Clique Convergent Surface Triangulations*

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Abstract

We show that almost all closed surfaces admit a Whitney triangulation whose underlying graph is clique convergent. The possible exceptions are the sphere, the projective plane, the torus and the Klein bottle. We also prove that any Whitney triangulation of the disk is clique null, provided that the degree of each interior vertex is at least 6.

Keywords: clique graphs, clique convergence, Whitney triangulations, clean triangulations.

1 Introduction

A *clique* of a (simple, connected) graph G is a maximal complete subgraph of G. We often identify induced subgraphs with their vertex sets; in particular, $v \in G$ means $v \in V(G)$. The *clique graph* $K(G)$ has all cliques of G as vertices, and $\{Q, Q'\} \in E(K(G))$ iff $Q \neq Q'$ and $Q \cap Q' \neq \varnothing$. Iterated clique graphs are inductively defined by $K^0(G) = G$ and $K^{n+1}(G) = K(K^n(G))$. The graph G is *K-divergent* if $\lim_{n\to\infty} |V(K^n(G))| = \infty$. We say that G is *K-convergent* if $K^i(G) \cong K^{i+p}(G)$ for some $i \geq 0$, $p \geq 1$; if i and p are minimal, they are the *transition index* and the *period* of G. Any graph is either K-convergent or K-divergent, but not both. A *K-null* graph G is such that $K^n(G)$ is the trivial graph K_1 for some n (so G is K-convergent). The *K-behaviour* of G can be either K-null, K-convergent but not K-null, or K-divergent. Extensive bibliography on clique graphs can be found in [8].

If $\mathcal{T}$ is a triangulation (simplicial decomposition) of a compact surface $\mathbb{X}$, and G is the underlying graph (1-skeleton) of $\mathcal{T}$, then any face (2-dimensional simplex) of $\mathcal{T}$ is a triangle of G. We call $\mathcal{T}$ a *Whitney triangulation* if any triangle of G is a face of $\mathcal{T}$. Thus a Whitney triangulation $\mathcal{T}$ is determined by its 1-skeleton G: the faces of $\mathcal{T}$ are just the triangles of G. This justifies identifying G with $\mathcal{T}$. Whitney triangulations have other names and have been studied before [1, 3, 5, 9, 10]; in particular, the description of their K-behaviour in the regular case has been completed in [5]. The tetrahedron K_4 is the underlying graph of an exceptional Whitney triangulation of the sphere: indeed, K_4 is more a 3-dimensional object than a 2-dimensional one. For instance: If G is a Whitney triangulation then, except for K_4, the cliques of G are precisely the faces of the triangulation.

The problem addressed in this work is: If G is a Whitney triangulation of some compact surface $\mathbb{X}$, to what extent the topology of $\mathbb{X}$ and the K-behaviour of G are related? For example, if G is K-null, Prisner proved [7] that $H_1(\mathbb{X}, \mathbb{Z}_2) = 0$, and it is even known [5] that $\pi_1(\mathbb{X}) = 1$, so $\mathbb{X}$ must be either the sphere or the disk. Another relation: If $G \neq K_4$ is regular and $\mathbb{X}$ is closed, then G is K-convergent if and only if the Euler characteristic $\chi(\mathbb{X})$ is negative [5]. This changes if we drop the regularity condition: We shall prove elsewhere that every closed surface $\mathbb{X}$ admits a K-divergent Whitney triangulation, but we will prove here that every such $\mathbb{X}$ also admits a K-convergent Whitney triangulation with the possible exceptions of the sphere, the projective plane, the torus and the Klein bottle.

These possible exceptions are interesting because they let live our conjecture that every Whitney tri-

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angulation $G \neq K_4$ of the sphere is K-divergent. A related conjecture is that every Whitney triangulation G of the disk is K-null, and in our last section we will prove this for the case in which the interior vertices have degree at least 6.

2 A Triangulation of $\mathbb{T}\#\mathbb{P}$

We denote by $\mathbb{T}$ and $\mathbb{P}$ the torus and the projective plane. Here we shall construct a K-convergent Whitney triangulation of the connected sum $\mathbb{T}\#\mathbb{P}$, the closed surface with Euler characteristic -1.

Given a graph G, the *neighbourhood* $N(x)$ of a vertex $x \in G$ is the subgraph of G induced by the set of adjacent vertices of x in G. Given two graphs G and H, we say that G is *locally* H if $N(x)$ is isomorphic to H for any vertex $x \in G$. Given a family of graphs $\mathcal{H} = \{H_1, H_2, \ldots\}$ we say that G is *locally* $\mathcal{H}$ if for each $x \in G$ there is an $H_i \in \mathcal{H}$ with $N(x) \cong H_i$. We denote by C_n and P_n the cycle and the path on n vertices. We say that G is *locally cyclic* if it is locally $\{C_n : n \geq 3\}$. The following characterization seems to be well known in the literature (see [6]) a proof can be found in [5].

Theorem 2.1 *G is the underlying graph of a Whitney triangulation of a closed surface (resp. compact surface) if and only if G is locally cyclic (resp. G is locally $\{C_n, P_m : n \geq 3, m \geq 2\}$).* $\square$

The projective plane with an open disk removed is just the Möbius band. We will construct our triangulation of $\mathbb{T}\#\mathbb{P}$ from a triangulation of the torus with a disk removed and one of the Möbius band.

We obtain our triangulation of the Möbius band by the circulant $M = C_9(1,4)$, i.e. $V(M) = \mathbb{Z}_9$ and $E(M) = \{\{u,v\} : u - v = \pm 1, \pm 4\}$. This graph is depicted in Figure 1.

Now consider the locally C_6 graph T in Figure 2 (identify opposite sides in orientable manner). T is obtained by taking $V(T) = \mathbb{Z}_5 \oplus \mathbb{Z}_5$ and $E(T) = \{\{u,v\} : u - v = \pm(1,0), \pm(0,1), \pm(1,-1)\}$. It is clearly a Whitney triangulation of the torus. The shaded triangle in Figure 2 is the disk that we will remove in order to take the connected sum.

Accordingly, take T and remove the vertex $(2,2)$ together with all its edges and also remove the edges $\{(1,3),(2,3)\}$, $\{(1,2),(2,1)\}$ and $\{(3,1),(3,2)\}$. Call the resulting graph T'.

We construct the sought graph H by taking the disjoint union of M and T' and identifying the

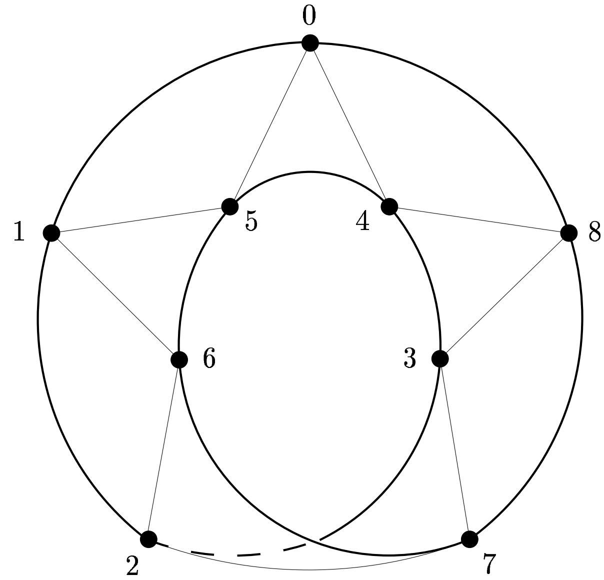

Figure 1: The Möbius band $M = C_9(1,4)$.

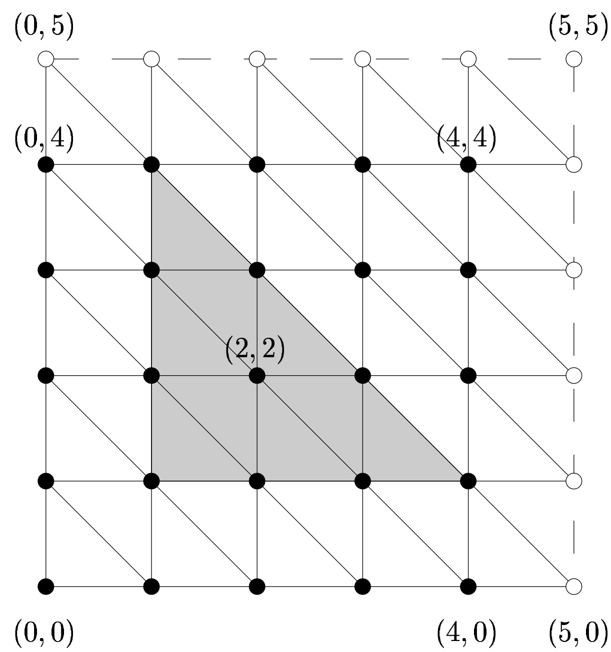

Figure 2: A torus T on $\mathbb{Z}_5 \oplus \mathbb{Z}_5$.

borders. More precisely, we identify the following pairs of vertices: $\{0,(1,4)\}$, $\{1,(2,3)\}$, $\{2,(3,2)\}$, $\{3,(4,1)\}$, $\{4,(3,1)\}$, $\{5,(2,1)\}$, $\{6,(1,1)\}$, $\{7,(1,2)\}$ and $\{8,(1,3)\}$. A direct verification shows that H is locally $\{C_6, C_8\}$ and therefore a Whitney triangulation of a surface. Since this is just the connected sum of $\mathbb{T}$ and $\mathbb{P}$, H is a Whitney triangulation of $\mathbb{T}\#\mathbb{P}$. Alternatively, just check that $|H| = V = 24$, $E = 75$ and $F = 50$, so $\chi(H) = V - E + F = -1$.

Now, a computer verification (we used GAP [2]) shows that H is indeed clique convergent (note that T is clique divergent by [4]) with transition index 6 and period 2. The sequence of orders of the iterated clique graphs of H is $24, 50, 74, 92, 100, 105, 106, 106, 106\ldots$

Thus we know that $\mathbb{T}\#\mathbb{P}$ admits a clique convergent Whitney triangulation. We will use this fact to construct clique convergent Whitney triangulations for most surfaces using triangular covering maps.

3 Triangular Covering Maps

If G is a graph and $v \in G$, the *closed neighbourhood* of v is $N[v] = N(v) \cup \{v\}$. If G and H are graphs, a *triangular covering map* $p : G \to H$ is just a local isomorphism, i.e. a mapping $p : V(G) \to V(H)$ such that the restriction $p_| : N[v] \to N[p(v)]$ is a graph isomorphism for any $v \in G$. All the fibers $p^{-1}(w)$ have the same cardinality: this is the *number of sheets* of p. We call p *finite* if it has a finite number of sheets. For an equivalent definition in terms of unique lifting properties and for more details we refer to [4].

Theorem 3.1 [4] *If $p : G \to H$ is a finite triangular covering map, there is another such map $p_K : K(G) \to K(H)$ with the same number of sheets. In particular, G and H have the same K-behaviour.* □

Theorem 3.2 *Each closed surface $\mathbb{X}$ with $\chi(\mathbb{X}) \le -1$ admits a clique convergent Whitney triangulation.*

Proof. Let us denote by $\mathbb{Y}$ the surface $\mathbb{Y} = \mathbb{T}\#\mathbb{P}$. We first observe that there is a finite (topological) covering map $\pi : \mathbb{X} \to \mathbb{Y}$. This is easy: If non-orientable, $\mathbb{X}$ is a sphere with one handle and $k \ge 1$ crosscaps ($\mathbb{X} = \mathbb{Y}$ for $k = 1$); now use the k-to-1 covering map from $\mathbb{T}$ to itself. If $\mathbb{X}$ is orientable, it is a sphere with $h \ge 2$ handles; now use the 2-to-1 covering map from the sphere to $\mathbb{P}$ and the $(h-1)$-to-1 covering map from $\mathbb{T}$ to itself.

Now consider the Whitney triangulation H of $\mathbb{Y}$ constructed in the previous section. We can look at H as an embedded graph in $\mathbb{Y}$: the vertices of H are points in $\mathbb{Y}$ and the edges of H are continuous curves in $\mathbb{Y}$; the components of $\mathbb{Y} - H$ are topological disks, and each triangle of H bounds precisely one of these disks. We now define the graph G (embedded in $\mathbb{X}$) by lifting along the covering map $\pi : \mathbb{X} \to \mathbb{Y}$: the vertex set of G is $V(G) = \cup\{\pi^{-1}(v) : v \in H\}$, and the edges of G are the curves $\gamma : I \to \mathbb{X}$ such that $\pi \circ \gamma$ is an edge of H.

Finally, define $p : G \to H$ as the restriction of π. As shown in [4], p is then a triangular covering map, which is finite because π is so. Since p is a local isomorphism, G is locally cyclic. By Theorem 3.1, G and H have the same K-behaviour, but as H is K-convergent so is G. □

The following problem immediately arises:

Problem 1 *Determine if there exist clique convergent Whitney triangulations for the sphere (not K_4), the projective plane, the torus and the Klein bottle.*

We have however a lot of experimental evidence suggesting that the sphere might not have such a triangulation.

Conjecture 2 [5] *Except for K_4, every Whitney triangulation of the sphere is clique divergent.*

Note that an affirmative answer to this conjecture implies, by Theorem 3.1, that also every Whitney triangulation of the projective plane is clique divergent. However, it seems that just the opposite is true for triangulations of the disk:

Conjecture 3 [5] *Every Whitney triangulation of the disk is clique null.*

We shall prove a weak version of this conjecture in the following section.

4 Triangulations of the Disk

Let $x, y \in G$. We say that *x is dominated by y* if $N[x] \subseteq N[y]$. However, we say that *x is dominated* (without specifying who is y) only when it is dominated by some $y \ne x$.

If $H \subseteq G$, we say that *G is dismantleable to H* if we can obtain H from G by successively removing dominated vertices. If G is dismantleable to a vertex, we just say that G is *dismantleable*. Prisner gave a different but (straightforwardly) equivalent definition of a dismantleable G, and proved the following:

Theorem 4.1 (Prisner [7]) *If G is dismantleable, then G is K-null.* $\square$

Theorem 4.2 *Let G be a Whitney triangulation of the disk. Let i and e be the numbers of interior and exterior vertices of G. Let $\bar{d}_i$ and $\bar{d}_e$ be the average degrees of the interior and exterior vertices. Then*

$$\bar{d}_e = \frac{i}{e}\left(6 - \bar{d}_i\right) - \frac{6}{e} + 4.$$

Proof. Counting the external face, by Euler: $\chi = V - E + F = 2$. Here $V = i + e$, $E = \frac{1}{2}\left(i \cdot \bar{d}_i + e \cdot \bar{d}_e\right)$ and if T is the number of triangles, we have $i \cdot \bar{d}_i + e \cdot \bar{d}_e = 3T + e$ (i.e. the sum of the degrees is equal to the number of faces counted as many times as the number of vertices belonging to them). Thus $F = T + 1 = \frac{1}{3}\left(i \cdot \bar{d}_i + e \cdot \bar{d}_e - e\right) + 1$ and then:

$$\begin{aligned} 2 &= i + e - \frac{i \cdot \bar{d}_i + e \cdot \bar{d}_e}{2} + \frac{i \cdot \bar{d}_i + e \cdot \bar{d}_e - e}{3} + 1 \\ &= i + \frac{2}{3}e - \frac{1}{6}\left(i \cdot \bar{d}_i + e \cdot \bar{d}_e\right) + 1. \end{aligned}$$

So we have $6 = 6i + 4e - i \cdot \bar{d}_i - e \cdot \bar{d}_e$, and it follows that $\bar{d}_e = \frac{i}{e}\left(6 - \bar{d}_i\right) - \frac{6}{e} + 4$. $\square$

Theorem 4.3 *Let G be a Whitney triangulation of the disk satisfying $\bar{d}_i \geq 6$. Then G has at least two exterior vertices with degree at most 3.*

Proof. If all but one of the external vertices had degree at least 4 and $\bar{d}_i \geq 6$, we would have:

$$\begin{aligned} \bar{d}_e &= \frac{i}{e}\left(6 - \bar{d}_i\right) - \frac{6}{e} + 4 \leq 4 - \frac{6}{e} \\ &< 4 - \frac{2}{e} = \frac{4(e-1)+2}{e} \leq \bar{d}_e \end{aligned}$$

which is a contradiction. $\square$

Theorem 4.4 *Let G be a Whitney triangulation of the disk such that each interior vertex has degree ≥ 6. Then G is dismantleable to any of its vertices.*

Proof. Let $u \in G$ be the vertex we want to dismantle G to. Thanks to the previous theorem there is an exterior vertex $v \in G$, $u \neq v$ with degree at most 3, but then the open neighbourhood of v induces a path of at most 3 vertices and v is dominated by a vertex w of this path. Then G is dismantleable to $G - v$. If $G - v$ is a triangulation of the disk, we finish by induction. Otherwise, $G - v$ is either an edge or it is the union of two subgraphs of $G - v$, say B and C, which share exactly one vertex, namely w.

If $G - v$ is an edge, it is clearly dismantleable to any of its vertices, so G is dismantleable to u. Otherwise B and C may be a pair of edges, an edge and a triangulation of the disk, or a pair of triangulations of the disk. Without loss of generality, we may assume that $u \in B$. Then, since C is dismantleable to w (by inductive hypothesis or because any edge is dismantleable to any of its vertices), we have that $G - v$ is dismantleable to B. Finally B is dismantleable to u either by inductive hypothesis or because B is an edge. $\square$

Theorem 4.5 *Let G be a Whitney triangulation of the disk such that every interior vertex has degree ≥ 6. Then G is K-null.*

Proof. By the previous theorem, G is dismantleable. Then use Theorem 4.1. $\square$

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