \subseteq G$ and then joining the free end of this path to all the vertices in $Q \subseteq V(P) \subseteq G$. If $q = 2p + 1$ is odd, attaching a q-handle between the vertices $x, y \in V(P) \subseteq H$ translates into attaching a path of length p to $x \in P \subseteq G$ and then joining the free end of this path to all the cliques in $y^* \subseteq V(K(P)) \subseteq G$, and also attaching a path of length p to $y \in P \subseteq G$ and then joining the free end of this path to all the cliques in $x^* \subseteq V(K(P)) \subseteq G$. The other possibilities in the vertex-clique construction (for instance attaching even handles between vertices of $V(P)$ or attaching handles joining vertices of $V(P)$ with vertices of $V(K(P))$) also translate to similar simple operations on G.

7.2 Applications

Theorem 7.2 *Any graph G is an induced subgraph of some involutive graph.*

PROOF. Escalante ([5], Satz 5) proved that G is an induced subgraph of a connected periodic Helly graph, so we can assume G to be connected, periodic and Helly. Take $B = BK(G)$. Then B is already good by 2.7, but $B^{[2]} = G \cup K(G)$ is disconnected. Choose any non-empty $J \subseteq V(G)$ and construct the connected, non-bipartite good graph H by attaching a 1-leg to each $x \in J \subseteq V(B)$. Then $G' = H^{[2]}$ is involutive by 5.4, and we know that G and $K(G)$ are induced subgraphs of G'. $\square$

Let us remark that the construction given above generalizes the one given (ohne Beweiss) by Escalante [5] in the second proof of his **Satz 7**: *Any graph G can be realized as an induced subgraph of some Helly self-clique graph.* Theorem 7.2 is a stronger version of Escalante's Satz 7, and it also follows from 5.5, but with with a greater-order involutive graph.

In their paper [14], Lim and Peng gave a family of self-clique graphs G_n for $n \geq 2$. Consider the star $P = K_{1,n}$ and its clique graph $K(P)$. Then G_n is obtained from $P \cup K(P)$ by joining with an edge each terminal vertex of P to the unique clique containing it. Notice that the terminal vertices of P are precisely the dominated ones, and that attaching a loop (0-leg) to each of them in $B = BK(P)$ we obtain a good H such that $H^{[2]} = G_n$, so G_n is involutive. This can be substantially generalized:

Proposition 7.3 *Assume that P is a Helly graph such that each dominated vertex is contained in a unique clique and each clique contains at most one dominated vertex (for instance, if P is a tree with $|V(P)| > 2$). Then a good H is obtained by attaching a loop to each dominated vertex of P in $B = BK(P)$, so $G = H^{[2]}$ is involutive by 5.4.*

PROOF. Considered as vertices of B, the dominated vertices of P have degree one. Since B is bipartite, N-Helly and the only loops of H lie at the free vertices of some pendant (and non-incident) edges, it is easy to show that H satisfies Gilmore's condition in 3.4, so it is N-Helly. Since H is clearly N-Sperner, it is good. $\square$

Let now P be a Helly graph and let $J \neq \varnothing$ be a subset of $V(P)$ containing all the dominated vertices of P. Consider any partition $J = S \cup \{x_1, x_{r+1}\} \cup \ldots \cup \{x_r, x_{2r}\}$ of J into a set of "single vertices" S and a family of $r \geq 0$ pairs $\{x_i, x_{i+r}\}$. Further, consider $|S| + r$ integers $(p_x)_{x \in S}$ and $(q_i)_{i=1}^{r}$ with $p_x \geq 1$ for all $x \in S$ and q_i odd for $i = 1, \ldots, r$. We also require that $q_i \geq 5$ if $\{x_i, x_{i+r}\} \in E(P)$ and $q_i \geq 3$ otherwise. Now construct the graph H by starting with $B = BK(P)$, attaching a p_x-leg to each $x \in S \subseteq V(B)$ and joining $x_i \in B$ to $x_{i+1} \in B$ by means of a q_i-handle for $i = 1, \ldots r$.

Proposition 7.4 *The graph $G = H^{[2]}$ is involutive.*

PROOF. Applying 7.1 $|S|+r$ times, H is N-Helly. Since it is also N-Sperner, H is good. Therefore, $G = H^{[2]}$ is an involutive graph by 5.4. $\square$

The two constructions of self-clique graphs given by Balakrishnan and Paulraja in [1] can be reformulated as particular cases of this construction when interpreted as a construction on B^2. However, [1] uses the rather restrictive assumptions that P is a block graph (every block of P is complete, hence a clique), each cutpoint of P lies in exactly two blocks and J is exactly the set of dominated vertices of P.

8 Chia-Type Self-Clique Graphs

Chia [4] characterized all self-clique graphs with at most one clique having more than two vertices. All Chia graphs are diamond-free (see §9) and they are easily seen to be involutive, but with our methods we can go farther with little effort.

Theorem 8.1 *Let G be a connected graph such that at most 3 cliques of G have more than 2 vertices. Then G is self-clique if and only if G is involutive.*

PROOF. Assume G to be self-clique with at most 3 large cliques as in the statement. It follows immediately from Szwarcfiter's characterization 2.2 that any non-clique-Helly graph has at least four cliques with at least three vertices, so G is Helly. By 4.3, $B = BK(G)$ is good and self-dual. Let $S = \{x_1, \ldots, x_s\}$ be the set of vertices of B with degree greater than 2. Obviously $s = |S| \leq 6$. For $x, y \in S$ with $x \neq y$ (resp. $x = y$) we denote by $P\{x, y\}$ the set of all paths (resp. cycles) joining x with y and not using any other vertex of S. Then every vertex in $B - S$ belongs to a unique path or cycle in a unique $P\{x, y\}$.

We claim that a permutation τ of S can be extended to an automorphism $\hat{\tau}$ of B if and only if, for every $x, y \in S$ and every $l \in \mathbb{N}$, the sets $P\{x, y\}$ and $P\{\tau(x), \tau(y)\}$ have exactly the same number of paths (or cycles) of length l. In order to show this let us first define the action of $\hat{\tau}$ on the paths and cycles of B: If $(P\{x, y\}, \ldots, P\{\tau^{n-1}(x), \tau^{n-1}(y)\})$ is the τ-orbit of $P\{x, y\}$, just select any length-preserving bijection between the sets $P\{\tau^m(x), \tau^m(y)\}$ and $P\{\tau^{m+1}(x), \tau^{m+1}(y)\}$ for $m = 0, \ldots, n-2$ and then select the only length-preserving bijection between $P\{\tau^{n-1}(x), \tau^{n-1}(y)\}$ and $P\{x, y\}$ that makes the composition of all these bijections equal to the identity in $P\{x, y\}$ (in case $n = 1$, you have to select only one bijection, make it the identity). Once we do this on each such τ-orbit, we know the action of $\hat{\tau}$ on all the paths and cycles of B and then it is obvious how to define the action of $\hat{\tau}$ on B. Note that the order of the constructed extension satisfies $o(\tau) = o(\hat{\tau})$.

It follows from the previous claim that, if σ is an automorphism of B and τ is a permutation on S such that every τ-orbit of the set $X = \{\{x, y\} : x, y \in S\}$ is contained in a σ-orbit of X, then τ can be extended to an automorphism $\hat{\tau}$

of B such that $o(\tau) = o(\hat{\tau})$.

Now consider a part-switching automorphism σ of B. If $o(\sigma) = 2^p(2m+1)$, then σ^{2m+1} is also part-switching, so we may assume σ to have order 2^p for some $p \geq 1$. Now every σ-orbit must have length 2^q for some $q \geq 1$ and of course, S is σ-invariant. Since $|S| \leq 6$, the σ-orbits of the elements of S must be of length 2 or 4. If every such orbit has length 2 take τ as the permutation induced by σ on S and extend it to an automorphism $\hat{\tau}$ of B of order $o(\hat{\tau}) = o(\tau) = 2$. Otherwise, the permutation induced by σ on S is of the form (x_1, x_2, x_3, x_4) (for $s = 4$) or $(x_1, x_2, x_3, x_4)(x_5, x_6)$ (for $s = 6$). Then take $\tau = (x_1, x_2)(x_3, x_4)$ or $\tau = (x_1, x_2)(x_3, x_4)(x_5, x_6)$ and just notice that each τ-orbit of the set $X = \{\{x, y\} : x, y \in S\}$ is contained in some σ-orbit of X. $\square$

A *large clique* is one with more than two vertices, and a *Chia-Type* graph is one with at most 3 large cliques. Thus the previous result says that Chia-type graphs are self-clique precisely when they are involutive. Notice that the Helly self-clique graphs G_i represented by their vertex-clique bipartite graphs B_i in Fig. 2 have only 4 large cliques and are not involutive. The following result characterizes the Chia-type self-clique graphs.

Theorem 8.2 *Assume that at most* 3 *cliques of a connected non-trivial graph G have more than* 2 *vertices. Then G is self-clique if and only if $G \cong B^2[X]$ for some connected bipartite graph $B = (X, Y)$ satisfying:*

(1) B is N-Sperner and has a part switching involution.
(2) $\delta(B) \geq 2$ and at most 6 *vertices have degree greater than* 2.
(3) Every hexagon of B has a chord.

PROOF. Assume that $G \cong B^2[X]$ for some bipartite B satisfying (1), (2) and (3). By 3.5, B is N-Helly, so B is good. By 2.7, $B \cong BK(G)$. By (1), B is self-dual, so G is self-clique by 4.4.

Let G be self-clique. By 8.1, G is involutive, so $B = BK(G)$ is N-Helly, N-Sperner and has a part-switching involution. Obviously, $\delta(B) \geq 2$ and at most 6 vertices have degree greater than 2. By 3.3(2), any induced hexagon in B would give us at least 8 vertices in B with degree greater than 2, so (3) also holds. $\square$

We shall now give a more precise characterization for *Chia self-clique graphs* (i.e. with at most one large clique). It is much simpler than the original characterization of Chia, which may be obtained easily from it.

Recall that a p-leg is path of length $p \geq 0$ with a loop attached to one end. By definition, a *lobster graph* is any graph G with $\delta(G) = 2$ which is constructed starting with a single vertex x and attaching to x any number of legs (at least one leg) and any number of cycles all sharing precisely the vertex x. By a cycle here we mean one with at least 3 vertices, but there can be a loop at x (at most one) if a 0-leg was attached to it.

Theorem 8.3 *G is a Chia self-clique graph if and only if $G \cong H^{[2]}$ for some lobster graph H satisfying:*

(1) Every cycle in H has even length at least 8.
(2) If H has a 0*-leg, it does not have a* 1*-leg nor a* 2*-leg.*
(3) If H has a 1*-leg, it is unique.*

PROOF. Let H be a lobster graph satisfying (1), (2) and (3). By 3.5, H is N-Helly. By 3.2, H is N-Sperner, so H is good. By 5.4, G is involutive. Clearly G has at most one large clique.

Let G be a Chia self-clique graph, and consider $B = BK(G)$ with its standard bipartition (X, Y). Let S be the set of those vertices of B with more than two neighbours. If $S = \varnothing$, then G is a cycle and H may be taken to be a lobster graph with no cycles and two legs (of lengths at least 1 and 2). Otherwise, $S = \{x_1, x_2\}$ and every other vertex in B is in a unique path or cycle in a unique $P\{x, y\}$ with $x, y \in S$ (see the proof of 8.1).

All the cycles in $P\{x_1, x_1\}$ and $P\{x_2, x_2\}$ are even and all the paths in $P\{x_1, x_2\}$ are odd. By 8.1 B has a part switching involution σ. Take $\tau = (x_1, x_2) = \sigma|_S$. Now the construction in the proof of 8.1 gives us an automorphism $\hat{\tau} : B \to B$ reversing every path in $P\{x_1, x_2\}$. Therefore, the quotient graph $H = B/\hat{\tau}$ is a lobster graph with all cycles of even length. By 5.1, $B \cong K_2 \times H$ and by 3.8, H is good. By 2.5 and 5.2, $G = B^2[X] \cong H^{[2]}$.

If a cycle in H has length 4 OR there is a 0-leg and a 1-leg, H is not N-Sperner. If H has a cycle of length 6 OR H has a 0-leg and a 2-leg OR H has two 1-legs (but not a 0-leg), H is not N-Helly. $\square$

Similar characterizations for involutive graphs with few large cliques may also be obtained easily.

Let us remark that the lobster graphs H in 8.3 can all be obtained by the vertex-clique construction starting with the vertex-clique bipartite graph B of some complete graph $P = K_n$. Indeed, if H has l non-trivial legs of lengths $p_1, ..., p_l$ and c cycles of lengths $q_1, ..., q_c$, we take $n = 2c + l$. It is then clear how to attach (to the vertices in $V(P) \subseteq B$) l legs of lengths $p_1 - 1, ..., p_l - 1$

and c handles of lengths $q_1 - 2, ..., q_c - 2$. If H has a 0-leg we must attach it to the only vertex in $V(K(P)) \subseteq B$.

9 Involutive Diamond-Free Graphs

An edge of a graph is *multicliqual* if it lies in more than one clique. The *graphs without multicliqual edges* were first studied by Lim and Peng in their paper [14]. We prefer the name and alternative definition given in [18]: A *diamond* is a square C_4 with a diagonal (or a $K_4 - e$) and the graph G is *diamond-free* if it is free of induced diamonds. Clearly, a graph is diamond-free if and only if it has not multicliqual edges.

Lemma 9.1 *A graph G is diamond-free if and only if $BK(G)$ has no squares.*

PROOF. There is a square (v, Q, v', Q') in $BK(G)$ iff the edge $\{v, v'\}$ is multicliqual. $\square$

Theorem 9.2 *Let B be a bipartite graph with minimum degree at least 2. The following are equivalent:*

(1) B contains neither squares nor hexagons.
(2) B is good and contains no squares.
(3) $B \cong BK(G)$ for some non-trivial periodic diamond-free graph G.

If this is the case, for any bipartition (X, Y) of B the graph $G = B^2[X]$ is diamond-free, Helly, periodic, and satisfies $B \cong BK(G)$.

PROOF. If (1) holds, the girth of B is at least 8 and B is good by 3.7, so (2) holds. Suppose that $\mathcal{H}$ is an hexagon of B. If $\mathcal{H}$ has a chord, B has a square. If $\mathcal{H}$ is induced, B has a square by 3.3(2). Thus, (1) and (2) are equivalent. The equivalence of (2) and (3) and the last claim follow by 2.7 and 9.1. $\square$

Theorem 9.3 *Let G be a non-trivial graph. Then G is diamond-free and involutive if and only if $G \cong H^{[2]}$ for some connected non-bipartite possibly loopy graph H such that:*

(1) $\delta(H) \geq 2$.
(2) There are no triangles, squares or hexagons in H.
(3) No pentagon of H has a loop.
(4) The distance between any two loops of H is at least 3.

PROOF. Since no graph on two vertices is involutive, let us assume that $|V(G)| \geq 3$. Let G be diamond-free and involutive, and let $B = BK(G)$. Since G has no dominated vertex and B is self-dual, $\delta(B) \geq 2$. By 5.4, $G \cong H^{[2]}$ for some possibly loopy, good connected and non-bipartite graph H such that $B \cong K_2 \times H$; in particular, 3.1 implies $\delta(H) \geq 2$. By 9.2, B contains neither squares nor hexagons, which implies conditions (2), (3) and (4) for H.

Let $G = H^{[2]}$ for some H as in the statement. By 3.6 H is good, so G is involutive and $B = BK(G) \cong K_2 \times H$ by 5.4. Since B is good by 3.8 and a square in B would contradict conditions (2) or (4), $G \cong B^2[X]$ is diamond-free by 9.2. □

Notice that the graph G_2 which is represented by its vertex-clique bipartite graph $B_2 = BK(G_2)$ in Fig. 2 is diamond-free, Helly self-clique, but not involutive.

Theorem 9.4 *Let G be a connected periodic Helly graph, and consider the square B^2 of $B = BK(G)$. The following are equivalent:*

(1) G is diamond-free.
(2) B^2 is a self-clique graph.
(3) B^2 is a Helly self-clique graph.
(4) B^2 is involutive.
(5) B^2 is clique-disk
(6) B° is good.

PROOF.

(1)⇒(6): If G is diamond-free, $g(B) \geq 8$ by 9.2, and then B° is good by 3.6. (6)⇒(5): If B° is good, $B^2 = (B^\circ)^{[2]}$ is clique-disk by 6.1, from which (5)⇒(4) also follows. (4)⇒(3): We know since the start of §5 that any involutive graph is Helly self-clique. (3)⇒(2) is trivial.

(2)⇒(1): Let us first introduce some subsets of $V(B^2)$. For any $v \in G$, let $\widehat{v} = \{v\} \cup v^* \subseteq V(B^2)$, and for $Q \in K(G)$, let $\widehat{Q} = Q \cup \{Q\} \subseteq V(B^2)$. Since G does not have dominated vertices by 2.4, all these subsets $\widehat{x}$ for $x \in V(B^2)$ are cliques of B^2, and they are all distinct. All our cliques $\widehat{x}$ have just one vertex either in $V(G)$ or in $V(K(G))$, and they induce in $K(B^2)$ a subgraph isomorphic to B^2. Since B^2 is self-clique, it has no more cliques, but if G were not diamond-free, B would have a square by 9.1, so B^2 would have a clique containing at least two vertices in each of $V(G)$ and $V(K(G))$. □

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