Weighted Cages

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Abstract

Cages (r-regular graphs of girth g and minimum order) and their variants have been studied for over seventy years. Here we propose a new variant, *weighted cages*. We characterize their existence; for cases $g = 3, 4$ we determine their order; we give Moore-like bounds and present some computational results.

1 Introduction

Cages [8] have been studied since 1947 when they were introduced by Tutte in [18]. They are regular graphs of a given girth with the smallest number of vertices for the given parameters. In 1963 Sachs [16] proved that for each $k \geq 2$ and each $g \geq 3$ there is a k-regular graph of girth g which implies that a cage exists for each such parameters. The smallest integer n for which there is a k-regular graph of girth g on n vertices is denoted by $n(k, g)$ and a k-regular graph of girth g with $n(k, g)$ vertices is called a (k, g)-cage. Several variations of the notion of cage have been studied in the literature including, among others: Biregular cages [1, 9], biregular bipartite cages [4, 11], vertex-transitive cages [14], Cayley cages [10], mixed cages [2, 3, 7] and mixed geodetic cages [17].

Standard terminology on graph theory used here, will be quickly reviewed in the next section.

In this work, we extend the notion of cage to weighted graphs. In general, a *weighted graph*, is a (simple, finite) graph $G = (V, E)$ together with a weight function $w : E(G) \to \mathbb{R}$, however, to keep the presentation as simple as possible,

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[4]Partially supported by PAPIIT-México under grant IN113324 and CONAHCyT: CBF 2023-2024-552.

[5]Partially supported by CONAHCYT, grant 799875.

[6]Partially supported by CONAHCYT, grant A1-S-45528.

[7]Supported by Centro de Modelamiento Matemático (CMM), FB210005, BASAL funds for centers of excellence from ANID-Chile.

we shall focus on weight functions of the form $w : E(G) \to \{1,2\}$. An edge with weight 1 is called a *light edge* while one with weight 2 is called a *heavy edge*.

Under these circumstances, each weighted graph G, *wgraph* for short, may be specified by a couple of graphs $L = L(G)$ and $H = H(G)$, which are the spanning subgraphs of G formed by the light edges and the heavy edges of G (respectively). Hence, we shall represent a wgraph G by $G = (L,H)$, where L and H are graphs such that $V(L) = V(H)$ and $E(L) \cap E(H) = \varnothing$.

In order to maintain the regularity aspect of the original notion we require L and H to be regular. Thus an (a,b)*-regular wgraph* is a wgraph $G = (L,H)$ where L is a-regular and H is b-regular. A *wcycle* in G is a cycle whose edges may be light or heavy and its *weight* is the sum of the weights of the edges composing it. The *girth* of a wgraph G is the minimum weight of its wcycles. Finally, by analogy with cages, we may define an (a,b,g)*-wgraph* as an (a,b)-regular wgraph of girth g, and an (a,b,g)*-wcage* as an (a,b,g)*-wgraph* of minimum order. We shall represent the order of an (a,b,g)-wcage by $n(a,b,g)$.

In this paper, we characterize their existence and, for the cases $g = 3,4$, we determine the value of $n(a,b,g)$; We also determine $n(a,b,g)$ for $a = 1,2$ when $g = 5,6$. We give Moore-like bounds and present some computational results.

An interesting feature of weighted cages is that, contrary to what happens with ordinary cages, $n(a,b,g)$ is not always monotone increasing in all its parameters, since we shall see that $n(3,1,4) = 8 > 6 = n(3,2,4)$ in Section 6 and that $n(4,1,5) = 20 > 19 = n(4,2,5)$ in Section 8.

We note that many of our results may be readily extended to weights of the form $w : E(G) \to \{w_1, w_2\} \subset \mathbb{N}$.

2 Terminology and Preliminaries

Our graphs are simple and finite. We use standard terminology for denoting the *set of vertices* and the *set of edges* of a graph X: $X = (V,E)$, $V = V(X)$ and $E = E(X)$. The *order* of a graph X is $|X| = |V(X)|$. An *edge* is an unordered pair of vertices $\{x,y\}$, which we may also write as xy. We write $x \simeq_X y$ for the *adjacent-or-equal relation* on a graph X. The *degree* of a vertex x in X is defined by $deg_X(x) = |\{xy : xy \in E(X)\}|$. The maximum degree is $\Delta(X) = \max\{deg_X(x) : x \in V(X)\}$. A graph X is r-regular if $deg(x) = r$, for all vertices $x \in V(X)$. The *distance* between vertices x and y in X is denoted by $dist_X(x,y)$. The *complete graphs* on n vertices are represented by K_n and the *complete balanced bipartite graphs* on n vertices are denoted by $K_{m,m}$, where $m = \frac{n}{2}$. Given graphs X and Y, some standard operations on graphs are: the *complement* of a graph $\overline{X} = (V(X), \overline{E(X)})$, where $\overline{E(X)} = \{\{x,y\} : x,y \in V(X), x \neq y \text{ and } \{x,y\} \notin E(X)\}$, the *square* of a graph $X^2 = (V(X), E(X^2))$, where $E(X^2) = \{\{x,y\} : 0 < dist_X(x,y) \leq 2\}$ and the *union* of graphs $X \cup Y = (V(X) \cup V(Y), E(X) \cup E(Y))$, while the *disjoint union* is $X \uplus Y = (V(X) \uplus V(Y), E(X) \uplus E(Y))$. Here we define the *difference* of graphs as $X - Y = (V(X), E(X) - E(Y)$, that is, the edges of Y, are removed from X, but not the vertices. The *girth* $g(X)$ of a graph, X, is the length of a shortest cycle in X. An (r,g)-*graph* is an r-regular graph of girth g. An (r,g)-*cage* is an (r,g)-graph of minimum order. The order of an (r,g)-cage is denoted by $n(r,g)$; when no such cage exists, we define $n(r,g) = \infty$ (this happens exactly when $r < 2$ or $g < 3$).

A *weighted graph* (*wgraph* for short) is $G = (L,H)$, where $L = L(G)$ is the *light-subgraph* of G and $H = H(G)$ is the *heavy-subgraph* of G; both L and H are ordinary graphs and we require that $V(L) = V(H)$ and $E(L) \cap E(H) = \varnothing$. Light edges have weight 1 and heavy edges have weight 2. A *wcycle* (*wpath*) in G is a cycle (path) whose edges may be light or heavy and its *weight* is the sum of the weights of the edges composing it. The *wdistance* between two vertices x and y in G is the minimum weight of a wpath in G joining x and y. Other terms like *wtree* and *subwgraph* will be used with the obvious meaning.

We say that $G = (L,H)$ is (a,b)-*regular* if L is a-regular and H is b-regular. The *girth*, $g(G)$, of a wgraph is the minimum weight of a wcycle in G. An (a,b,g)-*wgraph* G is an (a,b)-regular wgraph G of girth g and an (a,b,g)-*wcage* is an (a,b,g)-wgraph of minimum order. We define $n(a,b,g)$ as the order of an (a,b,g)-wcage (and we define $n(a,b,g) = \infty$ if there is no such (a,b,g)-wcage).

It should be clear that $n(a,b,g) = \infty$ whenever $a + b \le 1$ or $g < 3$. Also, it is immediate that $n(a,0,g) = n(a,g)$ and that $n(0,b,g) = n(b,\frac{g}{2})$ whenever g is even and $g \ge 6$ (otherwise, $n(0,b,g) = \infty$). A $(1,1,g)$-wcage must be a wcycle of weight g with alternating light and heavy edges, and hence:

$$n(1,1,g) = \begin{cases} \frac{2g}{3} & \text{if } g \ge 6 \text{ and } g \equiv 0 \mod 3, \\ \infty & \text{otherwise.} \end{cases}$$

We shall use *congruence module 2* very often, and hence we shall abbreviate "$x \equiv y \mod 2$" simply as "$x \equiv y$". It is a well know result (sometimes called *the first theorem of graph theory* or the *degree-sum formula*) that the sum of the degrees of a graph is even (and equals twice the number of edges). For an r-regular graph of order n, this means $nr \equiv 0$, and hence that there are no odd-regular graphs of odd order. This fact will be used very often in this paper and we shall refer to it simply as "*parity forbids*", as in: "parity forbids $r = 3$ and $n = 7$". We shall often need the following four lemmas:

Lemma 2.1. *Let $a, b \ge 0$ and $g \ge 3$. Then $n(a,b,g) \ge a + b + 1$. Moreover, if $ab \equiv 1$, then $n(a,b,g) \ge a + b + 2$.*

Proof. If there is no (a,b,g)-wcage, then, by definition, $n(a,b,g) = \infty$ and the inequalities hold. Otherwise, take an (a,b,g)-wcage $G = (L,H)$ and a vertex $x \in G$. Then x must have a neighbors in L and b neighbors in H, and therefore the closed neighborhood of x in G must have $a+b+1$ vertices. Thus $n(a,b,g) = |G| \ge a+b+1$. Parity forbids $n = a+b+1$ when $ab \equiv 1$, hence $n(a,b,g) \ge a+b+2$ in that case. □

Recall that a k-*factor*, F, of a graph X is a k-regular spanning subgraph of X. Thus a 1-factor is a *perfect matching* and a 2-factor is a collection of cycles that span all of X. A k-*factorization* of X is a decomposition of X into

k-factors, that is, a collection of k-factors $\{F_i\}_{i\in I}$, such that $E(F_i)\cap E(F_j)=\varnothing$ for all $i\neq j$ and $G=\bigcup_{i\in I}F_i$.

Lemma 2.2. *If $5\leq n\equiv 1$, there is a 2-factorization of K_n, $K_n=\bigcup_{i=1}^{\lfloor\frac{n}{2}\rfloor}F_i$, such that $F_1\cup F_2$ contains a triangle.*

Proof. Label the vertices of K_n with the elements of $\mathbb{Z}_n$. For $i\in\{1,2,\ldots,\lfloor\frac{n}{2}\rfloor\}$ define F_i as the spanning subgraph of K_n having edge set $E(F_i)=\{\{x,x+i\}:x\in\mathbb{Z}_n\}$. It is straightforward to verify that $\{F_i\}_{i=1}^{\lfloor\frac{n}{2}\rfloor}$ is a 2-factorization of K_n. A triangle in $F_1\cup F_2$ is induced by the vertices $\{0,1,2\}$. □

Lemma 2.3. *If $4\leq n\equiv 0$, there is a 1-factorization of K_n, $K_n=\bigcup_{i=0}^{n-2}\tilde{F}_i$, such that $\tilde{F}_0\cup\tilde{F}_1\cup\tilde{F}_2$ contains a triangle.*

Proof. Label one vertex of K_n as $*$ and label the remaining vertices with the elements of $\mathbb{Z}_{n-1}$. For $i\in\mathbb{Z}_{n-1}$, define $\tilde{F}_i$ as the spanning subgraph of K_n having edge set $E(\tilde{F}_i)=\{\{*,i\}\}\cup\{\{i+k,i-k\}:k\in\{1,2,\ldots,\frac{n-2}{2}\}\}$. It is straightforward to verify that $\{\tilde{F}_i\}_{i=0}^{n-2}$ is a 1-factorization of K_n. A triangle in $\tilde{F}_0\cup\tilde{F}_1\cup\tilde{F}_2$ is induced by the vertices $\{*,0,2\}$. □

Lemma 2.4. *If $m\geq 3$ there is a 1-factorization of $K_{m,m}$, $K_{m,m}=\bigcup_{i=0}^{m-1}\hat{F}_i$, such that $\hat{F}_0\cup\hat{F}_1\cup\hat{F}_2$ contains a 4-cycle.*

Proof. Let $\{X,Y\}$ be the bipartition of $K_{m,m}$. Label the vertices of X with $\{x_i:i\in\mathbb{Z}_m\}$ and the vertices of Y with $\{y_i:i\in\mathbb{Z}_m\}$. Define $\hat{F}_i$ as the spanning subgraph of $K_{m,m}$ with edge set $E(\hat{F}_i)=\{x_jy_{j+i}:j\in\mathbb{Z}_m\}$. It is straightforward to verify that $\{\hat{F}_i\}_{i=0}^{m-1}$ is a 1-factorization of $K_{m,m}$. A 4-cycle is induced in $\hat{F}_0\cup\hat{F}_1\cup\hat{F}_2$ by the vertices $\{x_0,y_1,x_1,y_2\}$. □

3 Existence of weighted cages

Given two graphs Z,Y, a *(weak) morphism*, $\varphi:Z\to Y$, is a function $\varphi:V(Z)\to V(Y)$, such that $z\simeq_Z z'$ implies $\varphi(z)\simeq_Y\varphi(z')$. Note that φ may map adjacent vertices into equal vertices. For $zz'\in E(Z)$ we define $\varphi(zz')=\{\varphi(z),\varphi(z')\}$ which may be singleton or an edge in Y. We also define $\varphi^{-1}(y)=\{z\in V(Z):\varphi(z)=y\}$ and $\varphi^{-1}(yy')=\{zz'\in E(Z):\varphi(zz')=yy'\}$.

Recall that a wcycle is a cycle composed by light and heavy edges and that its weight is the sum of the weights of its edges. An (a,b,g)-wcycle is a wcycle $C=(L,H)$ of weight g such that, for each $x\in V(C)$, $deg_L(x)\leq a$ and $deg_H(x)\leq b$. Hence, if C is an (a,b,g)-wcycle, then it is also an (a',b',g)-wcycle, whenever $a'\geq a$ and $b'\geq b$. For instance, a cycle of length g composed only of light edges is an $(a,0,g)$-wcycle, for each $a\geq 2$. Similarly, a cycle of length ℓ composed only of heavy edges is a $(0,b,2\ell)$-wcycle, for each $b\geq 2$. Also, any wcycle of weight g is a $(2,2,g)$-wcycle, but there are no (a,b,g)-wcycles when $a+b\leq 1$. Note that any (a,b,g)-wcage contains at least one (a,b,g)-wcycle.

We shall prove in this section that an (a, b, g)-wcage exists whenever an (a, b, g)-wcycle exists. The idea is very simple: Start by taking such an (a, b, g)-wcycle, extend it to achieve the light-regularity and then extend it again to achieve the heavy-regularity. The formal details, however, require a series of lemmas. Let us begin by characterizing the existence of (a, b, g)-wcycles:

Lemma 3.1. *Let $a, b \geq 0$ and $g \geq 3$, then an (a, b, g)-wcycle exists if and only if any of the conditions 1-4 holds:*

1. $a \geq 2$.
2. $a = 1$, $b \geq 2$, *and* $g \geq 5$.
3. $a = 1$, $b = 1$, $g \geq 6$ *and* $g \equiv 0 \mod 3$.
4. $a = 0$, $b \geq 2$, $g \geq 6$ *and* $g \equiv 0 \mod 2$.

Proof. Case 1: A wcycle can be formed using only light edges. Case 2: A wcycle can be formed either using only heavy edges (for even g, with $g \geq 6$) or using one light edge and $(g-1)/2$ heavy edges (for odd g, with $g \geq 5$). Case 3: Any such wcycle must alternate light and heavy edges; any two such consecutive edges in the wcycle contribute 3 to the weight of the wcycle and hence $g \equiv 0 \mod 3$. Also, the minimum of such wcycles has 4 edges and $g = 6$. Case 4: Any wcycle must contain only heavy edges and hence $g \equiv 0 \mod 2$. Also the minimum of such wcycles has 3 edges and $g = 6$.

It is straightforward to verify that these are all the cases in which an (a, b, g)-wcycle exists. □

Definition 3.2. *Given graphs X and Y we say that Z is a* semidirect product *of X and Y (written as $Z = X \rtimes Y$ or $Z = X \rtimes_\varphi Y$) whenever:*

1. *There is a morphism $\varphi : Z \to Y$*
2. $\varphi^{-1}(y) \cong X$, *for every $y \in V(Y)$.*
3. $|\varphi^{-1}(y_1 y_2)| = 1$, *for every $y_1 y_2 \in E(Y)$.*

Note that, given $Z = X \rtimes_\varphi Y$, we must have that φ is vertex- and edge-surjective, that $|Z| = |X||Y|$, and that Z is the disjoint union of $|Y|$ copies of X with some additional *external edges*, which only connect vertices from different copies of X in Z and such that given two such copies X_1 and X_2 of X in Z, there is at most one external edge connecting a vertex from X_1 to a vertex from X_2.

Lemma 3.3 (Extension Lemma)**.** *Let $d \geq 0$ be an integer. Let X be a graph with $\Delta(X) \leq d$. Define the* defect $D = d \cdot |X| - \sum_{x \in X} deg_X(x)$. *Let Y be a D-regular graph. Then there is a d-regular graph Z with $Z = X \rtimes Y$.*

Proof. Let us construct Z and φ as follows. First take $V(Z) = V(X) \times V(Y)$. Define $\varphi : Z \to Y$ by $\varphi(x, y) = y$. Add the following edges to Z:

$$\{(x,y)(x',y') : xx' \in E(X) \text{ and } y = y'\}.$$

At this point, we already have $\varphi^{-1}(y) \cong X$, for every $y \in V(Y)$.

Given an edge $yy' \in E(Y)$ select any pair of vertices $z = (x, y) \in \varphi^{-1}(y)$ and $z' = (x', y') \in \varphi^{-1}(y')$ satisfying $deg_Z(z) < d$ and $deg_Z(z') < d$ (if any). If the selection was possible, add the edge zz' to Z and mark the edge yy' as *used*. Repeat this procedure with the rest of the *unused* edges of Y until it is impossible to add more edges to Z. In this way, we just added at most one edge to Z for each edge of Y. Note that $deg_Z(z) \le d$ for all $z \in Z$. We claim that Z already possesses all the required properties.

Assume first that all edges of Y were used. Then to each copy $\varphi^{-1}(y)$ of X (for any $y \in Y$), we just added $deg_Y(y) = D$ new *external* edges (each ending in another copy of X). Then, recalling the definition of the defect D, the new degree sum of the all vertices $z = (x, y)$ in $\varphi^{-1}(y)$ is:

$$\sum_{z \in \varphi^{-1}(y)} deg_Z(z) = \sum_{x \in X} deg_X(x) + D = d \cdot |X|. \tag{1}$$

Since $deg_Z(z) \le d$ and $|\varphi^{-1}(y)| = |X|$, Equation (1) implies that $deg_Z(z) = d$, for all $z \in \varphi^{-1}(y)$. Since this happens for every y, it follows that Z is d-regular. It should be clear that all the conditions in Definition 3.2 are satisfied.

Finally, assume that some edge $yy' \in E(Y)$ could not be used. This means, without lost of generality, that every vertex $z \in \varphi^{-1}(y)$ already had degree d. But then $\sum_{z \in \varphi^{-1}(y)} deg_Z(z) = d \cdot |X| = \sum_{x \in X} deg_X(x) + D$, which mean that D additional external edges were added during the procedure to the vertices of $\varphi^{-1}(y)$. But $deg_Y(y) = D$ and hence all edges incident with y in Y were used. Therefore yy' was already used indeed. A contradiction. $\square$

Theorem 3.4. *For $a, b \ge 0$, $g \ge 3$, an (a, b, g)-wcage exists if and only if an (a, b, g)-wcycle exists.*

Proof. It suffices to show that an (a, b, g)-wgraph exists. Figure 1 illustrates this proof. Let G_0 be an (a, b, g)-wcycle, so $g(G_0) = g$.

Let $X_0 = L(G_0)$, the light-subgraph of G_0. Note that $g(X_0) \ge g$ and $\Delta(X_0) \le a$. Take $d = a$ and compute the defect $D = d \cdot |X_0| - \sum_{x \in X_0} deg_{X_0}(x)$ as in the Extension Lemma (3.3). Now, select any D-regular graph Y_0 of girth $g(Y_0) \ge g$, for instance: for $D = 0$ we may take $Y_0 = K_1$; for $D = 1$ take $Y_0 = K_2$; and for $D \ge 2$, we may take Y_0 as any (D, g)-cage.

Now we use the Extension Lemma with $d = a$, to get an a-regular graph $Z_1 = X_0 \rtimes_\varphi Y_0$ for some $\varphi : Z_1 \to Y_0$. Recall that Z_1 is the disjoint union of $|Y_0|$ copies of X_0, with some additional external edges. Now, in each of these copies of X_0 in Z_1 put back the heavy edges originally present in G_0 (if any) to obtain G_1 (i.e. Z_1 is a spanning subgraph of G_1).

We claim that $g(G_1) = g$. First note that the original wcycle G_0 is present in G_1, indeed, each copy of X_0 in Z_1 induce a copy of G_0 in G_1. Hence $g(G_1) \le g$.

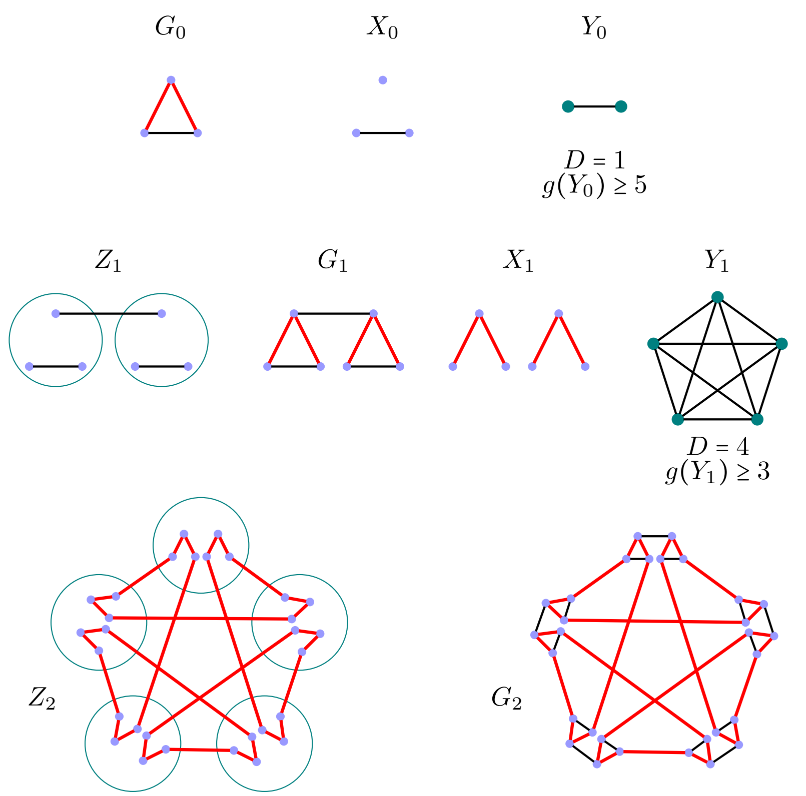

Figure 1: Construction in the proof of Theorem 3.4.

But, if there was a wcycle C in G_1 of weight $g' < g$, then, this wcycle must use external edges of Z_1 and, since $Z_1 = X_0 \rtimes_\varphi Y_0$ (see Definition 3.2), $\varphi(C)$ must be a closed walk in Y_0 which contain a wcycle C' in Y_0 of weight $g(C') \leq g(C) = g' < g$ which implies $g(Y_0) < g$, a contradiction. It follows that $g(G_1) = g$.

We now repeat the extension procedure for the heavy edges of G_1 to attain the desired heavy regularity:

Let $X_1 = H(G_1)$, the heavy-subgraph of G_1. Take $d = b$ and compute the defect $D = d \cdot |X_1| - \sum_{x \in X_1} deg_{X_1}(x)$. Select any D-regular graph Y_1 of girth $g(Y_1) \geq \lceil \frac{g}{2} \rceil$. Note that this time $g(Y_1) \geq \lceil \frac{g}{2} \rceil$ is enough since these edges are going to be the heavy edges of the final graph. Now use the Extension Lemma to get a b-regular graph $Z_2 = X_1 \rtimes Y_1$. In each copy of X_1 in Z_2 put back the light edges originally present in G_1 to obtain G_2 (i.e. Z_2 is a spanning subgraph of G_2). It should be clear as before, that $g(G_2) = g$ and that G_2 is (a, b)-regular. □

The general construction in the previous theorem gives bad general upper bounds. In several cases, we can get better upper bounds using the same ideas as shown in the Theorem 3.5.

The order of an (r,g)-cage, $n(r,g)$, is finite for $r \geq 2$ and $g \geq 3$, but for the constructions used in Theorem 3.5 we shall need this variant, $\tilde{n}(r,g)$, of $n(r,g)$ which is finite for $r \geq 0$, $g \geq 2$:

$$\tilde{n}(r,g) = \begin{cases} n(r,g) & \text{if } r \geq 2, g \geq 3, \\ r+1 & \text{if } 0 \leq r \leq 1 \text{ or } g = 2. \end{cases}$$

This function, $\tilde{n}(r,g)$, is the order of the smallest r-regular graph X of girth $g(X) \geq g$. It coincides with $n(r,g)$ when an (r,g)-cage exists (i.e. when $r \geq 2$ and $g \geq 3$), otherwise X is a complete graph on $r+1$ vertices and the girth of X is either ∞ (no cycles) or 3.

Theorem 3.5. *In the indicated cases, the following upper bounds hold.*

1. $n(a,b,g) \leq n(a,g) \cdot \tilde{n}(b \cdot n(a,g), \lceil \frac{g}{2} \rceil)$ *for* $a \geq 2,\ g \geq 3$.
2. $n(a,b,g) \leq n(b,\frac{g}{2}) \cdot \tilde{n}(a \cdot n(b,\frac{g}{2}), g)$ *for* $b \geq 2,\ g \geq 6$ *,* g *even.*
3. $n(a,b,g) \leq 2 \cdot n(a,g)$ *for* $a \geq 2,\ b = 1,\ g \leq 6$.
4. $n(a,b,g) \leq 2 \cdot \tilde{n}(b,3)$ *for* $a = 1,\ b \geq 1,\ g = 6$.

Proof. We shall use the Extension Lemma 3.3 and ideas similar to those in the proof of Theorem 3.4. But in order to get better bounds, whenever possible (cases 1, 2 and 3), we start with a cage and not just with a wcycle. In this way, we can guarantee the girth and one of the regularities, and then we obtain the desired wcage by using only one extension operation. In the last case, the girth is guaranteed not by the initial graph but by the construction itself.

(1) Let G_0 be an (a,g)-cage. Since $a \geq 2$ and $g \geq 3$, G_0 does exist. In addition, $g(G_0) = g$, this guarantees the girth of the wgraph that will be constructed.

Let $X_0 = H(G_0)$, the heavy-subgraph of G_0, which is a discrete graph. Take $d = b$ and then the defect $D = b \cdot |X_0| - 0$ as in the Extension Lemma. Now, select Y_0 to be a D-regular graph with girth $g(Y_0) \geq \lceil \frac{g}{2} \rceil$ and minimal order $|Y_0| = \tilde{n}(D, \lceil \frac{g}{2} \rceil)$. Use the Extension Lemma to get a b-regular graph $Z_1 = X_0 \rtimes_\varphi Y_0$ as in the Theorem 3.4.

Now consider the edges of Z_1 to be heavy edges and, in each copy of X_0, put back the light edges originally present in G_0. Let us name the resulting wgraph as G_1. Since $g(G_0) = g$, as in the proof of Theorem 3.4, we have that $g(G_1) = g$. Furthermore, each vertex of G_1 is incident with a light and b heavy edges. Hence, G_1 is an (a,b,g)-wgraph of order $n(a,g) \cdot \tilde{n}(D, \lceil \frac{g}{2} \rceil) = n(a,g) \cdot \tilde{n}(b \cdot n(a,g), \lceil \frac{g}{2} \rceil)$.

(2) Let G_0 be a $(b,\frac{g}{2})$-cage. Since $b \geq 2$ and $g \geq 6$, G_0 does exist. The edges of G_0 will produce the heavy edges of the constructed wgraph. Let $D = a \cdot |G_0|$ and let Y_0 be a D-regular graph with girth $g(Y_0) \geq g$ and minimal order $|Y_0| = \tilde{n}(D,g)$. Y_0 is the light-subgraph of the sought graph. As before, we can construct G_1, an (a,b,g)-wgraph of order $n(b,\frac{g}{2}) \cdot \tilde{n}(a \cdot n(b,\frac{g}{2}), g)$.

(3) Construct an (a,g)-cage with weight 1 on its edges, take two copies and complete it with a matching of heavy edges. We claim that no wcycles with a weight less than 6 are formed, since the new wcycles contain at least two heavy edges of the matching, and the rest are light ones, which are also at least two, therefore, the new wcycles have weight at least 6.

(4) Construct a b-regular graph of girth at least 3 and then consider its edges to be heavy. Take two disjoint copies of that, and complete it with a matching of light edges, taking care that at least one wcycle of weight 6 is formed. As before, no wcycles of weight less than 6 are formed. □

4 Moore-like bounds

Much in the way of Moore's lower bounds for cages [13], we may also provide lower bounds for wcages. As in the classic case, we construct a wtree which must be an induced subwgraph of any wcage of some given parameters, and where all the vertices must be different to avoid creating wcycles of weight less than g. The result is Theorem 4.1 and this section is devoted to prove it.

Assume first that g is odd. Start with a root vertex and create a wtree of depth (wdistance from the root) $h = \lfloor (g-1)/2 \rfloor = (g-1)/2$ whose inner vertices have a and b light incident edges and heavy incident edges, respectively. All of these vertices must be different since, otherwise we would form a wcycle of weight at most $2h = g-1 < g$, which are forbidden. Any (a, b, g)-wcage must contain this wtree as an induced subwgraph, and hence the order of the wtree is a lower bound for $n(a, b, g)$.

Since we have light and heavy edges, we should create the several levels of the wtree, considering the wdistance of the vertices from the root (the root is at level 0), and hence heavy edges skip two levels at a time as in Figure 2. We shall consider two kinds of vertices, the light vertices (in green) and the heavy vertices (in red): Vertices are light or heavy depending on the weight of the edge connecting them to their respective parents. The root vertex is not of any of these kinds, but we shall see that, for counting purposes, it can be considered light or heavy depending on the case at hand.

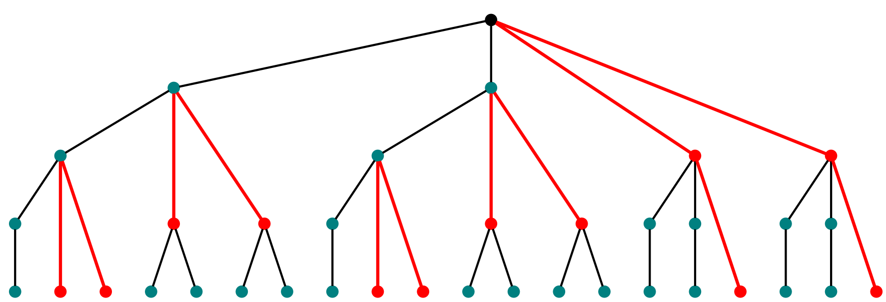

Figure 2: Moore-like wtree for $a = 2, b = 2, g = 9$.

We define L_i as the number of light vertices at level i and H_i as the number of heavy vertices at level i. Then it should be clear that the recurrences for light and heavy vertices at level i are:

$$\begin{array}{rcl} L_i & = & (a-1)L_{i-1} + aH_{i-1}, \\ H_i & = & (b-1)H_{i-2} + bL_{i-2}. \end{array} \tag{2}$$

And that, the base cases are:

$$\begin{array}{ll} L_0 = 1, & H_0 = 0, \\ L_1 = a, & H_1 = 0. \end{array} \tag{3}$$

Note that in this case the root vertex is considered light ($L_0 = 1$), since it affects the number of heavy vertices at level 2 according to the recurrence for H_i in (2), but it does not affect the light vertices at level 1 since those are determined by the base cases in (3) and not by the recurrences.

For future reference, let us name this lower bound.

$$M_1 := M_1(a,b,g) := \sum_{i=0}^{(g-1)/2} (L_i + H_i) \quad \text{using (2) and (3), } g \text{ is odd.} \tag{4}$$

Now assume g is even. As before we can construct a wtree, and again, its depth must be at most $h = \lfloor (g-1)/2 \rfloor = (g-2)/2$ to guarantee that all of the vertices are different (assuming there are no wcycles of weight less than g). Although we can not add an additional full level preserving this guarantee, we can indeed add an additional level but only to the subwtree of one of the children of the root and still guarantee that all the vertices are different, this is true since $h + (h+1) = (g-2)/2 + (g-2)/2 + 1 = g - 1 < g$. Since we have light and heavy edges, this can be done in two different ways as shown in figures 3(a) and 3(b). There, the child of the root selected to have an additional level of descendants is marked with a square box. Any (a,b,g)-wcage must contain both of these wtrees as induced subwgraphs and hence the orders of these wtrees are both lower bounds for $n(a,b,g)$.

In order to count the vertices of these wtrees, we may proceed as before, but the additional partial levels would require us to resort to two additional sets of recurrences to compute how many light and heavy edges are present in the last two levels of the subwtrees that were expanded, so we can finally count the number of vertices in the additional partial levels. A better idea is to move the selected childs one level up, as in figures 3(a') and 3(b'). In this way, all the leaves are aligned and the recurrences in (2) hold good for all levels ($i \geq 2$) and all cases. Then we only have to check which the new base cases are. It should be clear that the new base cases when the selected child is light (as in Figure 3(a')), are:

$$\begin{array}{ll} L_0 = 2, & H_0 = 0, \\ L_1 = 2(a-1), & H_1 = 0. \end{array} \tag{5}$$

Also, the base cases when the selected child is heavy (as in Figure 3(b')) are:

$$\begin{array}{ll} L_0 = 0, & H_0 = 1, \\ L_1 = a, & H_1 = 1. \end{array} \tag{6}$$

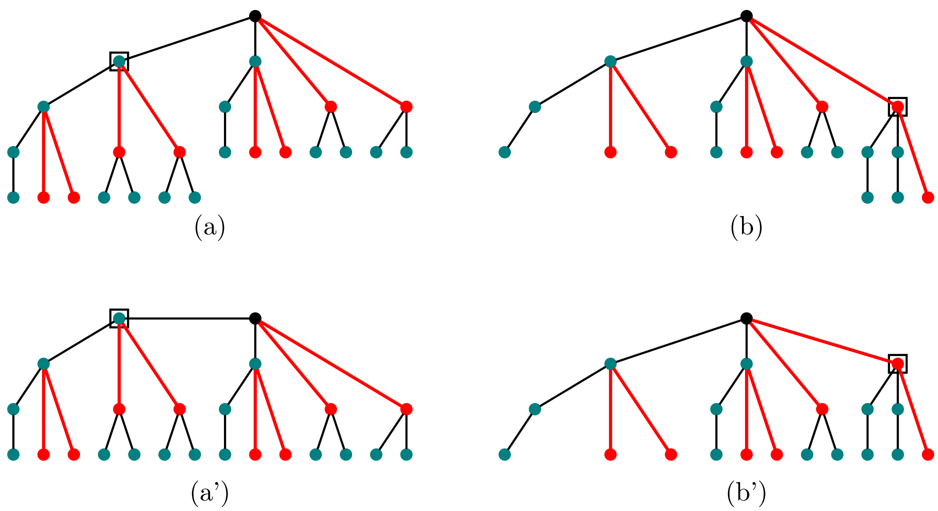

Figure 3: Moore-like wtrees for $a = 2, b = 2, g = 8$.

Note that the root vertex is considered light ($L_0 = 2$) in (5) and heavy ($H_0 = 1$) in (6) this affects the number of heavy vertices at level 2 as computed with the recurrences (2). Also, the selected child in both cases moves only one level up, so the selected child is at level 0 in Figure 3(a') and it is at level 1 in Figure 3(b') in agreement with equations (5) and (6). Let us name these two new lower bounds:

$$M_2 := M_2(a,b,g) := \sum_{i=0}^{(g-2)/2} (L_i + H_i) \quad \text{using (2) and (5), } g \text{ is even.} \tag{7}$$

$$M_3 := M_3(a,b,g) := \sum_{i=0}^{(g-2)/2} (L_i + H_i) \quad \text{using (2) and (6), } g \text{ is even.} \tag{8}$$

Let $n = n(a,b,g)$. Note that parity forbids both $an \equiv 1$ and $bn \equiv 1$, hence we can add one to each lower bound whenever the bound itself is odd and either a or b is odd. Hence, for $i \in \{1,2,3\}$, we define:

$$M_i^+ = \begin{cases} M_i + 1 & \text{if } M_i \text{ is odd and either } a \text{ or } b \text{ is odd,} \\ M_i & \text{otherwise.} \end{cases}$$

Therefore, in this section we have proven that $n(a,b,g) \geq M_1^+$, for odd g, and that $n(a,b,g) \geq \max\{M_2^+, M_3^+\}$, for even g. However, we have found in practice that almost always we have that $M_2 \geq M_3$ (and hence that $M_2^+ \geq M_3^+$). The only relevant exception we have found is $M_3(1,2,10) = 15 > 14 = M_2(1,2,10)$ (other exceptions occur when the (a,b,g)-wcage does not even exist). Hence we prefer to state the theorem that we have proven in this section as follows:

Theorem 4.1. *Let $a \geq 1$, $b \geq 1$, $g \geq 3$. Then we have:*

$$\begin{array}{lcll} n(a,b,g) & \geq & M_1^+ & \text{when } g \text{ is odd,} \\ n(a,b,g) & \geq & M_2^+ & \text{when } g \text{ is even.} \\ n(a,b,g) & \geq & M_3^+ = 16 & \text{when } a = 1, b = 2, g = 10. \end{array}$$

We shall collectively denote these lower bounds (M_1^+, M_2^+ and M_3^+, according to the cases as in the previous Theorem) simply by $n_0(a,b,g)$. Hence the previous Theorem says that $n(a,b,g) \geq n_0(a,b,g)$. Note that the standard Moore lower bound for ordinary cages is $n_0(r,g) = n_0(r,0,g)$, and that the standard Moore trees are the same as the trees in this section in the case $b = 0$. Whenever we have an (a,b,g)-wgraph of order n, we shall say that its *excess* is $n - n_0(a,b,g)$. It is straightforward to verify that these lower bounds give the following closed formulas (we used GAP [12] for the required symbolic computations):

$$\begin{array}{ll} g = 3: & M_1 = a + 1 \\ g = 5: & M_1 = a^2 + b + 1 \\ g = 7: & M_1 = a^3 - a^2 + 2ab + a + b + 1 \\ g = 9: & M_1 = a^4 - 2a^3 + 3a^2b + 2a^2 + b^2 + 1 \\ g = 11: & M_1 = a^5 - 3a^4 + 4a^3b + 4a^3 - 3a^2b + 3ab^2 - 2a^2 + b^2 + a + 1 \end{array}$$

$$\begin{array}{ll} g = 4: & M_2 = 2a \\ g = 6: & M_2 = 2a^2 - 2a + 2b + 2 \\ g = 8: & M_2 = 2a^3 - 4a^2 + 4ab + 4a \\ g = 10: & M_2 = 2a^4 - 6a^3 + 6a^2b + 8a^2 - 4ab + 2b^2 - 4a + 2 \\ g = 12: & M_2 = 2a^5 - 8a^4 + 8a^3b + 14a^3 - 12a^2b + 6ab^2 - 12a^2 + 4ab + 6a \end{array}$$

These lower bounds are not great for $g = 3$ or $g = 4$ as we also have the lower bound $n(a,b,g) \geq a + b + 1$ from Lemma 2.1, which often surpasses both of these bounds. In the following two sections we shall determine $n(a,b,g)$ for $g = 3, 4$. After that (sections 7 and 8), we shall see that these lower bounds are much better for $g = 5, 6$, and that they are relevant for $g \geq 7$.

5 Weighted cages of girth 3

We shall prove Theorem 5.1 which characterizes $n(a,b,3)$. Recall by Lemma 2.1 that, in general, we have that $n(a,b,g) \geq a + b + 1$ and that, when $ab \equiv 1$, we have $n(a,b,g) \geq a + b + 2$. Hence, Theorem 5.1 says that these lower bounds are sharp except for the first two conditions in the Theorem. Note that a wcycle of girth $g = 3$ must use only light edges and hence the heavy edges can be placed freely in our wgraph never affecting the already minimal girth of the wgraph.

A frequently used idea is that if $n = a + b + 1$ and L is already a-regular of girth $g = 3$, then its complement $H = \overline{L}$ can always be used to obtain the desired wgraph $G = (L, H) = (L, \overline{L})$.

Theorem 5.1. *For each $a \geq 0$ and $b \geq 0$ we have that*

$$n(a,b,3) = \begin{cases} \infty & \text{if } a < 2 \\ 6 & \text{if } a = 2 \text{ and } b \in \{1,2\} \\ a+b+1 & \text{if } a = 2 \text{ and } b \notin \{1,2\} \\ a+b+1 & \text{if } a \geq 3 \text{ and } ab \equiv 0 \\ a+b+2 & \text{if } a \geq 3 \text{ and } ab \equiv 1 \end{cases}$$

Proof. **Case 1 [$a < 2$]:** Immediate form Lemma 3.1.

Case 2 [$a = 2$ and $b \in \{1,2\}$]: Let G be an $(a,b,3)$-wcage and let L its light-subgraph. Since $a = 2$, L is a disjoint union of cycles. Since $g = 3$ one of these cycles must be a triangle. Since $b > 0$, we need at least two cycles in L and since cycles have length at least 3, it follows that $n(2,b,3) \geq 6$ in this case. It should be clear that the required heavy edges can always be added to the disjoint union of two triangles. Hence $n(2,b,3) = 6$ for $b \in \{1,2\}$.

Case 3 [$a = 2$ and $b \notin \{1,2\}$]: Take $n = a + b + 1 = b + 3$. For $b = 0$ a triangle G will work. For $b \geq 3$, we can take L as the disjoint union of a triangle and a cycle of length b. Then $G = (L, \overline{L})$ is the required wgraph.

Case 4 [$a \geq 3$ and $ab \equiv 0$]: Take $n = a + b + 1$.

Assume first that $a \equiv b \equiv 0$. Then $n \equiv 1$, $a \geq 4$ and $n \geq 5$. Take F_i as in Lemma 2.2 and take $L = \bigcup_{i=1}^{\frac{a}{2}} F_i$. Then $G = (L, \overline{L})$ is the required wgraph.

Assume now that $a \not\equiv b$. Then $n \equiv 0$ and $n \geq 4$. Take $\tilde{F}_i$ as in Lemma 2.3 and $L = \bigcup_0^{a-1} \tilde{F}_i$. Then $G = (L, \overline{L})$ is the required wgraph.

Case 5 [$a \geq 3$ and $ab \equiv 1$]: In this case, we have $n \geq a+b+2$ by Lemma 2.1, but we can indeed provide a wgraph on $a + b + 2$ vertices with the required parameters: Take $n = a + b + 2 \equiv 0$ and take $\tilde{F}_i$ as in Lemma 2.3. Now take $L = \bigcup_{i=0}^{a-1} \tilde{F}_i$ and $H = \bigcup_{i=a}^{n-3} \tilde{F}_i$. Since $n - 3 = a + b - 1$, H is b-regular and $G = (L, H)$ is the required wgraph. □

6 Weighted cages of girth 4

In this section we prove Theorem 6.4 that determine the values $n(a,b,4)$ for each $a \geq 0$ and $b \geq 0$. Besides the lower bounds in Lemma 2.1, we also have the bound $n(a,b,4) \geq M_2 = 2a$ from page 12. Hence, Theorem 6.4 says that $n(a,b,4)$ stays close to these bounds except when $a < 2$. This time we have to avoid triangles in $L = L(G)$ and also, we have to guarantee a wcycle of weight 4 in G, which may be formed by four light edges or by two light edges and a heavy edge. Once this is achieved, we can add heavy edges freely to G without changing the girth of G. Also, if L is already a-regular of girth 4 and order $n = a + b + 1$, then we can always get the required wgraph by taking $G = (L, \overline{L})$.

Lemma 6.1. *If $3 \leq a \leq b$ and $a \equiv b$, then $n(a,b,4) \leq a + b + 2$.*

Proof. Let $n = a + b + 2 \equiv 0$ and $m = \frac{n}{2} \geq a + 1$. Note that $m \leq b + 1$. Let X, Y be the parts of the complete bipartite graph $K_{m,m}$ considered in Lemma 2.4 and also let $\hat{F}_i$ as in that lemma. Take $L = \bigcup_{i=0}^{a-1} \hat{F}_i$. Clearly L is a-regular, of girth 4 and order n. For H, take all possible edges within X and all possible edges within Y. At this point, H is already $(m-1)$-regular, since $m \leq b+1$, H could already be b-regular, but if not, the extra edges may obtained by adding to H the edges of $\bigcup_{i=a}^{m-2} \hat{F}_i$. Since $(m-1) + (m-2-a+1) = 2m - a - 2 = b$, H is now b-regular and $G = (L, H)$ is the required wgraph. □

Lemma 6.2. *If $3 \leq a \leq b, a \equiv 0$ and $b > \frac{3a}{2} - 2$ then $n(a, b, g) = a + b + 1$.*

Proof. As before, it will suffice to construct an a-regular graph L of girth 4 and order $n = a + b + 1$. By our hypotheses, we have $b \geq \frac{3a}{2} - 1$ and hence $n \geq \frac{5a}{2}$. Let r and k be integers such that $n = \frac{5a}{2} + \frac{a}{2}r + k$ with $r \geq 0$ and $0 \leq k < \frac{a}{2}$.

Assume first that $r = 0$. Figure 4 shows a diagram of our construction. There, each node represents an independent set of vertices of the indicated cardinality ($\frac{a}{2}$ for the round nodes and $\frac{a}{2} + k$ or $\frac{a}{2} - k$ for the others). A solid line in the diagram, means to add all possible edges between the corresponding independent sets. The dashed line, means to add edges between the corresponding independent sets in such a way as to form an $\frac{a}{2}$-regular bipartite graph among them. It is straightforward to verify that the just constructed graph L is a-regular, of girth 4 and of order $n = \frac{5a}{2} + k$.

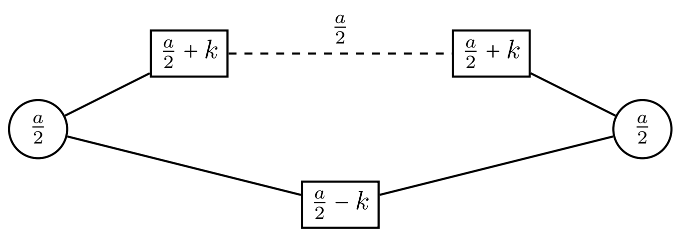

Figure 4: Construction of (a,b,4)-wcages for $n = \frac{5a}{2} + k$.

Assume now, that $r \geq 1$. Figure 5 shows a diagram of our construction. As before, our $n = \frac{5a}{2} + \frac{a}{2}r + k$ vertices are partitioned into independent sets indicated by the nodes in the diagram: round nodes contain $\frac{a}{2}$ vertices and the other node contains k vertices, the number of gray nodes must be r (and hence there is at least one) always forming a path as indicated in the diagram (hence, Figure 5 illustrates the case $r = 4$). Again, the solid lines means to add all possible edges there and the dashed lines means to add edges in such a way as to form an s-regular bipartite graph there (the value of s is indicated by the number near the dashed line). It is then straightforward to verify that the just constructed graph L is a-regular, of girth 4 and of order $n = \frac{5a}{2} + \frac{a}{2}r + k$.

□

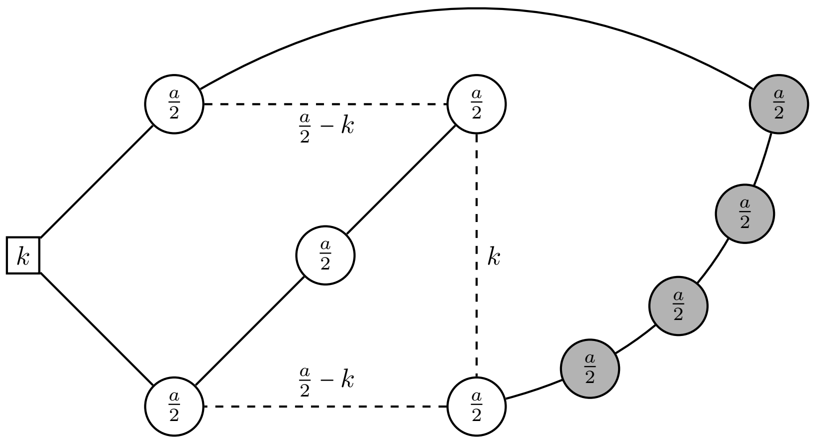

Figure 5: Construction of (a,b,4)-wcages for $n = \frac{5a}{2} + \frac{a}{2}r + k$ with $r = 4$.

Lemma 6.3. *Let $a \equiv 0$ and $n \equiv 1$ with $2a < n < \frac{5a}{2}$. Then every a-regular graph L on n vertices has a triangle.*

Proof. Let L be a triangle-free a-regular graph with n vertices, n an odd integer and $2a < n < \frac{5a}{2}$.

Let x and y be two adjacent vertices and let $A_x = N(x) \setminus \{y\}$ and $A_y = N(y) \setminus \{x\}$. Then, $A_x \cap A_y$ is empty. Hence, $I := V \setminus (A_x \cup A_y \cup \{x, y\})$ has $n - 2a \leq \frac{a}{2} - 1$ vertices, since $|A_x| = |A_y| = a - 1$. Moreover, $|I|$ is odd and each vertex $u \in I$ has at least $\frac{a}{2} + 2$ neighbors not in I.

We first prove that for each vertex u in I, $N(u) \cap A_x$ is empty or $N(u) \cap A_y$ is empty. For the sake of a contradiction, let us assume without loss of generality that a vertex $u \in I$ has k neighbors in I, l_x neighbors in A_x and l_y neighbors in A_y, with $l_x \geq l_y \geq 1$. Then, $a = k + l_x + l_y$. Let $w \in N(u) \cap A_y$. Then

$$N(w) \subseteq \{y\} \cup (A_x \setminus N(u)) \cup (I \setminus N(u)).$$

Thus, as $|A_x| = a - 1$, we get that

$$a \leq 1 + |A_x| - |A_x \cap N(u)| + |I| - |I \cap N(u)| = a - l_x + |I| - k.$$

Hence, $l_x + k \leq |I|$. But, since $l_x \geq l_y$ and $k + l_x + l_y = a$, we get the contradiction $2|I| \geq a$.

This property allows us to split I into the sets I_x and I_y, where I_x (resp. I_y) contains all vertices in I having a neighbor in A_x (resp. A_y).

Let $u \in I_x$. Then, u has no neighbors in A_y and it has at most $\frac{a}{2} - 2$ neighbors in I. Hence, it has at least $\frac{a}{2} + 2$ neighbors in A_x. Therefore, the set I_x is independent. A similar argument shows that the set I_y is independent as well.

For sets X and Y, let $e(X,Y)$ denotes the number of edges between X and Y. We have that

$$a|I_x| = e(I_x, A_x) + e(I_x, I_y),$$
$$a|I_y| = e(I_y, A_y) + e(I_y, I_x),$$

and

$$(a-1)^2 = e(A_x, A_y) + e(A_x, I_x) = e(A_x, A_y) + e(A_y, I_y).$$

These equalities imply that $|I_x| = |I_y|$ which is not possible when I is odd. □

Theorem 6.4. *For each $a \geq 0$ and $b \geq 0$ we have that*

$$n(a,b,4) = \begin{cases} \infty & \text{if } a < 2 \\ 2a & \text{if } a = 2 \text{ and } b = 0 \\ a+b+1 & \text{if } a = 2 \text{ and } b \geq 1 \\ 2a & \text{if } 3 \leq a > b \text{ and } ab \equiv 0 \\ 2a+2 & \text{if } 3 \leq a > b \text{ and } ab \equiv 1 \\ a+b+1 & \text{if } 3 \leq a \leq b \text{ and } a \not\equiv b \\ a+b+2 & \text{if } 3 \leq a \leq b, a \equiv b \equiv 1 \\ a+b+2 & \text{if } 3 \leq a \leq b, a \equiv b \equiv 0 \text{ and } b \leq \frac{3a}{2} - 2 \\ a+b+1 & \text{if } 3 \leq a \leq b, a \equiv b \equiv 0 \text{ and } b > \frac{3a}{2} - 2. \end{cases}$$

Proof. Recall that $n(a,b,4) \geq a+b+1$ and that $n(a,b,4) \geq 2a$, so constructions of these orders immediately determine $n(a,b,4)$.

Case 1 [$a < 2$]: Immediate from Lemma 3.1.

Case 2 [$a = 2$, $b = 0$]: Immediate.

Case 3 [$a = 2$, $b \geq 1$]: Take $n = a+b+1 \geq 4$ and L as a cycle of length n. Take $G = (L, \overline{L})$. The girth 4 in G is guaranteed by any two consecutive edges in the cycle and the corresponding heavy chord. Thus G is the required wgraph.

Case 4 [$3 \leq a > b$ and $ab \equiv 0$]: Take $n = 2a$ and $L = K_{a,a}$. Since $a \geq 3$, L already has girth 4. Let X and Y be the independent parts of L on a vertices each. Since $a > b$, we can always put a b-regular graph in each of X and Y: it is immediate for $a = 3$; use Lemma 2.3 when $4 \leq a \equiv 0$ or use Lemma 2.2 when $5 \leq a \equiv 1$, $b \equiv 0$. Let H be the disjoint union of these two b-regular graphs, then $G = (L, H)$ is the sought wgraph.

Case 5 [$3 \leq a > b$ and $ab \equiv 1$]: If there is a wgraph G with the required parameters and $|G| = 2a$, by Turán's Theorem, we must have $L = L(G) = K_{a,a}$. But then, since $a \equiv b \equiv 1$, parity forbids to add the required heavy edges to the parts X, Y of L to obtain G. Also, since $a \equiv 1$, parity forbids $|G| = |L| = 2a+1$. Hence $n(a,b,4) \geq 2a+2$ in this case. Let $n = 2a+2$, let M be a matching of $K_{a+1,a+1}$ and take $L = K_{a+1,a+1} - M$. Then $4 \leq a+1 \equiv 0$ and by Lemma 2.3 we can add the required heavy edges to the parts X, Y of L to obtain a b-regular H. Hence $G = (L, H)$ is the required wgraph.

Case 6 [$3 \leq a \leq b$ and $a \not\equiv b$]: Take $n = a+b+1 \equiv 0$ and $m = \frac{n}{2} > a$. Take $\hat{F}_i$ as in Lemma 2.4. Define $L = \bigcup_{i=0}^{a-1} \hat{F}_i$, then L is a-regular of girth 4. Hence $G = (L, \overline{L})$ is the required wgraph.

Case 7 [$3 \le a \le b$ and $a \equiv b \equiv 1$]: Parity forbids $|G| = a+b+1$. Hence $n(a,b,4) = a+b+2$ by Lemma 6.1.

Case 8 [$3 \le a \le b$, $a \equiv b \equiv 0$ and $b \le \frac{3a}{2} - 2$]: Assume first that $n = a+b+1$ and that $G = (L,H)$ is an $(a,b,4)$-wcage on n vertices. Note that $n \equiv 1$. By our hypotheses, we have that $n = a+b+1 \le a + \frac{3a}{2} - 2 + 1 < \frac{5a}{2}$ and that $n = a+b+1 \ge 2a+1 > 2a$. Hence, by Lemma 6.3, L has a triangle, which is a contradiction. It follows that $n(a,b,4) \ge a+b+2$ and by Lemma 6.1, that $n(a,b,4) = a+b+2$.

Case 9 [$3 \le a \le b$, $a \equiv b \equiv 0$ and $b > \frac{3a}{2} - 2$]: Immediate from Lemma 6.2. □

We find interesting the following reinterpretation of the results in this section:

Theorem 6.5. *For each $a \ge 3$ there is an a-regular graph with girth four and n vertices if and only if any of the following cases holds.*

1. *$n \equiv 0$ and $n \ge 2a$ or,*
2. *$n \equiv 1$ and $a \equiv 0$ and $n \ge \frac{5a}{2}$.*

Proof. Let a, n as in the statement. Assume L is an a-regular graph of girth 4 and order n (if it exists). Since $n(a,b,4) \ge 2a$, no such L exists for $|L| < 2a$.

Assume first that $n \equiv 0$ and $n \ge 2a$, then take $m = \frac{n}{2}$ and $\hat{F}_i$ as in Lemma 2.4, now $L = \bigcup_{i=0}^{a-1} \hat{F}_i$ is the required graph. Note that parity forbids $n \equiv a \equiv 1$. Assume next that $n \equiv 1$, $a \equiv 0$ and $2a < n < \frac{5a}{2}$, then, by Lemma 6.3, L does not exist. Finally, if $n \equiv 1$, $a \equiv 0$ and $n \ge \frac{5a}{2}$, take $b = n-a-1$. Then $b \ge \frac{3a}{2} - 1 > \frac{3a}{2} - 2$. By Lemma 6.2, there is an (a,b,g)-wgraph on $a+b+1$ vertices. Then $L = L(G)$ is the required graph. □

7 Weighted cages of girth 5 and 6

Contrary to the cases $g = 3$ and $g = 4$, our Moore-like bounds in Theorem 4.1 are very good for $g = 5$ and $g = 6$. Indeed we shall see in the next section that for these values of g, $n(a,b,g)$ coincides with the corresponding Moore-like bound for all the finite values that we could compute, except for $n(4,1,5) = 20 > 18 = M_1^+(4,1,5)$. The following theorem proves that this is indeed the case at least for $a = 1,2$:

Theorem 7.1. *If $n(a,b,5) < \infty$ and $a \in \{1,2\}$, then $n(a,b,5) = M_1^+(a,b,5)$. Also, if $n(a,b,6) < \infty$ and $a \in \{1,2\}$ then $n(a,b,6) = M_2^+(a,b,6)$.*

For the reader's convenience and using the polynomials in page 12, we restate the previous theorem in the following equivalent form:

Theorem 7.2. *The following relations hold:*

$n(1,b,5) = b+2$	for $2 \le b \equiv 0$,
$n(1,b,5) = b+3$	for $3 \le b \equiv 1$,
$n(2,b,5) = b+5$	
$n(1,b,6) = 2b+2$	for $b \ge 1$,
$n(2,b,6) = 2b+6$,	

Proof. Since the values match the lower bounds $M_1^+(a,b,5)$ or $M_2^+(a,b,6)$, it will suffice to give constructions matching these values.

Case 1 [$a = 1$**,** $g = 5$ **and** $2 \le b \equiv 0$**]:** Take $n = b + 2 \equiv 0$ and take $L = \tilde{F}_0$ (see Lemma 2.3). Then $G = (L, \overline{L})$ guarantees $n(a,b,g) \le b + 2$.

Case 2 [$a = 1$**,** $g = 5$ **and** $3 \le b \equiv 1$**]:** Take $n = b + 3 \equiv 0$, $L = \tilde{F}_0$ and $H = \bigcup_{i=1}^{n-3} \tilde{F}_i$. Then $G = (L, H)$ guaranties $n(a,b,g) \le b + 3$.

Case 3 [$a = 2$ **and** $g = 5$**]:** Take $n = b + 5$. Note that n may be even or odd. Take $L = C_n$, the n-cycle. Let $H = \overline{L^2}$ (in this case, $L^2 \cong C_n(1,2)$ is the circulant on n vertices with jumps 1 and 2). Now $G = (L, H)$ guaranties $n(a,b,g) \le b + 5$.

Case 4 [$a = 1$**,** $g = 6$ **and** $b \ge 1$**]:** Take $n = 2b + 2 \equiv 0$ and $m = \frac{n}{2} = b + 1$. Take $H = K_m \uplus K_m$ and let L be a matching between these two complete subgraphs. Then $G = (L, H)$ guaranties $n(a,b,g) \le 2b + 2$.

Case 5 [$a = 2$ **and** $g = 6$**]:** Take $n = 2b + 6 \equiv 0$ and $m = \frac{n}{2} = b + 3$. Take $H = \overline{C_m} \uplus \overline{C_m}$ and let L be a $2m$-cycle zigzagging between these two complements of cycles, taking care that no two consecutive edges of L join two adjacent vertices in H. Then $G = (L, H)$ guaranties $n(a,b,g) \le 2b + 6$. □

8 Experimental results

We used computerized, exhaustive searches using backtracking with symmetry reductions to obtain the values of $n(a,b,g)$ in the following tables. The experimental results for the cases $g = 3$ and $g = 4$ coincide with the characterizations in the respective sections, and hence they are omitted here. We also omit the cases $a = 0$ and $b = 0$ since those were already characterized in the preliminaries section. Blank squares are unknown values. In all these cases the computed values differ by either 0, 2 or 4 from the respective Moore-like bounds in Theorem 4.1. When the difference is 2 the number in the table is in boldface and black, when the difference is 4 the number in the table is in blue. We used GAP [12] and YAGS [5] for these computations.

$n(a,b,5)$

$a\backslash b$	1	2	3	4	5	6	7	8
1	∞	4	6	6	8	8	10	10
2	6	7	8	9	10	11	12	13
3	12	12	14	14	16	16		
4	**20**	19	20	21				

$n(a,b,8)$

$a\backslash b$	1	2	3	4
1	∞	10	**16**	**20**
2	16	24		

$n(a,b,6)$

$a\backslash b$	1	2	3	4	5	6	7	8
1	4	6	8	10	12	14	16	18
2	8	10	12	14	16	18	20	
3	16	18	20	22	24			
4	28	30	32					

$n(a,b,9)$

$a\backslash b$	1	2	3
1	6	**14**	**24**
2	**24**		

$n(a,b,7)$

$a\backslash b$	1	2	3	4	5
1	∞	**10**	**14**	**18**	**22**
2	**14**	**19**			

$n(a,b,10)$

$a\backslash b$	1	2	3
1	∞	16	**28**
2	**32**		

Besides the values in the tables, we also computed $n(1,2,11) = 24 = M_1^+ + 4$, $n(1,2,12) = 26 = M_2^+$ and $n(5,2,6) = 46 = M_2^+$.

9 General constructions for (a,b,g)-wcages

Let X be an (r,g')-cage. Assume that X has an a-factor F. Then $G = (F, X-F)$ is an $(a, r-a, g)$-wgraph for some girth g with $g' \leq g \leq 2g'$, thus $n(a, r-a, g) \leq n(r, g')$ in this case. Assume further that F is an a-factor of girth $g(F) \geq g'+1$, then we have $g \geq g'+1$: This is true since a cycle of length g' in X can not be a cycle of F and hence every cycle in G must contain at least one heavy edge. Moreover, if X contains both a a-factor F with $g(F) \geq g'+1$ and a cycle C of girth g' with $|E(C) - E(F)| = 1$, then $g = g'+1$.

A case of special interest is when X is Hamiltonian. In this case, the Hamiltonian cycle F is a 2-factor an certainly $g(F) \geq g'+1$ whenever $r \geq 3$. It follows that $G = (F, X-F)$ is a $(2, r-2, g)$-wgraph for some g with $g'+1 \leq g \leq 2g'$.

These constructions can be applied in many cases to obtain upper bounds for (a,b,g)-wcages. At least in the following cases, this method matches the experimental results in the previous section and hence, the produced wgraphs are indeed wcages:

1. Petersen's graph: $n(3,5) = 10$, gives $n(1,2,8) = 10$.
2. Heawood's graph: $n(3,6) = 14$ gives $n(2,1,7) = 14$ and $n(1,2,9) = 14$.
3. McGee's graph: $n(3,7) = 24$ gives $n(2,1,9) = 24$ and $n(1,2,11) = 24$.

Moreover, in the following cases the constructions give interesting (a, b, g)-wgraphs of small excess:

1. Hoffman-Singleton graph: $n(7,5) = 50$ gives a $(5,2,6)$-wgraph on 50 vertices (but $n(5,2,6) = 46$).

2. Tutte-Coxeter Graph: $n(3,8) = 30$ gives a $(1,2,12)$-wgraph on 30 vertices, (but $n(1,2,12) = 26$).

Recall that a *Moore cage* X is an (r, g')-cage that attains the Moore bound, and that the *Moore bound* is $n_0(r, g') = n_0(r, 0, g')$ as described after Theorem 4.1. Recall that this bounds come from the trees described in Section 4, which in the case $a = r$, $b = 0$ gives the standard Moore trees.

Theorem 9.1. *Assume $r \geq 3$, $g' \equiv 0$ and that there is an (r, g')-cage which is a Hamiltonian Moore cage, then we have that:*

$$n(2, r-2, g) \leq n_0(r, g')$$

for some g with $g' + 1 \leq g \leq \frac{3}{2}g' - 1$.

Proof. Let X be an (r, g')-cage which is a Hamiltonian Moore cage, with $g' \equiv 0$ and $r \geq 3$. Let C be a Hamiltonian cycle of X. Starting with any edge $xy \in X$, we can construct its Moore tree T within X, which consist of xy and two subtrees of T, T_x and T_y, which are rooted at x and y respectively. Each of these trees have depth $\frac{g'-2}{2}$. Since X is a Moore cage, this tree T is a spanning subgraph of X and the rest of the edges of X connect leaves in T_x to leaves in T_y.

Then we can construct the wgraph $G = (C, G - C)$, which is a $(2, r-2, g)$-wgraph, for some girth g satisfying $g \geq g' + 1$. We take an edge $xy \in C$ and construct the Moore tree T in X starting with it. Then C must pass by xy and go down from there to some leaf $\hat{x}$ of T_x and some leaf $\hat{y}$ of T_y. Let P_x and P_y be the corresponding paths in T that go from x to $\hat{x}$ and from y to $\hat{y}$.

Let $y_1, y_2, \ldots, y_{r-1}$ be the neighbours of y in T_y, assume without loss that y_1 is in C. Consider the subtrees T_{y_i} of T rooted at y_i for $i = 1, 2, \ldots, r-1$. All of these trees have height $\frac{g'-4}{2}$. Notice that $\hat{y}$ is a leaf of T_{y_1}.

If $\hat{x}$ and $\hat{y}$ are adjacent in X, clearly $\hat{x}\hat{y}$ is not in C, and then we have a cycle of weight $g' + 1$ on G. Suppose that $\hat{x}$ and $\hat{y}$ are not adjacent in X. Since the girth of X is g' and $\hat{x}$ is has degree r, $\hat{x}$ must be adjacent to exactly one leaf of each of $T_{y_1}, T_{y_2}, \ldots, T_{y_{r-1}}$. Hence, there is a neighbour, y', of $\hat{x}$ among the leaves of T_{y_1}. Let P'_{y_1} be the path in T_1 from y_1 to y'. Then there is a cycle $C' = P_x \cup xy \cup yy_1 \cup P'_{y_1} \cup \hat{x}y'$ of length g' in X with at least $\frac{g'+2}{2}$ edges in C and at most $\frac{g'-2}{2}$ not in C. Hence, the girth of G is bounded by the weight of C' as follows $g \leq \frac{g'+2}{2} + 2\left(\frac{g'-2}{2}\right) = \frac{3}{2}g' - 1$. We conclude that $g' + 1 \leq g \leq \frac{3}{2}g' - 1$. □

The previous theorem is still true if we replace $g' \equiv 0$ with $g' \equiv 1$ and the upper bound with $g \leq \frac{3}{2}g' - \frac{1}{2}$. However, besides the hypothetical $(57,5)$-cage, the only Hamiltonian Moore cages of odd girth with $r \geq 3$ are the complete graphs and the Hoffman-Singleton graph [8]. The complete graphs only give easy bounds already established in Theorem 6.4, and the Hoffman-Singleton graph gives a bad upper bound: $n(2,5,6) = 16 < 50 = n(7,5)$.

It is a well known observation that all Moore $(r,6)$-cages are incidence graphs of projective planes of order $(r-1)$ (see for instance [6]). Also, we know from [15] that all of them are Hamiltonian. Furthermore, it is also well known that projective planes, and hence Moore cages of girth 6, exists when $(r-1)$ is a prime power and that $n_0(r,6) = 2(r^2 - r + 1)$. Hence, the previous Theorem gives us the following corollary for girths $g \in \{7,8\}$:

Corollary 9.2. *Let $(r-1)$ be a prime power then:*

$$n(2, r-2, g) \leq 2(r^2 - r + 1)$$

for some g with $7 \leq g \leq 8$.

Moreover, let X be an $(r,6)$-cage. If X has a 6-cycle with all the edges on the Hamiltonian cycle except one, we have that:

$$n(2, r-2, 7) = 2(r^2 - r + 1),$$

otherwise:

$$n(2, r-2, 8) = 2(r^2 - r + 1).$$

We point out that it is a folklore conjecture that all cages are Hamiltonian except for the Petersen graph and hence these results should have wide applicability. For instance, besides the uses that we already mentioned above (Heawood, McGee), we can also apply them to the Tutte-Coxeter $(3,8)$-cage on 30 vertices to obtain a $(2,1,9)$-wgraph of order 30 (but $n(2,1,9) = 24$). Also, Benson $(3,12)$-cage on 126 vertices gives us a $(2,1,13)$-wgraph of order 126. This last is the best upper bound that we know for $n(2,1,13)$ and the Moore-like lower bound is $n_0(2,1,13) = 66$, our algorithms can only provide the lower bound $n(2,1,13) \geq 68$.

References

[1] M. Abreu, G. Araujo-Pardo, C. Balbuena, D. Labbate and G. López-Chávez. *Biregular cages of girth five.* Electron. J. Combin. **20** (2013) 1–14. `doi:10.37236/2594`.

[2] G. Araujo-Pardo, C. De la Cruz and D. González-Moreno. *Mixed cages: monotonicity, connectivity and upper bounds.* Discrete Math. **345** (2022) Paper No. 112792, 11. `doi:10.1016/j.disc.2021.112792`.

[3] G. Araujo-Pardo, C. Hernández-Cruz and J.J. Montellano-Ballesteros. *Mixed cages.* Graphs Combin. **35** (2019) 989–999. `doi:10.1007/s00373-019-02050-1`.

[4] G. Araujo-Pardo, R. Jajcay, A. Ramos-Rivera and T. Szőnyi. *On a relation between bipartite biregular cages, block designs and generalized polygons.* J. Combin. Des. **30** (2022) 479–496. doi:10.1002/jcd.21836.

[5] C. Cedillo, R. MacKinney-Romero, M.A. Pizaña, I.A. Robles and R. Villarroel-Flores. *YAGS - Yet Another Graph System, Version 0.0.5*, 2019. http://xamanek.izt.uam.mx/yags/.

[6] G. Chartrand and L. Lesniak. *Graphs & Digraphs.* Chapman & Hall/CRC, 3rd edition, 1996. (First edition: 1979).

[7] G. Exoo. *On mixed cages.* Discrete Math. Theor. Comput. Sci. **25** (2023) Paper No. 14, 17. doi:10.46298/dmtcs.11057.

[8] G. Exoo and R. Jajcay. *Dynamic cage survey.* Electron. J. Combin. **DS16** (2013) 55. doi:10.37236/37.

[9] G. Exoo and R. Jajcay. *Biregular cages of odd girth.* J. Graph Theory **81** (2016) 50–56. doi:10.1002/jgt.21860.

[10] G. Exoo, R. Jajcay and J. Širáň. *Cayley cages.* J. Algebraic Combin. **38** (2013) 209–224. doi:10.1007/s10801-012-0400-2.

[11] S. Filipovski, A. Ramos-Rivera and R. Jajcay. *On biregular bipartite graphs of small excess.* Discrete Math. **342** (2019) 2066–2076. doi:10.1016/j.disc.2019.04.004.

[12] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10*, 2019. (http://www.gap-system.org).

[13] A.J. Hoffman and R.R. Singleton. *On Moore graphs with diameters* 2 *and* 3. IBM J. Res. Develop. **4** (1960) 497–504. doi:10.1147/rd.45.0497.

[14] R. Jajcay and J. Širáň. *Small vertex-transitive graphs of given degree and girth.* Ars Math. Contemp. **4** (2011) 375–384. doi:10.26493/1855-3974.124.06d.

[15] F. Lazebnik, K.E. Mellinger and O. Vega. *Embedding cycles in finite planes.* The Electron. J. Combin. **20** (2013) 1–17. doi:10.37236/3377.

[16] H. Sachs. *Regular graphs with given girth and restricted circuits.* J. London Math. Soc. **38** (1963) 423–429. doi:10.1112/jlms/s1-38.1.423.

[17] J. Tuite and G. Erskine. *On networks with order close to the Moore bound.* Graphs Combin. **38** (2022) Paper No. 143, 34. doi:10.1007/s00373-022-02535-6.

[18] W.T. Tutte. *A family of cubical graphs.* Proc. Cambridge Philos. Soc. **43** (1947) 459–474. doi:10.1017/s0305004100023720.