Tercer Taller Latinoamericano de Clanes en Gráficas

28 al 31 de octubre de 2008
Guanajuato, Gto., México.

Presentación

Tercer Taller Latinoamericano de Clanes en Gráficas
Centro de Investigación en Matemáticas A.C.
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La teoría de las gráficas ha tenido una expansión importante en los últimos años, en parte por sus numerosas aplicaciones dentro y fuera de las matemáticas y en parte debido a su propia dinámica interna. Dentro de esta área de conocimiento, la teoría de las gráficas de clanes ha venido desarrollándose vigorosamente en años recientes y ha empezado a establecerse como una teoría sólida con un número creciente de técnicas y resultados cada vez más sofisticados. Aplicaciones de la teoría han empezado a emerger en áreas como el estudio de la propiedad del punto fijo en órdenes parciales y la gravitación cuántica de bucles.

El Tercer Taller Latinoamericano de Clanes en Gráficas reunirá a una porción significativa de la comunidad de teoría de las gráficas y combinatoria de Latinoamérica, con participantes de Argentina, Brasil, Chile y México, además de invitados latinoamericanos, de Estados Unidos y Suiza. El Taller contará con la participación de la mayoría de los expertos mundiales en gráficas de clanes y servirá para avivar la colaboración entre ellos además de difundir sus técnicas y resultados entre los investigadores y estudiantes interesados.

Queremos agradecer el apoyo del Centro de Investigación en Matemáticas A.C. (CIMAT), del Departamento de Ingeniería Eléctrica de la Universidad Autónoma Metropolitana (DIE-UAM) y del Instituto de Matemáticas de la Universidad Nacional Autónoma de México (IMATE-UNAM). En especial, queremos agradecer al CIMAT por hospedar generosamente este taller en sus instalaciones.

El Comité Organizador:

Hernán González, Instituto de Matemáticas UNAM, México.
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Programa

Los números indican la página en donde se puede consultar el resumen de la plática

Todas las pláticas se realizarán en el Auditorio del CIMAT.

	Martes	Miércoles	Jueves	Viernes
09:20-09:30	Inauguración			
09:30-10:30	G. Salazar [1]	L. Alcón [2]	C.P. de Mello [3]	E. Prisner [3]
10:30-11:00	Café			
11:00-11:30	M. Matamala [6]	M. Gutierrez [12]	G. Araujo [17]	F. Bonomo [28]
11:30-12:00	F. Bonomo [8]	B. Llano [12]	R. Strausz [18]	F. Oliveira [30]
12:00-12:30	Café			
12:30-13:00	R.Villarroel [9]	V. Pedrotti [14]	M. Manrique [21]	M.E. Frías [32]
13:00-13:30	P. Larrión [10]	J. Szwarcfiter [16]	L. Faria [22]	M. Dourado [34]
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17:30-18:30	Brindis			
20:00-22:00			Cena	

Lista de Pláticas

Pláticas Invitadas

The rectilinear crossing number of K_n: closing in (or are we?)

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Keywords: Crossing number, Rectilinear crossing number.

The problem of determining the rectilinear crossing number of the complete graphs K_n is an open classical problem in discrete geometry. A major breakthrough was achieved in 2003 by two teams of researchers working independently (Abrego and Fernandez-Merchant; and Lovasz, Vesztergombi, Wagner and Welzl), revealing and exploiting the close ties of this problem to other classical problems, such as the number of convex quadrilaterals in a point set, the number of $(\leq k)$–sets in a point set, the number of halving lines, and Sylvester's Four Point Problem. Since then, we have seen a sequence of improvements both from the lower bound and from the upper bound sides of the problem, and nowadays the gap between these bounds is very small. Our aim in this talk is to review the state of the art of these problems.

From Chordal to Helly circular-arc graphs

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Abstract

We consider simple, finite, undirected graphs. Given a graph G, $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set of G.

The *intersection graph* of a family $\mathcal{F}$ of non-empty sets is obtained by representing each set by a vertex and connecting two vertices by and edge if and only if the corresponding sets intersect.

A *complete* set of G is a subset of $V(G)$ inducing a complete subgraph. A *clique* is a maximal complete set. The family of cliques of G is denoted by $\mathcal{C}(G)$. The *clique graph of* G, $K(G)$, is the intersection graph of $\mathcal{C}(G)$. The weighted clique graph of G, $K^w(G)$, is the complete graph with the members of $\mathcal{C}(G)$ as vertices, and each edge QQ' weighted by $\mid Q \cap Q' \mid$.

A *chordal graph* is a graph such that every cycle with more than three vertices has a chord, i.e. an edge joining nonconsecutive vertices of the cycle. The following needed background of chordal graphs is based in references [1, 2, 3, 4, 8] and is also contained in books [6, 7].

A *tree* is a connected graph without cycles. A *subtree* of a tree T is a subset of $V(T)$ inducing a connected subgraph which is also called a subtree of T. A graph G is chordal if and only if there exists a tree T and a family $\mathcal{F}$ of subtrees of T such that G is the intersection graph of $\mathcal{F}$. The pair $(T, \mathcal{F})$ is called a *tree representation* of G.

A *clique-tree* of G is any tree T whose vertices correspond to the cliques of G, that is $V(T) = \{Q, Q \in C(G)\}$, and, for any vertex $v \in V(G)$, the set $Q_v = \{Q \in C(G)/v \in Q\}$ is a subtree of T.

It is clear that if T is a clique-tree of G, then $(T, (Q_v)_{v \in V(G)})$ is a tree representation of G, even more,

Theorem 1. *A graph G admits a tree representation if and only if G admits a clique tree.*

The following theorem provides two alternative manners of determining if a graph G admits a clique tree.

Theorem 2. *For any chordal graph G the following statements are equivalent.*

1. *T is a clique tree of G.*
2. *T is a maximum spanning tree of $K^w(G)$.*
3. *T is a spanning tree of $K^w(G)$ with weight*

$$w(T) = (\sum_{Q \in \mathcal{C}(G)} \mid Q \mid) - \mid V(G) \mid .$$

An arc of a cycle C is a subset of consecutive vertices of C. A graph G is said a *circular-arc graph* if there exists a cycle C and a family $\mathcal{A}$ of arcs of C such that G is the intersection graph of $\mathcal{A}$. The pair $(C, \mathcal{A})$ is called a circular-arc representation of G.

A set family satisfies the Helly property if any pairwise intersecting subfamily has non empty intersection.

Any family of subtrees of a tree has the Helly property, but not any family of arcs of a cycle has the Helly property. Graphs admitting a Helly circular-arc representation, i.e. a circular-arc representation such that the family of arcs has the Helly property, are called *Helly circular-arc graphs*.

A *clique-cycle* of a graph G is any cycle C whose vertices correspond to the cliques of G, that is $V(C) = \{Q, Q \in C(G)\}$, and, for any vertex $v \in V(G)$, $Q_v = \{Q \in C(G)/v \in Q\}$ is an arc of C. It is clear that if C is a clique-cycle of G, then $(C, (Q_v)_{v \in V(G)})$ is a Helly circular-arc representation of G.

The following theorem, proved in [5], is equivalent to Theorem 1 for Helly circular-arc graphs.

Theorem 3. *A graph G admits a Helly circular-arc representation if and only if G admits a clique cycle.*

In this work, we prove the following theorem which is equivalent to Theorem 2 for Helly circular-arc.

Theorem 4. *For any Helly circular-arc graph G the following statements are equivalent.*

1. *C is a clique cycle of G.*
2. *C is a maximum spanning cycle of $K^w(G)$.*
3. *C is a spanning cycle of $K^w(G)$ with weight*

$$w(T) = (\sum_{Q \in \mathcal{C}(G)} \mid Q \mid) - \mid V(G) \mid + \mid U(G) \mid,$$

where $U(G)$ is the set of universal vertices of G.

We obtain general results which have Theorems 2 and 4 as corollaries. We investigate other classes of intersection graphs for which these results can be applied.

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The maximum number of edges in the line graph

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Keywords: Line graph, squares of degrees.

Let $\mathcal{G}(v, e)$ be the set of all simple graphs with v vertices and e edges and let $P_2(G) = \sum d_i^2$ denote the sum of the squares of the degrees, $d_1, \ldots, d_v$, of the vertices of G.

In this talk we consider the problem of maximizing the number of edges of the line graph $L(G)$ of a graph G in $\mathcal{G}(v, e)$. We first show that the problem is equivalent to finding the maximum value of $P_2(G) = \sum d_i^2$ over all graphs in $\mathcal{G}(v, e)$.

We then provide a complete solution to the problem by showing the following:

The maximum value of $P_2(G)$ for $G \in \mathcal{G}(v, e)$ occurs at one or both of two special graphs in $\mathcal{G}(v, e)$; the quasi-star graph or the quasi-complete graph. For each pair (v, e), we determine which of these two graphs has the larger value of $P_2(G)$.

Moreover, we classify all graphs G in $\mathcal{G}(v, e)$ that achieve equality. That is, we find all other graphs in $\mathcal{G}(v, e)$ for which the maximum value of $P_2(G)$ is attained.

On colouring of graphs

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Keywords: total colouring, edge colouring, split graphs, indifference graphs.

A total colouring of a graph G is a colouring of its vertices and edges such that no adjacent vertices, no adjacent edges, and no incident vertices and edges get the same color. An edge colouring of G is a partial case of total colouring when only the edges are colored. The minimum number of colours needed in each case is called total-chromatic number ($\chi_T(G)$) and chromatic index ($\chi'(G)$), respectively. Clearly, $\chi'(G) \geq \Delta(G)$ and $\chi_T(G) \geq \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. It is well known that to decide whether $\chi'(G) = \Delta(G)$ or $\chi_T(G) = \Delta(G) + 1$ is NP-complete. Moreover, these problems remain NP-complete for several classes. In this talk, we will discuss results on total and edge colourings of some classes of graphs.

Why Cliques?

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Keywords: cliques, bicliques, dicliques.

There are certain variants of cliques, as bicliques in bipartite or general graphs, or dicliques in digraphs. Different from the clique case, these concepts also come in two flavors, the induced and non-induced case. Now it is well-known that cliques, or rather complete graphs, play a very special role in intersection graph theory. Examples are the Erdös-Goodman-Posa result on edge-clique coverings and intersection graphs, or Krausz's-type Theorems for characterizing special types of intersection graphs. In this talk I will try to shed some light on the question whether these variants of cliques mentioned above have also such a strong relation to a certain intersection model, and if they have, whether the corresponding classical theorems have also counterparts in these models.

Contribuciones

Edge colorings induced by vertex labelings

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Keywords: Graph labeling, bipartite graphs.

1 Introduction

Let $G = (V, E)$ be an undirected connected graph. A labeling of G is an injective function $\varphi : V \to \mathbb{N}$. Its *line function* $\hat{\varphi} : E \to \mathbb{N}$ is defined by $\hat{\varphi}(uv) = |\varphi(u) - \varphi(v)|$.

A labeling φ is a *graceful labeling* of $G = (V, E)$ if it satisfies the following two conditions.

(1) $\varphi(V) \subseteq [|E|] := \{0, \ldots, |E|\}$;

(2) $\hat{\varphi}$ is a bijective function.

Graceful labelings have been extensively studied in the literature (see [2] for a dynamic survey). They were introduced by Rosa as tools for dealing with decompositions of complete graphs into copies of given trees. In this context, Rosa conjectured that every tree admits a graceful labeling raising the now famous *Graceful Conjecture* [4].

While this conjecture is still wide open, it has been attacked by considering subfamilies of trees, and proving that they admit a graceful labeling, or by modifying condition (1) or condition (2), and by proving that every tree admits a labeling that satisfies the modified (weaker) conditions. In these situations the considered labelings are not longer graceful labelings.

In this work we follow the latter approach. On one hand, instead of considering condition (2), we shall consider the following alternative conditions for function φ and its line function $\hat{\varphi}$.

- $\hat{\varphi}$ is an edge-coloring.
- φ is *extreme:* For every $v \in V$, $\varphi(v) = \max\{\varphi(u) : uv \in E\}$ or $\varphi(v) = \min\{\varphi(u) : uv \in E\}$.
- φ is *threshold:* there is an integer k, such that for each edge uv,

$$\min\{\varphi(u), \varphi(v)\} < k \leq \max\{\varphi(u), \varphi(v)\}.$$

When φ is threshold or extreme, then $\hat{\varphi}$ is an edge coloring. Threshold and extreme labelings were considered as an additional requirement to graceful labeling in [3] and in [1]. While it is known that not every tree admits a graceful threshold labeling [3], Cahit conjectured that every tree has a graceful extreme labeling [1]. In this work we consider families of labelings less restrictive than graceful colorings. The family $\mathcal{TH}$ of all threshold labelings, and the family $\mathcal{E}$ of all extreme labeling. To the best of our knowledge, no work has been done along this line, where global condition (2) is replaced by any of these local conditions.

On the other hand, we shall explore labelings which does not satisfy Condition (1). It is clear that a strong restriction when constructing graceful labelings for a given graph $G = (V, E)$, is that the set of differences of $[|E|]$ is *tight*: There are exactly $|E|$ different values for $\hat{\varphi}$. Hence, it would be more easy for a graph to admit a labeling φ satisfying condition (2) and whose image is a set of non negative integers B other than $[|E|]$. By instance, the latter statement easily holds for a set B whose set of differences given by $B - B := \{|u - v| : u, v \in B, u \neq v\}$ is *complete*, that is, it has $\binom{|B|}{2}$ elements.

In order to assess previous ideas we consider, for a class of labelings $\mathcal{F}$, two classes of graphs: $\mathcal{F}-$*robust* graphs and $\mathcal{F}-$*optimal* graphs. A graph G on n vertices is $\mathcal{F}-$robust if for every set B of n elements it admits a labeling in $\mathcal{F}$ whose image is B. A graph G on n vertices is $\mathcal{F}-$optimal if it admits a labeling in $\mathcal{F}$ whose image is $[n-1]$. We denote by $o\mathcal{F}$ the class of all graphs that are $\mathcal{F}-$optimal and by $r\mathcal{F}$ the class of all graphs that are $\mathcal{F}-$robust. Clearly, $r\mathcal{F} \subseteq o\mathcal{F}$. We say that $\mathcal{F}$ is *stable* when $r\mathcal{F} = o\mathcal{F}$ and *unstable* otherwise.

Our first result is the following characterization of bipartite graphs in terms of previously defined labelings.

Theorem 1. *Given a graph $G = (V, E)$, the following statements are equivalents.*

1. *G is bipartite.*
2. *G is $\mathcal{TH}-$robust.*
3. *G is $\mathcal{TH}-$optimal.*
4. *G admits a threshold labeling.*
5. *G is $\mathcal{E}-$robust.*
6. *G is $\mathcal{E}-$optimal.*
7. *G admits an extreme labeling.*

Corollary 2. *The class $\mathcal{TH}$ and the class $\mathcal{E}$ are stables.*

2 Interval bipartite graphs

Theorem 1 gives a complete characterization of bipartite graphs in terms of locally restricted labelings, while the Graceful Conjecture can be seen as a characterization of trees in terms of globally restricted labelings.

Hence, it is reasonable to explore further restrictions on locally restricted labelings. An *interval* of a set of integers B is any subset $I = [x, y]$ of B with $x, y \in B$, such that for each $z \in B$, if $x \leq z \leq y$, then $z \in I$. We say that φ is an *interval* labeling of $G = (V, E)$ if $\forall v \in V$, the set $\{\varphi(u) : uv \in E\}$ is an interval of $\varphi(V)$. Let $\mathcal{I}$ be the class of interval labeling. One can see that not every bipartite graph admits an interval labeling. However, we can give an structural characterization of those bipartite graphs that do admit.

Theorem 3. *For a given graph $G = (V, E)$, the following statements are equivalents.*

1. *G is bipartite with parts $X = \{v_1, \ldots, v_n\}$ and $Y = \{v_{n+1}, \ldots, v_{n+m}\}$ such that if v_i and v_j are adjacent with u, for some u, then for each k with $i \leq k \leq j$, the vertex v_k is adjacent with u.*
2. *G is $\mathcal{I}-$robust.*
3. *G is $\mathcal{I}-$optimal.*
4. *G admits an interval labeling.*

Corollary 4. *The class $\mathcal{I}$ is stable.*

3 Trees

In this section we study graceful labelings that are interval labelings. It is not hard to see that a connected optimal graceful colorable graph must be a tree.

Let Ag denote the class of graceful labelings in A, for $A \in \{\mathcal{I}, \mathcal{TH}, \mathcal{E}\}$.

We now show that $\mathcal{I}g$ is stable. Moreover, we show that $\mathcal{I}g-$robust coincides with the class of trees admiting a dominating path (usually called caterpillars).

Theorem 5. *For a connected graph G the following are equivalents.*

(i) G is a caterpillar.

(ii) G is $\mathcal{I}g-$robust.

(iii) G is $\mathcal{I}g-$optimal.

(iv) G is a tree and it admits a graceful interval labeling.

Corollary 6. *The class $\mathcal{I}g$ is stable.*

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On $L(2,1)$-labeling of block graphs

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Keywords: block graphs, distance-two labeling problem, graph coloring.

1 Introduction

The distance-two labeling problem of graphs was proposed by Griggs and Roberts in 1988 (c.f. [6]), and it is a variation of the frequency assignment problem introduced by Hale in 1980 [7]. Suppose we are given a number of transmitters or stations. The $L(2,1)$-labeling problem is to assign frequencies (nonnegative integers) to the transmitters so that "close" transmitters receive different frequencies and "very close" transmitters receive frequencies that are at least two frequencies apart.

Let G be a simple, finite, undirected graph with vertex set $V(G)$. Let $\Delta(G)$ denote the maximum degree of a vertex of G, $d_G(u,v)$ denote the distance in G between vertices u and v, and $\omega(G)$ denote the maximum size of a clique of G.

An $L(2,1)$-labeling of a graph G is a function $f : V(G) \to \mathbb{N}_0$ such that $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$ and $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$. For a nonnegative integer k, a k-$L(2,1)$-labeling is an $L(2,1)$-labeling such that no label is greater than k. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling. It is not difficult to see that $\lambda(G) \geq \Delta(G) + 1$, and $\lambda(G) \geq 2\omega(G) - 2$. The $L(2,1)$-labeling problem has been studied widely. Griggs and Yeh [6] showed that the $L(2,1)$-labeling problem is NP-complete for general graphs. They proved that $\lambda(G) \leq \Delta^2(G) + 2\Delta(G)$ and conjectured that $\lambda(G) \leq \Delta^2(G)$ for general graphs different from K_2. Chang and Kuo [2] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G)$ and gave a linear-time algorithm for the $L(2,1)$-labeling problem on cographs. Král and Škrekovski [8] proved that $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 1$ for graphs different from K_2. For further studies on the $L(2,1)$-labeling and some generalizations, see [1, 3, 4, 5, 9, 10].

A *block* of a graph is a maximal 2-connected component. An *end-block* is a block containing exactly one cutpoint. A *block-cutpoint tree* of a graph G is a tree whose vertices are the cutpoints of G plus one vertex (not a cutpoint) for each end-block of G, and such that two vertices are adjacent if and only if they belong to the same block of G. A graph is a *block graph* if it is connected and every block is a complete.

Block graphs with $\omega(G) = 2$ are trees. Griggs and Yeh [6] showed that $\Delta(G) + 1 \leq \lambda(G) \leq \Delta(G) + 2$ for trees, and Chang and Kuo [2] gave a polynomial-time algorithm for the $L(2,1)$-labeling problem on this class of graphs. However, there is no simple characterization distinguishing the cases $\lambda = \Delta + 1$ and $\lambda = \Delta + 2$. For the special case of paths, it is not difficult to see that $\lambda(P_1) = 0$, $\lambda(P_2) = 2$, $\lambda(P_3) = \lambda(P_4) = 3$ and $\lambda(P_n) = 4$ for $n \geq 5$.

The aim of this work is to study the $L(2,1)$-labeling problem on block graphs. We find upper bounds for $\lambda(G)$ in the general case, and we tight those bounds for some particular cases with $\omega(G) = 3$.

2 Main results

Theorem 1. *Let G be a block graph with maximum degree Δ and maximum clique size ω. Then $\lambda(G) \leq \max\{\Delta + 2, \min\{3\omega - 2, \Delta + \omega\}\}$.*

Corollary 2. *Let G be a block graph with maximum degree Δ and maximum clique size 3. If $\Delta \geq 5$ then $\lambda(G) \leq \Delta + 2$, if $\Delta \leq 4$ then $\lambda(G) \leq 7$.*

These bounds are tight for $\Delta = 4$ and $\Delta \geq 5$, and they are attained by the central and rightmost graphs of Figure 1, respectively. We can improve the bound for $\Delta = 3$.

Theorem 3. *Let G be a block graph with maximum degree 3 and maximum clique size 3. Then $\lambda(G) \leq 6$.*

The bound is tight, and it is attained by the leftmost graph of Figure 1.

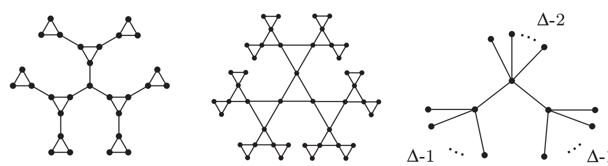

Figure 1: Examples showing tightness of the bounds.

Theorem 4. *Let G be a block graph with maximum degree 4 and maximum clique size 3. If G does not contain the leftmost graph in Figure 2, then $\lambda(G) \leq 6$.*

The computational complexity of finding $\lambda(G)$ on a block graph G is open, even when $\omega(G) = 3$. Nevertheless, the proofs of the previous theorems are constructive, and lead to algorithms to produce $L(2,1)$-colorings for graphs with the showed upper bounds.

2.1 Paths of triangles

We will call *path of triangles* to a block graph G such that $\omega(G) = 3$ and the block-cutpoint tree of G is a path. Examples of paths of triangles can be seen in Figure 2. Note that, since $\omega(G) = 3$, then $\lambda(G) \geq 4$.

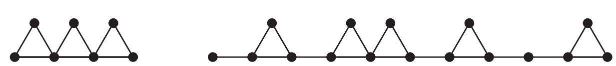

Figure 2: Paths of triangles.

For these kind of graphs we prove that $\lambda(G) \leq \Delta(G) + 2$ and give a complete characterization for each possible value of λ.

Theorem 5. *Let G be a path of triangles with maximum degree Δ. Then $\lambda(G) \leq 6$. Moreover, $\lambda(G) = 6$ if and only if G contains the leftmost graph in Figure 2, and $\lambda(G) = 4$ if and only if G does not contain any of the graphs in Figure 3.*

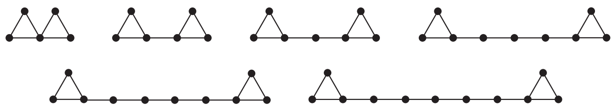

Figure 3: Paths of triangles with $\lambda = 5$.

This characterization leads to an efficient algorithm to compute $\lambda(G)$ and an optimum $L(2,1)$ coloring on paths of triangles.

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Posets, clique graphs and their homotopy type

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Keywords: clique graphs, posets, homotopy type, simplicial complex.

The aim of this talk is to present the main results of the paper [4]

Let P denote a finite partially ordered set, and G denote a finite simple graph. The poset P has an associated simplicial complex $\Delta(P)$ where the simplices are the totally ordered subsets of P (see for example [2, 5]). And also to the graph G we can associate a complex $\Delta(G)$ with simplices the complete subgraphs of G. In this way we can attach topological concepts to both posets and graphs, and for example, we will say that P and G are homotopy equivalent, and write $P \simeq Q$, whenever the geometrical realizations of $\Delta(P)$ and $\Delta(G)$ are homotopy equivalent.

We will use well-known techniques and results from poset topology (such as Quillen's Theorem) to analyze the homotopy type of complexes of the form $\Delta(G)$. Given any poset P we define two graphs $\Omega(P)$ and $\mho(P)$ as follows: The vertex set of $\Omega(P)$ is the set of minimal elements of P, and two minimal elements are declared adjacent whenever they have a common upper bound. The graph $\mho(P)$ is defined dually with the maximal elements of P as its vertices. For example, if G is a graph and P is the poset of complete subgraphs of G ordered by inclusion, one has that $\Omega(P)$ is G and that $\mho(P)$ is $K(G)$, the clique graph of G.

Under relatively mild hypothesis on P, it can be proven that $\Omega(P)$ has the same homotopy type as $\mho(P)$. In this way, we obtain a generalization of the result contained in [3], finding a more general condition that implies that a graph G is homotopy equivalent to its clique graph $K(G)$. Furthermore, as a composition of two graph operators studied in [1], we obtain a graph operator $G \mapsto H(G)$, that presevers clique-Hellyness and dismantlability (this follows trivially from [1]), but we also show that H has the property that $G \simeq H(G) \simeq K(H(G))$ for any graph G.

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The Fundamental Group of the Clique Graph

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Keywords: clique graph, bipartite graph, fundamental group.

1 Introduction

Our graphs are simple, finite and connected.

Any graph G can be seen as a topological space via its *complex of completes* $\Delta(G)$. The vertices of this simplicial complex are those of G, and its simplices are the vertex sets of the complete subgraphs of G. Rather than making explicit mention of the complex $\Delta(G)$ or its geometric realization, we directly apply topological concepts and constructions to the graph G.

We are mainly interested in clique graphs. The study of the clique graph operator under the topological viewpoint of the complex of completes was initiated by Prisner in [9, 8] and has been further pursued in [1, 2, 3, 4, 5, 6].

Among other things, Prisner proved in [9] that the first modulo 2 homology groups of any graph and its clique graph are always isomorphic:

$$H_1(K(G), \mathbb{Z}_2) \cong H_1(G, \mathbb{Z}_2)$$

We have proved recently [7] that even the fundamental group remains invariant under the clique graph operator:

$$\pi_1(K(G)) \cong \pi_1(G)$$

2 Main results

As a matter of fact, we deduce the clique-invariance of the fundamental group from a more general result:

Theorem 1. *Let $B = (X, Y)$ be a connected bipartite graph. Then B^2, $B^2[X]$ and $B^2[Y]$ have isomorphic fundamental groups.*

Here B^2 denotes the square of B, and $B^2[X]$ and $B^2[Y]$ are the subgraphs of B^2 induced by the parts X and Y of B.

Recall that a *complete edge cover* of a graph G is a family $\mathcal{F} = (G_i : i \in I)$ of complete subgraphs of G such that any vertex and any edge of G lie in some G_i. In other words, G is the union of its complete subgraphs G_i, $i \in I$.

Applying Theorem 1 to the bipartite incidence graph of vertices of G and members of $\mathcal{F}$ we obtain:

Theorem 2. *Let $\mathcal{F}$ be a complete edge cover of the graph G, and let H be the intersection graph of $\mathcal{F}$, then, $\pi_1(H) \cong \pi_1(G)$.*

That $\pi_1(K(G)) \cong \pi_1(G)$ then follows in the case that $\mathcal{F}$ is the family of cliques of G, but Theorem 2 can also be applied to other graph operators. Recall that the *line graph* $L(G)$ is the intersection graph of the edges of G. The *graph of completes* $C(G)$ is the intersection graph of the completes of G. For $m \geq 2$, the *m-simplex graph* $\nabla_m(G)$ is the intersection graph of the subset of all inclusion-maximal elements in the set of all completes of cardinality at most m of G (see [8]). By Theorem 2 we have:

Theorem 3. *Let the graph operator $\mathcal{O}$ be a composition $\mathcal{O} = \mathcal{O}_1 \circ \mathcal{O}_2 \circ \cdots \circ \mathcal{O}_n$ where each $\mathcal{O}_i$ is one of L, C, K, or ∇_m ($m \geq 2$). Then $\pi_1(\mathcal{O}(G)) \cong \pi_1(G)$ for each graph G.*

The *total graph* $T(G)$ has $V(G) \cup E(G)$ as vertex set and, in addition of all edges of G and $L(G)$, $T(G)$ has also all edges of the form ve where $v \in G$, $e \in L(G)$ and $v \in e$. Applying directly Theorem 1 we get:

Theorem 4. *Any non-trivial graph G has the same fundamental group as $L(G)$ and $T(G)$*

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Falsos gemelos en grafos cordales: Prohibidos minimales de subclases UV, DV y RDV

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1 Introducción

Los *grafos cordales* fueron definidos como aquellos que no poseen ciclos inducidos de 4 vértices o más. Gavril [1] probó que todo grafo cordal, es el grafo de intersección de una familia de subárboles de algún árbol. Más formalmente, un grafo G es cordal si y sólo si existe un árbol T y una familia $\mathcal{F} = (F_v)_{v \in V(G)}$ de vértices de T tal que cada F_v induce un subárbol de T y $v \neq w$ son adyacentes si y sólo si $F_v \cap F_w \neq \emptyset$. (T, F) es una *representación* de G.

Especificando condiciones sobre las representaciones se han definido distintas subclases de los grafos cordales [2]. Así, los grafos UV son los cordales para los cuales existe un T tal que cada F_v induce un camino del T. Los grafos DV son los cordales para los cuales existe un T orientable de modo que los subárboles son caminos dirigidos. Los grafos RDV son los cordales que poseen algún T orientable enraizado de modo que los subárboles son caminos dirigidos.

Las tres clases UV, DV y RDV son hereditarias, o sea todo subgrafo inducido de un grafo de la clase está en la clase, por lo cual pueden ser caracterizadas por familias de prohibidos minimales. Un grafo G es prohibido minimal para una clase A si $G \notin A$ pero para todo v vértice de G se tiene que $G - v \in A$.

En el presente trabajo, para cada clase UV, DV y RDV se estudian los grafos prohibidos minimales con vértices falsos gemelos. Se dice que dos vértices son falsos gemelos si no son adyacentes y tienen los mismos vecinos. Se prueba que no hay ningún grafo de este tipo prohibido minimal para DV ni para RDV mientras que para UV los únicos son los cometas impares (ver figura).

2 Prohibidos minimales y falsos gemelos

Si G es un grafo, se llama $\mathcal{C}(G)$ a la *familia de cliques* de G y $\mathcal{DC}(G)$ a la familia dual de ésta, es decir, $\mathcal{DC}(G) = (C_v)_{v\in V(G)}$ donde C_v es el conjunto de cliques de G a los cuales v pertenece. Es fácil verificar que cualquier grafo G es el grafo de intersección de $\mathcal{DC}(G)$. Se ha probado que un grafo G es UV (resp.DV, resp. RDV) si y sólo si existe un árbol T cuyos vértices son los cliques de G y tal que $(T, \mathcal{DC}(G))$ es una representación UV (resp.DV, resp. RDV) de G. Se dice que T es un árbol clique de G y que $(T, \mathcal{DC}(G))$ es una representación canónica de G [2].

Un vértice $s \in V(G)$ es *simplicial* si $N(s)$ es un completo de G. Un vértice simplicial se dice *esencial* si $N(s)$ está contenido en más de un clique de G. Luego se tiene que s es un simplicial escencial de G si y sólo si $C(G-s) = C(G) - H_s$ siendo H_s el único clique de G que contiene a s. Dado un grafo G se dice que dos vértices x e y de G son *gemelos* si $N[x] = N[y]$ y que x e y son *falsos gemelos* si x e y no son adyacentes y $N(x) = N(y)$.

Observar que dado que un grafo cordal no posee ciclos inducidos de 4 vértices, se tiene que si x e y son falsos gemelos de un grafo cordal, entonces x e y deben ser vértices simpliciales.

Claramente, si $(T, (C_v)_{v\in V(G)})$ es una representación canónica de G se tiene que x e y son vértices gemelos si y sólo si $C_x = C_y$. Es fácil observar que si G es un grafo prohibido minimal para UV o DV o RDV entonces G no puede poseer vértices gemelos y todo simplicial de G debe ser esencial. En cambio, existen grafos prohibidos minimales para la clase UV con falsos gemelos.

Para probar esto veamos el rol que juegan los vértices simpliciales esenciales en las representaciones canónicas de un grafo cordal. Sea G un grafo cordal, x es un vértice simplicial esencial de G y $G' = G - x$. Dado que x es simplicial esencial, como ya fue observado $C(G') = C(G) - H_x$, con lo cual si C'_v es el conjunto de cliques de G' que poseen a v, se tiene que $C'_v = C_v - \{H_x\}$, observar que si vx no pertenece a $E(G)$, entonces $C'_v = C_v$. Como es natural las representaciones canónicas de G y G' están vinculadas, más específicamente veamos que en algunos casos una representación UV de G' puede conducir a una UV de G.

Si G un grafo cordal con x e y vértices falsos gemelos, sabemos que ambos son vértices simpliciales de G. Supongamos que x es un simplicial esencial de G y que G' es un grafo UV. Sea $(T', (C'_v)_{v\in V(G')})$ una representación UV de G', construiremos un grafo auxiliar $\Pi_{T'}$ asociado a T' del siguiente modo: los vértices de $\Pi_{T'}$ son el vecindario en T' de H_y, sea $V(\Pi_{T'}) = \{C_1, ..., C_n\}$ y $C_iC_j \in \Pi_{T'}$ si y sólo si existe un $v_{ij} \in C_i \cap C_y \cap C_j$.

Teorema 1. *Sean G un grafo cordal, x e y falsos gemelos de G y x simplicial esencial de G. Si $(T', (C'_v)_{v\in V(G')})$ es una representación UV (resp. DV, resp. RDV) de $G' = G - x$ y $\Pi_{T'}$ es bipartido entonces G es UV (resp. DV, resp. RDV).*

Demostración:
Idea: Sea $\Pi_{T'} = (A, B)$ se construye un árbol T particionando el vértice H_y en H_y, H_x. Las ramas de T' incidentes en H_y que tienen vértices de A continuarán en T incidentes en H_y, en cambio las ramas de T' incidentes en H_y que tienen vértices de B serán en T incidentes a H_x.□

Para concluir construiremos 3 grafos (ver figura).

- $\mathcal{T}_l$: *árbol cometa de l puntas*, es un árbol con $l + 2$ vértices y un vértice de grado l.
- $\mathcal{E}_l$: *estrella de l puntas*, es un grafo completo K_l y l vértices de grado 2.
- $\mathcal{H}_l$: *cometa de l puntas*, es el grafo compuesto por una estrella $\mathcal{E}_l$ y dos vértices x e y falsos gemelos, siendo sus vecinos los vértices de K_l.

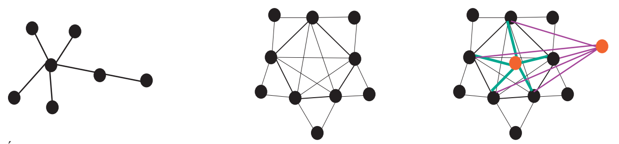
Es simple probar que $\mathcal{H}_l$ con l impar mayor o igual a 3, es un grafo prohibido minimal para la clase UV y prohibido no minimal para las clases DV y RDV. Mientras que $\mathcal{E}_l$ con l impar mayor o igual a 3, es prohibido minimal para las clases DV y RDV. El siguiente Corolario dice que no hay otros grafos prohibidos minimales con falsos gemelos para la clase UV.

Corolario 2. *Si G un grafo prohibido minimal para la clase UV, x e y falsos gemelos de G entonces G es un cometa de l puntas con l impar mayor o igual que 3.*

Demostración: Dado que G es un grafo prohibido minimal para la clase UV x e y son simpliciales esenciales de G. Por otro lado $G' = G - x$ es un grafo UV, sea $(T', (C'_v)_{v\in V(G')})$ una representación UV de G'. Por el Teorema 1, $\Pi_{T'}$ no puede ser bipartido, luego tiene algún ciclo impar.

Sea $L = C_1, .., C_l$ un ciclo impar de $\Pi_{T'}$. Observar que existen $v_{12}, v_{23}, .., v_{l1}$ vértices de G induciendo un completo. Por otro lado, existen z_i vértices en $C_i - C_y$ para cada $i \in \{1, ..., l\}$, tales que z_i es adyacente a $v_{i,j}$. Luego $v_{12}, v_{23}, .., v_{l1}, z_1, .., z_l, x, y$ inducen un cometa de l puntas $\mathcal{H}_l$ en G. Por la minimalidad es claro que $G = \mathcal{H}_l$. □

Corolario 3. *Los grafos cordales prohibidos minimales para las clases RDV o DV no posseen falsos gemelos.*



Árbol cometa de 5 puntas   Estrella de 5 puntas   Cometa de 5 puntas

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The Erdős' problem on absorbent sets by monochromatic directed paths in m-colored tournaments

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Keywords: m-colored tournaments, absorbent sets, kernels by monochromatic directed paths.

In this talk, we consider the following problem due to Erdős: for each $m \in \mathbb{N}$, is there a (least) positive integer $f(m)$ so that every finite m-colored tournament contains an absorbent set S by monochromatic directed paths of $f(m)$ vertices? In particular, is $f(3) = 3$? We prove several bounds for absorbent sets of m-colored tournaments under certain conditions on the number of colors of the arcs incident to every vertex from its in-neighborhood (resp. ex-neighborhood). In particular, we establish the validity of Erdős' problem for 3-colored tournaments with this condition. It is also proven that a 3-colored tournament containing no heterochromatic directed triangles with at most bichromatic ex-neighborhoods (resp. in-neighborhoods) has a kernel by monochromatic directed paths. Some other results are valid for m-colored quasi-transitive digraphs and, as they are a generalization of tournaments, we obtain some special instances for which Erdős' problem is satisfied.

K_r-packing of P_4-sparse graphs[1]

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Keywords: K_r-packing, P_4-sparse graph, modular decomposition.

Abstract

The K_r-packing problem is that of finding the maximum number of pairwise disjoint cliques of size r in a graph. This problem is NP-hard for general graphs when $r \geq 3$, and even for split graphs when $r \geq 4$. Guruswami et al. proposed a polynomial time algorithm for cographs (when r is fixed). In this work we extended this algorithm to P_4-sparse graphs, keeping the same time complexity.

1 Introduction

The K_r-packing problem is that of finding the maximum number of pairwise disjoint cliques of size r in a graph. Note that, for $r = 2$ the problem is exactly the maximum matching problem, which has a well-known polynomial time algorithm, but, for $r \geq 3$, this problem is NP-hard for general graphs. Even for restricted graph classes, such as line and total graphs ($r \geq 3$), and split graphs ($r \geq 4$), the problem remains NP-hard. Guruswami et al. proposed a polynomial time algorithm for cographs (when r is fixed) [3]. In this work we extend this algorithm to P_4-sparse graphs.

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Throughout this paper, $G = (V(G), E(G))$ denotes a simple, finite, and undirected graph, and we use standard graph terminology [1]. A subset M of $V(G)$ is called a module of G if there is no $\{a, b\} \subseteq M$ and $c \in (V(G) \setminus M)$ such that $\{a, c\} \in E(G)$ and $\{b, c\} \notin E(G)$. A module M of G is said to be strong if there is no module N of G, $N \neq M$, such that $N \setminus M \neq \emptyset$, $M \setminus N \neq \emptyset$, and $N \cap M \neq \emptyset$.

The modular decomposition tree (MDT) of a graph G has one node for each strong module of G. The parent of a node, related to strong module M, is the node associated with the smaller strong module that properly contains M. Hence, the MDT represents inclusions of strong modules, from isolated vertices (leafs) to the module $V(G)$ (the root). If N is a non-leaf node of the MDT and M is its corresponding module in G, then N is called serial (parallel), if $\overline{G[M]}$ ($G[M]$) is not connected. Otherwise, the node is called neighborhood.

A graph is P_4-sparse [4] if the subgraph induced by each module corresponding to a neighborhood node in its MDT is isomorphic to a spider [2], which is a graph whose vertex set may be partitioned into three sets K, S, and H, such that: (1) $|K| = |S| \geq 2$, but H can be empty; (2) K is a clique and S is a stable set; (3) $\{i, j\} \in E(G), \forall i \in K, \forall j \in H$; (4) $\{i, j\} \notin E(G), \forall i \in S, \forall j \in H$; (5) $d(i) = 1$ (thin spider) or $d(i) = |S| - 1$ (thick spider), $\forall i \in S$; and (6) $d(j) = |H| + 1$ (thin spider) or $d(j) = |H| + |S| - 1$ (thick spider), $\forall j \in K$.

2 K_r-packing of cographs

In this section we recall the following results given by Guruswami et al. [3], that proposed an algorithm to compute in polynomial time a K_r-packing of a cograph, using a dynamic programming technique.

To describe these results, some definitions are necessary. A graph G is $(n_1, n_2, n_3, \dots, n_r)$-packed if there exists a partition P of $V(G)$ such that there are n_i parts in P which are cliques of size i in G, for all $1 \leq i \leq r$. This obviously implies that $|V(G)| = \sum_{i=1}^{r} i n_i$. The partition P is called a $(n_1, n_2, n_3, \dots, n_r)$-packing of G. The K_r-packing problem asks for the maximum value of n_r such that there is a $(0, 0, 0, \dots, n_r)$-packing of G.

Consider the function $f(G, n_3, n_4, \dots, n_r)$ defined as $\max\{n_2 : G \text{ has a } (0, n_2, n_3, n_4, \dots, n_r)\text{-packing}\}$, and it is undefined if G has no such packing for any value of n_2. Note that, if we compute $f(G, 0, 0, \dots, n_r)$ for $n_r \geq 0$, we solve the K_r-packing problem for G. A cograph is a graph whose MDT has only serial and parallel nodes. So, using the following algorithms we can compute $f(G, n_3, n_4, \dots, n_r)$ recursively for any cograph G.

If N is a parallel node of the MDT of G, M is the associated module, and $M_1, M_2 \dots, M_k$ are the modules associated to the children of N, then $G[M] = G[M_1] \cup G[M_2] \cup \dots \cup G[M_k]$. To compute $f(G[M], n_3, n_4, \dots, n_r)$, we apply repeatedly an algorithm that computes f on a graph $G' \cup G''$. The algorithm returns the maximum of $f(G', n_3', n_4', \dots, n_r') + f(G'', n_3'', n_4'', \dots, n_r'')$, for every integers $n_i' \geq 0$ and $n_i'' \geq 0$ such that $n_i = n_i' + n_i''$, for each $3 \leq i \leq r$.

Now, if N is a serial node, then $G[M] = G[M_1] + G[M_2] + \dots + G[M_k]$. By a similar argument, it suffices to apply the algorithm that computes f on a graph $G' + G''$. The algorithm returns the maximum of $n_{2,0} + n_{2,1} + n_{2,2}$, for all integers $n_{i,j}$ such that: (1) for $1 \leq i \leq r$, $n_i = \sum_{j=0}^{i} n_{i,j}$ where $n_{i,j} \geq 0$ for $0 \leq j \leq i$; (2) $n_j' = \sum_{i=j}^{r} n_{i,j}$ and $n_j'' = \sum_{i=j}^{r} n_{i,i-j}$ for $1 \leq j \leq r$; (3) $f(G', n_3', \dots, n_r') \geq n_2'$ and $f(G'', n_3'', \dots, n_r'') \geq n_2''$; (4) $n' = \sum_{j=1}^{r} j n_j'$ and $n'' = \sum_{j=1}^{r} j n_j''$; and (5) $\sum_{i=2}^{r} n_{i,j} = 0$ or $\sum_{i=2}^{r} n_{i,0} = 0$.

3 K_r-packing of P_4-sparse graphs

To decide the K_r-packing problem for a P_4-sparse graph, we need to solve the function f on spiders using an algorithm similar to the one for joint graphs. Let G be a spider and K, S, and H be the partition of $V(G)$ as defined in Section 1. If P is a $(n_1, n_2, n_3, \dots, n_r)$-packing of G, then each $K_i \in P$ either has a vertex in S or is a subset of $K \cup H$. Hence, we can define, for $1 \leq i \leq r$ and $0 \leq j \leq i$, the integer $n_{i,j}$ as the number of $K_i \in P$ such that $|K_i \cap K| = j$ and $|K_i \cap H| = i - j$, and the integer n_i^S as the number of $K_i \in P$ such that $K_i \cap S \neq \emptyset$. Moreover, we define $n_j' = \sum_{i=j}^{r} n_{i,j}$ as the number of $K_i \in P$, such that $K_i \cap S = \emptyset$ and $|K_i \cap K| = j$; and $n_j'' = \sum_{i=j}^{r} n_{i,i-j}$ as the number of $K_i \in P$, such that $K_i \cap S = \emptyset$ and $|K_i \cap H| = j$. From the following lemmas, we can construct an algorithm for the K_r-packing problem for a spider graph.

Lemma 1. *A thin spider G has a $(n_1, n_2, \dots, n_r)$-packing P if, and only if, there exist non-negative integers n_1^S, n_2^S, and $n_{i,j}$, for $1 \leq i \leq r$ and $0 \leq j \leq i$, such that: (1) $n_2'' \leq f(G[H], n_3'', \dots, n_r'')$; (2) $|S| = n_1^S + n_2^S$, $|K| = n_2^S + \sum_{i=1}^{r} i n_i'$, and $|H| = \sum_{i=1}^{r} i n_i''$; and (3) $n_i = \sum_{j=0}^{i} n_{i,j}$ for $3 \leq i \leq r$ and $n_i = n_i^S + \sum_{j=0}^{i} n_{i,j}$ for $i \in \{1, 2\}$.*

Lemma 2. *If G is a thin spider and P is a $(n_1, \dots, n_r)$-packing of G, then there is another $(n_1, \dots, n_r)$-packing, P', of G, such that every K_2 in P' either is contained in H or has one vertex in S and the other in K.*

Lemma 3. *A thick spider G has a $(n_1, \ldots, n_r)$-packing if, and only if, there exist non-negative integers $n_{i,j}$, for $1 \leq i \leq r$, $0 \leq j \leq i$, and n_i^S, for $1 \leq i \leq r$, such that: (1) $n_2'' \leq f(G[H], n_3'', \ldots, n_r'')$; (2) $|S| = \sum_{i=1}^{r} n_i^S$, $|K| = \sum_{i=2}^{r}(i-1)n_i^S + \sum_{i=1}^{r} in_i'$, and $|H| = \sum_{i=1}^{r} in_i''$; (3) $n_i^S = 0$, for $i > |K|$; and (4) $n_i = n_i^S + \sum_{j=0}^{i} n_{i,j}$, for $1 \leq i \leq r$.*

Lemma 4. *If a thick spider G has a $(n_1, n_2, \ldots, n_r)$-packing, then there is another partition P', which is also a $(n_1, n_2, \ldots, n_r)$-packing of G, but every K_2 in P' either is contained in H or has a vertex in S.*

Now we are ready to describe the algorithm that finds K_r-packing for a P_4-sparse graph G. The function f is computed in each node of the MDT of G, processing serial and parallel nodes as in Section 2. For neighborhood nodes, f is computed as the maximum of $n_{2,0} + n_2^S$ over all integers $n_{i,j}$ and n_i^S satisfying the conditions given in lemmas 1 and 3 and the condition $\sum_{i=2}^{r} n_{i,i} = 0$ or $\sum_{i=2}^{r} n_{i,0} = 0$.

The expression maximized is due to lemmas 2 and 4. The additional condition comes from the fact that $G[K \cup H] = G[K] + G[H]$ and from the lemma: *If P is a $(n_1, n_2, \ldots, n_r)$-packing of $G = G' + G''$, then there exists a $(n_1, n_2, \ldots, n_r)$-packing P' covering precisely the same vertices as P does and P' does not contain C' and C'' such that $C' \subseteq V(G')$ and $C'' \subseteq V(G'')$.*[2]

The MDT of any graph is obtained in linear time [5]. We also can identify if a graph is spider, as well as identify the partition of the spider in the three sets K, S, and H, in linear time [2]. Since the number of possibilities evaluated for spiders is a subset of the possibilities evaluated for joint graphs, the time complexity of the proposed algorithm is also polynomial.

We conclude this note observing that MDT can be applied to solve the K_r-packing problem for other graphs that have neighborhood nodes in its MDT well characterized, such as P_4-tidy graphs [2].

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The K-Behavior of Circular-Arc Graphs

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1 Introduction

In this paper we determine the K-behavior of circular-arc graphs. That is, we decide if the clique graph of a circular-arc graph G is K-convergent, K-divergent or K-null. Furthermore, when G is K-convergent, we characterize the graph to which G K-converges. The method leads to a linear time algorithm for this decision problem.

The *iterated clique graph* is defined by $K^0(G) = G$ and $K^{i+1}(G) = K(K^i(G))$. The analysis of the *K-behaviour* of a clique graph is one of the main topics about iterated clique graphs. A graph G is K-*null* if $K^i(G)$ is the trivial graph, for some i. Say that G is K-*periodic* with period i if $K^i(G) = G$ for some $i > 0$. When the period is 1 the K-periodic graph is called *self-clique*. A graph is K-*convergent* when it is K-null or $K^i(G)$ is K-periodic for some $i \geq 0$. If G is not K-convergent, then $|V(K^i(G))|$ is unbounded when $i \to \infty$; in this case G is K-*divergent*. For the general case, the problem of determining the K-behavior of a graph is not known even if it is computable. Nevertheless, polynomial-time algorithms to decide the K-behavior of a few classes are known. This is the case for cographs [6], P_4-tidy graphs [2] and complete multipartite graphs [10]. Clique-Helly graphs K-converge to graphs with period either 1 or 2 [4], interval graphs are K-null and octahedra of dimension at least 3 K-diverge.

The *k-th power* of a graph G is the graph G^k that has the same vertices as G and two vertices are adjacent whenever their distance is lower than or equal to k. The *neighborhood* $N_G(v)$ of a vertex v is the set of its adjacent vertices, and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. When there is no ambiguity, we may simply write $N(v)$ or $N[v]$. A vertex v is *universal* when $N[v] = V(G)$. Vertex v *dominates* vertex w when $N[w] \subseteq N[v]$, and they are *twins* when w also dominates v. A *dismantling* of a graph G is the subgraph obtained from G by iteratively removing any vertex v, which is dominated by some vertex $w \neq v$, in the subgraph so far obtained. It is not hard to see that the dismantling is unique up to isomorphism. In [5] it is proved that the K-behavior is the same for a graph and its dismantling. We remark that for general graphs, the dismantling of a graph can be computed in polynomial-time.

A *circular-arc* (CA) model $\mathcal{M}$ is a pair $(C, \mathcal{A})$, where C is a circle and $\mathcal{A}$ is a collection of arcs of C. When traversing the circle C, we will always choose the clockwise direction. If s, t are points of C, write (s, t) to mean the arc of C defined by traversing the circle from s to t. Call s, t the *extremes* of (s, t), while s is the *beginning point* and t the *ending point* of the arc. For $A \in \mathcal{A}$, write $A = (s(A), t(A))$. The *extremes* of $\mathcal{A}$ are those of all arcs in $\mathcal{A}$.

When no arc of $\mathcal{A}$ contains any other, $\mathcal{M}$ is a *proper* circular-arc (PCA) model. When every set of pairwise intersecting arcs share a common point, $\mathcal{M}$ is called a *Helly* circular-arc (HCA) model. A CA (PCA) (HCA) graph is the intersection graph of a CA (PCA) (HCA) model.

Clique graphs of Helly circular-arc graphs have been considered in [3]. Previous results on the K-behavior of circular-arc graphs have been before presented in [1]. In fact, in [1] it has been proved that an HCA graph G is K-periodic if and only if G is isomorphic to C_n^k with $n > 3k$. Moreover, in the same paper it has been proved that K-periodic Helly circular-arc graphs are always self-clique.

We extend the above results, as follows. First we employ the dismantling of G to observe that its K-behavior follows from the results of [7] and [9]. We then conclude that G is K-null if and only if its dismantling is K-null; G K-converges to a graph which is not trivial if and only if its dismantling is C_n^k, with $n > 3k$; and G is K-divergent otherwise. Next, we prove that a K-convergent CA graph always K-converges to its dismantling. Furthermore, we characterize the K-convergent CA graphs, which are not K-null.

2 Main theorems

Our method to decide the K-behavior of CA graphs is based on the two theorems below. The first of them characterizes the dismantling of a circular-arc graph. We conclude that the dismantling of a circular-arc graph is a PCA graph and contains no dominated vertices. The second theorem specifies exactly when does the dismantling K-converge.

Theorem 1 ([9]). *: Let G be a non-complete graph. Then the following are equivalent:*

- **(i)** *G is isomorphic to C_n^k for some pair of values n, k.*
- **(ii)** *G is a PCA graph without dominated vertices.*
- **(iii)** *G has a unique PCA model with arcs $A_1, \ldots, A_n$ where $t(A_i)$ lies immediately after $s(A_{i+k})$.*
- **(iv)** *G has a PCA model where every beginning point is followed by an ending point.*

Theorem 2 ([8]). *: Graph C_n^k is K-convergent if and only if it is complete or $n > 3k$.*

These two theorems can be used to actually deciding the K-behavior of a general CA graph. A circular-arc graph is K-null if its dismantling is K-null; it K-converges to a graph which is not K-null if its dismantling is C_n^k with $n > 3k$; or it K-diverges otherwise. However, much more can be said about the graph to which G K-converges when it does, because this graph is self-clique and thus unique.

Theorem 3. *A circular-arc graph G is K-convergent to a non-trivial graph if and only if G is a non-interval HCA graph admitting a model with no two arcs covering the circle.*

Finally, we characterize the graphs to which a circular-arc graph K-converges, when it does.

Theorem 4. *If a circular-arc graph K-converges then it K-converges to its dismantling.*

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A classification of defensive alliances in 6-regular graphs

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Keywords: Alliances, induced graphs.

A *defensive alliance* is a set of vertices satisfying that each vertex has at least as many neighbors in the alliance (including itself) than neighbors not belonging to the alliance.

In a d-regular graph, a defensive alliance is a set of vertices that induces a subgraph with minimum degree at least $\lfloor \frac{d}{2} \rfloor$ and maximum degree at most d. We are interested in the following problem: which graphs can a critical defensive alliance induce?

The answer is known for degree $d \leq 5$. For 6-regular graphs, it turns out to be complex. We study alliances in graphs of degree 6, and of given cardinality $k \leq 8$. Even in these restricted cases, there is not an easy description of such alliances.

Because of the complexity of the problem, in [1], we restrict the problem to a family of very symmetric graphs, the well known circulant graphs. In this paper we proved that the alliance number in a circulant graph of degree 6 is at most 8 and we use the results exhibit in this work to characterize all of them.

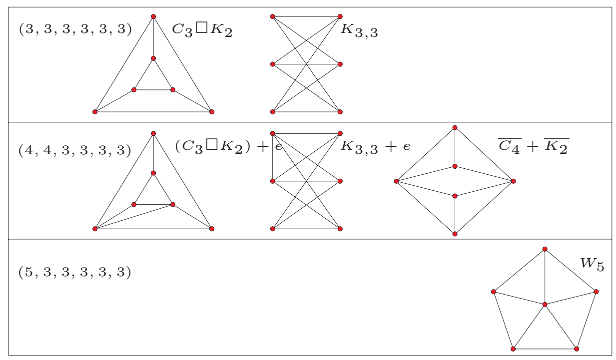

Figure 1: The graphs induced by critical defensive alliances of cardinality 6 in a 6-regular graph, with their associated degree sequence.

The following definition is taken from [3].

Definition 1 (Defensive alliance). *A non-empty set* $S \subseteq V$ *is a* defensive alliance *of* G *if, for every* $v \in S$,

$$|N_S[v]| \geq |N_{\overline{S}}(v)|. \qquad (1)$$

There exists many kinds of alliances (see [2, 3]), in this note we are only interested in defensive alliances. We say that a defensive alliance is *critical* if none of its proper subsets is a defensive alliance. In fact, the known results for regular graphs of degree $d \leq 5$ allow us to completely characterize critical alliances for these graphs, (for instance in [4] the authors study alliances in cubic graphs):

- If G is 1-regular, the critical alliances are exactly the singletons.
- The critical alliances in a 2-regular or 3-regular graph are exactly the edges.
- The critical alliances in a 4-regular or 5-regular graph are exactly the induced cycles.

Given a d-regular graph, G, we are concerned in characterize critical alliances in G, i.e., if S is a critical alliance in G, which graphs could the subgraph of G induced by S, $\langle S\rangle$, be isomorphic to? Unfortunately, there is no simple characterization of the alliances d-regular graphs if $d > 5$. In this work we will concentrate on alliances of given cardinality and $d = 6$.

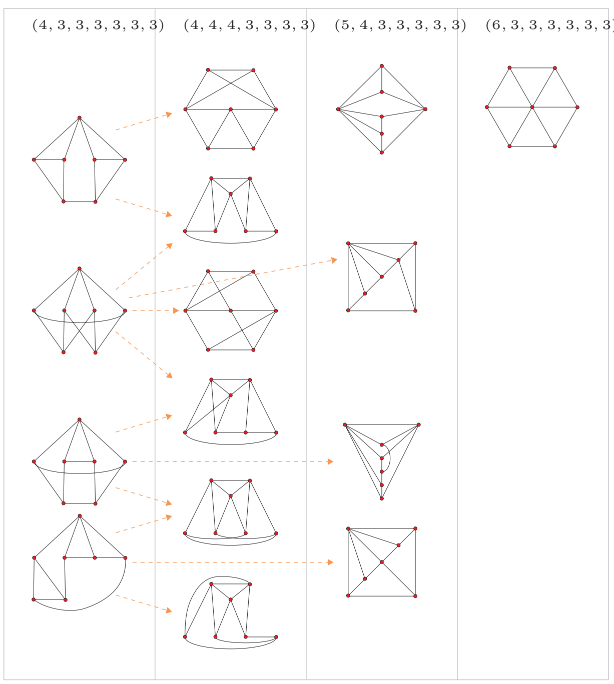

Figure 2: The graphs induced by critical defensive alliances of cardinality 7 in a 6-regular graph G. The arrows indicate the subgraph relation.

Proposition 2. *Let G be a 6-regular graph and S a critical defensive alliance of cardinality 6 of G then $\langle S\rangle$ is one of the following graphs:*

$$C_3\Box K_2,\ K_{3,3},\ (C_3\Box K_2)+e,\ K_{3,3}+e,\ \overline{C_4}+\overline{K_2},\ W_5.$$

A representation of these graphs is given in Figure 1.

Proposition 3. *Let G be a 6-regular graph and S a critical defensive alliance of cardinality 6 of G then $\langle S\rangle$ is one of the 15 graphs in Figure 2.*

If G is a 6-regular graph, the set of graphs that a defensive alliance of G of cardinality 8 can induce contains exactly 65 graphs (in [1] appears a nice figure of all them).

We can easily extend the results to 7-regular graphs because any critical defensive alliance in a 6-regular graph is a critical defensive alliance in a 7-regular graph.

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El índice pseudoacromático de la gráfica completa

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Keywords: pseudoachromatic index, complete graph, projective plane.

Una coloración $\varsigma : E \to [k]$ de las aristas de una gráfica $G = (V, E)$ se dice que es *completa* si todo par de colores es incidente en algún vertice; es decir, si para todo par $i, j \in \binom{k}{2}$ existen aristas $e, f \in E$ tales que $e \cap f \neq \emptyset$, $\varsigma(e) = i$ y $\varsigma(f) = j$. En particular, una coloración propia con $\chi'(G)$ colores —el índice cromático de G— es necesariamente una coloración completa.

El *índice pseudoacromático* de una gráfica G, denotado aqui por $\psi(G)$, es el máximo número de colores que se pueden usar en una coloracón completa de las aristas de G. Claramente, $\chi'(G) \leq \psi(G)$.

En este trabajo estamos interesados en estimar el índice pseudoacromático de la gráfica completa y lo denotaremos por $\psi(n) = \psi(K_n)$. En particular, podemos calcular precisamente $\psi(n + q + 1)$ siempre que $n = q^2 + q + 1$ y $q = 2^\beta$.

Como podran adivinar algunos colegas, la expresión $n = q^2 + q + 1$ surge de la existencia de un plano proyectivo finito. Es la estructura de líneas del proyectivo Π_q —cualesquiera dos líneas son incidentes— la que nos permite exhibir una coloración completa... y es un milagro que un par de cotas superiores para $\psi(n)$, muy naturales en este contexto, nos permitan demostrar que esas coloraciones son óptimas. Explicitamente, demostraremos el siguiente

Teorema. *Si* $q = 2^\beta$ *y* $n = q^2 + q + 1$, *entonces*

$$\psi(n + q + 1) = q^3 + 2q^2 + 3q.$$

En la figura, se puede apreciar una coloración de las aristas de K_7 con 10 colores que se puede modificar y extender a una coloración de K_{10} con 22 colores. Durante la plática exhibiremos esta coloración en detalle.

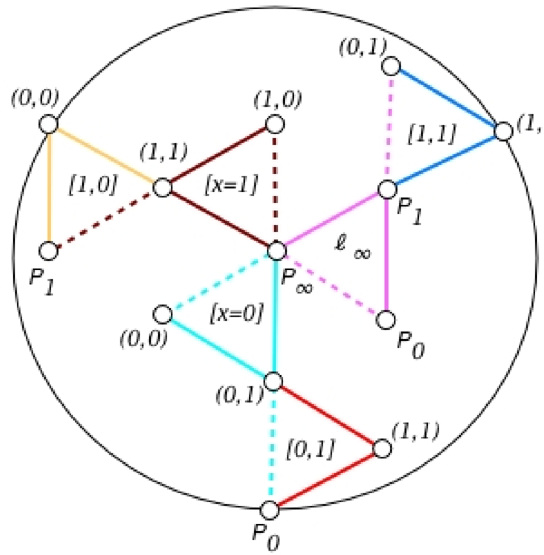

Figura. *Coloreando* K_7 *con el plano de Fano* Π_2 *como patrón.*

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Level Hypergraphs-abstract

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In the present work we introduce a way of relating any hypergraph with a simpler one, which keeps its edge-structure but has often much less vertices. This allows us to obtain results in several branches of hypergraph theory.

Given a hypergraph $H = (E_1, ..., E_m)$, its level-hypergraph is the result of identifying all vertices which belong to exactly the same edges. This new hypergraph has the same edge-structure as the original one, but may have less vertices. The tool makes it possible to emulate known theorems regarding bounds for important numbers (like the transversal number, the matching number, etc.) given in terms of order or rank; the new results apply to different classes of hypergraphs and are stated in terms of edge-structure.

Definition: Given a hypergraph $H = (E_1, ..., E_m)$, we define a partition $P = \{P_1, ..., P_l\}$ of $V(H)$ in the following way: $\{x, y\} \subset V(H)$ is contained in an element of P iff x and y belong to exactly the same edges of H. We call this the *natural partition* of $V(H)$, and the partition defined over each edge $E \in H$ as $\{P_i \in P \mid P_i \subset E\}$ is the *natural partition* of E. The elements of P are called *levels of* H, and the levels of H contained in an edge E of H are called *levels of* E.

Definition: Given a hypergraph $H = (E_1, ..., E_m)$ with natural partition $P = \{P_1, ..., P_l\}$, the *level hypergraph* $L_H = (E'_1, ..., E'_m)$ of H is the hypergraph resulting from deleting every vertex but one from each level P_i of P. In other words, we consider a set $S = \{x_1, ..., x_l \mid x_i \in P_i \ \forall i \in \{1, ..., l\}\}$ and take $L_H = H[S]$. It is clear that L_H is well defined, that is, it does not matter which vertex from each level is kept, for all of them play an equivalent role.

Since every edge E of H has at least one vertex, it contains at least one level, so it induces an edge E' in L_H; then both H and L_H have the same number of edges. In the same way, E_i and E'_i contain the same number of levels and $E_i \cap E_j \neq \emptyset$ iff $E'_i \cap E'_j \neq \emptyset$. This implies that H is simple iff L_H is simple, and that H has repeated edges iff L_H does too.

Observe that every vertex belonging to a given level P_i of H has the same degree, as well as the vertex corresponding to that level in L_H. In particular, $\Delta(H) = \Delta(L_H)$ and $\delta(H) = \delta(L_H)$.

Given an edge E of a hypergraph H, the corresponding edge of L_H will be called E', and an edge of L_H will always be written with an apostrophe. We will use the same symbol for a level of H and its corresponding level in L_H. We consider $V(L_H) = \{x_1, ..., x_l \mid x_i \in P_i \ \forall i \in \{1, ..., l\}\}$ and call x_i the *representative* of P_i.

Let H be a hypergraph and $D(H)$ its dual hypergraph. Then $D(L_H)$ is like $D(H)$ without repeated edges. Indeed, $|H| = |L_H|$ and for each vertex in $V(H)$ there is a vertex in $V(L_H)$ belonging to the same edges, so $D(H)$ and $D(L_H)$ have the same structure, but $D(H)$ may have more (repeated) edges. On the other hand, the only way of generating repeated edges in $D(H)$ is having two vertices in the same level of H, so $D(L_H)$ has no repeated edges.

Proposition 1: Let H be a hypergraph and let L_H be its level hypergraph. Then:

1. H is regular iff L_H is regular.
2. H is regularisable iff L_H is regularisable.
3. H is quasi-regularisable iff L_H is quasi-regularisable.

In cases 2 and 3, the integer used to get the regular hypergraph is the same for an edge in H and for the edge it induces in L_H.

Proposition 2: Let H be a hypergraph. H is balanced (totally balanced) iff L_H is balanced (totally balanced).

Proposition 3: Let H be a hypergraph. H is unimodular (strongly unimodular) iff L_H is unimodular (strongly unimodular).

Now follow some theorems obtained from classical results using level hypergraphs. The original theorems are marked with the symbol [$*$].

Theorem 1' [$*$]: A hypergraph is balanced iff its induced subhypergraphs are two colourable.

Theorem 1: A hypergraph H is balanced iff the induced subhypergraphs of its level-hypergraph L_H are two colourable.

Proof: The theorem follows from Proposition 2 and Theorem 1'. This result makes it easier to decide wether a given hypergraph is balanced or not.

Theorem 2' [$*$]: A hypergraph of rank $r \leq 3$ is unimodular iff it is balanced.

Theorem 2: A hypergraph such that every edge has at most 3 levels is unimodular iff it is balanced.

Proof: Let H be a hypergraph. If H is unimodular, then it is balanced. Now suppose H is balanced and no edge of H has more than three levels. Then L_H is balanced of rank $r \leq 3$, so from Theorem 2' it is unimodular. Proposition 3 implies that H is unimodular.

Definition: Let H be a hypergraph. $T \subset V(H)$ is a *transversal* of H iff it meets all its edges. A transversal T of H is *minimal* iff for every transversal T' of H, $T \subset T' \Rightarrow T = T'$. A transversal T of H is *minimum* iff for every transversal T' of H, $|T| \leq |T'|$. τ represents the cardinality of a minimum transversal, and τ' represents the maximum cardinality of a minimal transversal.

A transversal of L_H is also a transversal of H, and a transversal T of H induces a transversal of L_H by deleting all vertices but one from $P_i \cap T$ for each level P_i such that $|P_i \cap T| > 1$. Moreover, a minimal transversal of H has no more than one vertex per level, for two vertices in the same level belong to exactly the same edges; on the other hand, if an edge E'_i of L_H is met by a given set $S \subset V(H)$, then the corresponding edge E_i of H is met by S; it follows that every set $T \subset V(H)$ is a minimal (minimum) transversal of H iff T is a minimal (minimum) transversal of L_H, taking $T \cap P_i$ as x_i, the representative of P_i, whenever $T \cap P_i \neq \emptyset$. So we have $\tau'(H) = \tau'(L_H)$ and $\tau(H) = \tau(L_H)$.

Definition: Let H be a hypergraph. A set $S \subset V(H)$ is *strongly independent* iff $|S \cap E_i| \leq 1$.

A strongly independent set S in L_H is a strongly independent set in H, since L_H is an induced subhypergraph of H. Conversely, a strongly independent set S in H meets any level at most once, since it meets any edge at most once. By taking $S \cap P_i$ as x_i we have that S is a strongly independent set in L_H, for any edge $E'_i \in L_H$ and any set $A \subset V(H)$ satisfy $|E'_i \cap A| \geq 2 \Rightarrow |E_i \cap A| \geq 2$.

Notice that the concept of independence ($S \subset V(H)$ is *independent* iff there is no edge $E \in H$ such that $E \subset S$) does not translate well to level hypergraphs, since we may have $S \cap P \neq \emptyset$ and $(V(H) \setminus S) \cap P \neq \emptyset$ for a given level P of H.

Theorem 3' [$*$]: Let H be a balanced hypergraph. Then H has a good k-colouring for every $k \geq 2$.

Theorem 3"[$*$]: Let H be a balanced hypergraph such that $|E| = r$ for every edge $E \in H$. Then $V(H)$ may be partitioned in r pairwise disjoint strongly independent transversals.

Proof: This follows from Theorem 3', since given a good r-colouring every color class is a strongly independent transversal.

Theorem 3: Let H be a balanced hypergraph such that every edge $E \in H$ has r' levels. Then H has r' pairwise disjoint strongly independent transversals.

Proof: From Proposition 2 we have that L_H is a balanced, uniform hypergraph of rank r', so from Theorem 3" L_H has r' pairwise disjoint strongly independent transversals, which are as well strongly independent transversals of H.

Definition: Let H be a hypergraph. A *matching* in H is a set of pairwise disjoint edges of H, and $\nu(H)$ denotes the maximum cardinality of a matching in H.

Since $E_i \cap E_j \neq \emptyset$ iff $E'_i \cap E'_j \neq \emptyset$, we have that $\nu(H) = \nu(L_H)$.

Theorem 4' [$*$]: Let $H = (E_1, ..., E_m)$ be a linear hypergraph without repeated loops and let $n = |V(H)|$. Then $\nu(H) \geq \frac{m}{n}$.

Theorem 4: Let $H = (E_1, ..., E_m)$ be a level-linear hypergraph without repeated one-level edges. Let n' be the total amount of levels in H. Then $\nu(H) \geq \frac{m}{n'}$.

Proof: Let H be as asked. Then L_H is a linear hypergraph without repeated loops with n' vertices and m edges. The result follows from Theorem 4'.

Theorem 4 gives a better value than Theorem 4', for $n' \leq n$. It applies to a different class of hypergraphs: linear is asking more than one-level intersections, but no repeated loops is asking less than no repeated one-level edges.

Theorem 5' [$*$]: Every regular r-uniform hypergraph H such that $|V(H)| = n$ satisfies $\frac{n}{r^2 - r + 1} \leq \nu(H)$.

Theorem 5: Every simple regular hypergraph H with n' levels and such that every edge has r' levels satisfies $\frac{n'}{(r')^2 - r' + 1} \leq \nu(H)$.

Proof: Take H satisfying the conditions asked. Then L_H is a regular r'-uniform hypergraph with $|V(L_H)| = n'$, so from Theorem 5' $\frac{n'}{(r')^2 - r' + 1} \leq \nu(L_H) = \nu(H)$.

Those are some results obtained by means of level hypergraphs. As we stated at the beginning, the tool is useful for getting a theorem in terms of edge-structure from one refering to rank or order, and allows as well other kinds of results.

L-coloring bicolored 3-uniform hypergraphs lists sized k with $2k-1$ colors

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A *hypergraph* $\mathcal{H}$ is a non-empty collection $E(\mathcal{H})$ of nonempty finite sets called *edges*. The union of the edges of $E(\mathcal{H})$, denoted by $V(\mathcal{H})$, is the *set of vertices* of $\mathcal{H}$. A hypergraph is r*-uniform* if each edge has cardinality r. We say that L is a *list* of colors to $\mathcal{H}$ if L is a function that for each vertex v of $V(\mathcal{H})$ defines a set L(v) of colors. We say that $\mathcal{H}$ is L-*colorable* if it is possible to color each vertex $v \in V(\mathcal{H})$ with a color of L(v) in such a way that no edge is contained in any color class. If L is constant, $|\mathrm{L}(v)| = k$, and $\mathcal{H}$ is L-colorable, then we say that $\mathcal{H}$ is k*–colorable*, in this case if $k = 2$, then $\mathcal{H}$ is said to be a *bipartite* hypergraph, and $\mathcal{H}$ is *bicolored* if it is given a bipartition (V_1, V_2) for $V(\mathcal{H})$ defined by 2 color classes. Lovász [4] proved that is NP-complete deciding whether a hypergraph $\mathcal{H}$ is 2-colorable. The hypergraph $\mathcal{H}$ is called $k-$*choosable* if k is the smallest integer such that, $\mathcal{H}$ is L-colorable for every list L, where L(v) has size k, for every $v \in V(\mathcal{H})$. The Π_2^p class [5] of problems consists of the decision problems which a certificate to the answer Yes can be checked in polynomial time by a non-deterministic Turing machine. Erdös, Rubin and Taylor [3] exhibited a polynomial time algorithm to decide whether a graph is 2–choosable, and proved that if $k \geq 3$, then deciding whether a graph is $k-$choosable is a Π_2^p-complete problem [5].

Recently, Dinur, Regev, and Smyth [2] proved that given a positive integer $k \geq 2$, and a 3-uniform bipartite hypergraph $\mathcal{H}$, it is NP-hard to color $\mathcal{H}$ with k or less colors. Let $k \geq 2$ be a positive integer, $\mathcal{H} = (V, E)$ be a 3-uniform bipartite hypergraph, (V_1, V_2) be a bipartition for the vertices of V and a list L, where $\left|\bigcup_{v \in V} \mathrm{L}(v)\right| \leq 2k-1$, such that for each vertex $v \in V$, $|\mathrm{L}(v)| = k$. In this article we prove that it is NP-complete to decide whether $\mathcal{H}$ is L-colorable. Therefore, it is a Π_2^p problem to decide whether a bicolored 3-uniform hypergraph $\mathcal{H}$ is $k-$choosable, where $k \geq 2$ and at most $2k-1$ colors are allowed. Let k be a positive integer. We consider the decision problem:

<u>k-LIST BICOLORED 3-UNIFORM HYPERGRAPH</u> (<u>k-LB3UH</u>)
<u>INSTANCE</u>: $J = (\mathcal{H}_k, V_1, V_2, \mathrm{L}, p)$, where $p \geq k$ is a positive integer, $\mathcal{H}_k = (V, E)$ is a bipartite 3-uniform hypergraph, (V_1, V_2) a bipartition for V for which each edge of E has a vertex of V_1 and a vertex of V_2, and a function $\mathrm{L}: V \longrightarrow \mathrm{P}(\{1, 2, 3, \ldots, p\})$, such that $\mathrm{P}(\{1, 2, 3, \ldots, p\})$ is the collection of subsets of $\{1, 2, 3, \ldots, p\}$, and $|\mathrm{L}(v)| = k$ is a list with k colors.

<u>QUESTION</u>: Is $\mathcal{H}$ L-colorable?

2-LB3UH is NP-complete with $2 = k \leq p \leq 3 = 2k-1$

First we describe the construction of a particular instance $J = (\mathcal{H}_2 = (V, E),\ V_1, V_2, \mathrm{L}, 3)$ of 2–LB3UH from an instance $I = (U, C)$ of 3SAT. The vertex set V contains the set $V_{aux} = \{a_1, a_2, \ldots, a_{16}, b_1,\ b_2,\ \ldots,\ b_{16}\}$. For each variable $u_i \in U$, $1 \leq i \leq n$, there are 2 vertices u_i and $\overline{u_i}$, belonging to V, corresponding to the literals of the variable u_i. For each clause $c_j = (x_1 \vee x_2 \vee x_3)$, $1 \leq j \leq m$, there are 3 vertices c^j_{x1}, c^j_{x2}, c^j_{x3}, belonging to V, corresponding to the literals x_1, x_2, and x_3 of the clause c_j.

The edge collection E is partitioned into the collections E_1, E_2, E_3, E_4, and E_5:

$E_1 = \{\{u_i, \bar{u}_i, a_1\}\ ,\ \{u_i, \bar{u}_i, b_1\}, i \in \{1, 2, 3, \ldots, n\}\}$;

$E_2 = \Big\{\{c^j_{x1}, c^j_{x2}, c^j_{x3}\}, \{\{a_1, c^j_{x1}, c^j_{x2}\}, \{a_1, c^j_{x1}, c^j_{x3}\}, \{a_1, c^j_{x2}, c^j_{x3}\}$, for each $c_j = (x_1 \vee x_2 \vee x_3) \in C\}$;

$E_3 = \{\{a_{4i+1}, a_{4i+2}, a_{4i+3}\}, \{a_{4i+1},\ a_{4i+2},\ a_{4(i+1)}\}, \{a_{4i+1},\ a_{4i+3},\ a_{4(i+1)}\},\ \{a_{4i+2},\ a_{4i+3},\ a_{4(i+1)}\}, \{a_{4i+1},\ a_{4i}, a_{4i-1}\},\ \{a_{4i+1},\ a_{4i},\ a_{4i-2}\}, \{a_{4i+1},\ a_{4i-1},\ a_{4i-2}\},\ i \in\ \{0, 1, 2, 3\}\}$;

$E_4 = \{\{b_{4i+1}, b_{4i+2}, b_{4i+3}\},\ \{b_{4i+1},\ b_{4i+2},\ b_{4(i+1)}\}, \{b_{4i+1}, b_{4i+3},\ b_{4(i+1)}\},\ \{b_{4i+2},\ b_{4i+3},\ b_{4(i+1)}\}, \{b_{4i+1},\ b_{4i},\ b_{4i-1}\},\ \{b_{4i+1},\ b_{4i},\ b_{4i-2}\}, \{b_{4i+1},\ b_{4i-1}, b_{4i-2}\}, i \in\ \{0, 1, 2, 3\}\}$;

The only part that depends on which literal x_i occurs in which clause c_j is the collection of edges $E_5 = \left\{\{c^j_{xi}, x_i, a_1\},\ \text{if } x_i \text{ is a literal of the clause } c_j\right\}$. The list L of colors satisfies $L(a_1) = L(a_2) = L(a_3) = L(a_4) = L(a_9) = L(a_{14}) = L(a_{15}) = L(a_{16}) = L(b_6) = L(b_7) = L(b_8) = L(b_9) = L(b_{10}) = L(b_{11}) = L(b_{12}) = [0, 2]$; $L(b_1) = L(b_2) = L(b_3) = L(b_4) = L(b_9) = L(b_{13}) = L(b_{14}) = L(b_{15}) = L(b_{16}) = L(a_6) = L(a_7) = L(a_8) = [1, 2]$; If v is other vertex of H then, $L(v) = [0, 1]$. We exhibit a bipartition to $V(H)$ into 3 steps. In the first step set vertices $\{a_1, a_2, a_5, a_6, a_9, a_{10}, a_{13}, a_{14}, b_1, b_2, b_5, b_6, b_9, b_{10}, b_{13}, b_{14}\}$ to partition 1. In the second step, for every clause $c = (x \vee y \vee z)$ of C select vertices c^j_x and c^j_y to partition 1 and c^j_z to partition 2. In the third step set the remaining vertices of H to partition 2.

We prove that if $\mathcal{H}_2$ is L-colorable, then colors 0 and 1 are assigned, respectively, to vertices a_1, and b_1; We use this fact to prove that there is a satisfiable truth assignment for $I = (U, C)$, if and only if $\mathcal{H}_2$ is L-colorable.

We offer in Figure 1 an example of construction of an instance $J = (\mathcal{H}_2, V_1, V_2,\text{L}, 3)$ of 2–LB3UH.

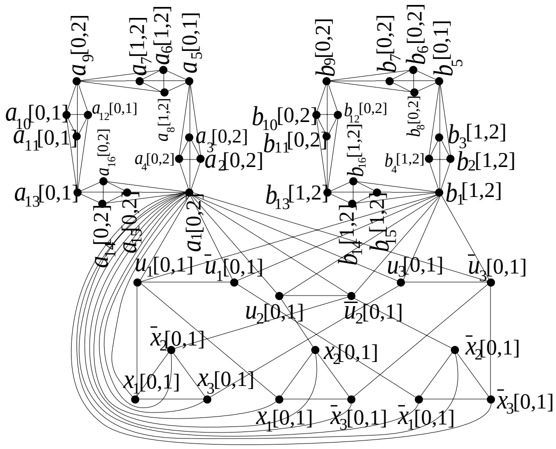

Figure 1: Instance $J = (\mathcal{H}_2, V_1, V_2,\text{L}, 3)$ of 2–LB3UH obtained from the 3SAT instance $I = (U, C) = (\{u_1, u_2, u_3\}, \{(u_1 \vee \overline{u}_2 \vee u_3), (u_1 \vee u_2 \vee \overline{u}_3), (\overline{u}_1 \vee \overline{u}_2 \vee \overline{u}_3)\})$.

k–LB3UH is NP-complete, with $3 \leq k \leq p \leq 2k - 1$

We use the NP-completeness of 2–LB3UH as induction basis to prove the general case that k–LB3UH is NP-complete for $k \geq 3$ and $p \leq 2k - 1$.

First we consider the instance $J = (\mathcal{H}_2 = (V, E), V_1, V_2,\text{L}, 3)$ of 2–LB3UH obtained from the 3SAT instance $I = (U, C)$. For the purpose of defining $\mathcal{H}_k$, we describe the list sized k bipartite hypergraph $\mathcal{A}_k(k-1)$ with $2k-1$ colors. The property key of hypergraph $\mathcal{A}_k(k-1)$ is that it is L-colorable and for every L-coloring of $\mathcal{A}_k(k-1)$, a special vertex v_k must be colored with color $k-1$. Hence, we use induction in $J = (\mathcal{H}_{k-1} = (V, E),\ V_1, V_2,\text{L}, 3)$ and 2 copies of hypergraphs $\mathcal{A}_k(c)$ and $\mathcal{A}_k(c')$ to take the corresponding 4 vertices v_k's as universal vertices for $\mathcal{H}_{k-1}$, in the sense that there is an extra edge containing each vertex of $\mathcal{H}_{k-1}$ and each pair of the 4 vertices v_k's, where c and c' are two extra colors not used in the $k-1$ sized lists of the induction step hypergraph $\mathcal{H}_{k-1}$ to get the general case of $\mathcal{H}_k$. Since color c cannot be assigned to the vertices of $\mathcal{H}_{k-1}$, we increase with c the lists of $\mathcal{H}_{k-1}$ from size $k-1$ to k, and the total number of colors from $2(k-1)-1$ to $2k-1$ by adding c, c' to the set of $2(k-1)-1$ colors.

Conclusions and open problems

We prove that for every $k \geq 2$ the decision problem k–LB3UH is NP-complete even for a fixed $p \leq 2k - 1$. We notice that k–LB3UH is a polynomial decision problem if $p = k$, since one can assign in each list the color given by the bipartition (V_1, V_2). We left as an open problem to determine the minimum p, $2 \leq k < p \leq 2k - 1$ such that k–LB3UH is NP-complete.

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Sobre el Clan-Comportamiento de las Triangulaciones de Superficies Compactas

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Un *clan* de una gráfica es una subgráfica completa maximal. La *gráfica de clanes* $K(G)$ es la gráfica de intersección de todos los clanes de G. Se definen las *gráficas iteradas de clanes* de manera recursiva: $K^0(G) = G$ y $K^{n+1}(G) = K(K^n(G))$. Decimos que una gráfica es *clan-divergente* si la sucesión de órdenes de sus gráficas iteradas de clanes crece sin cota, caso contrario decimos que es *clan-convergente*. El *clan-comportamiento* de una gráfica G es clan-convergente o clan-divergente según lo sea G. El clan-comportamiento ha sido estudiado en diversos trabajos, por ejemplo [21, 20, 3, 6, 7, 10, 19, 2, 5]. Las gráficas de clanes han sido consideradas para el estudio de las Redes Sociales [8], la Propiedad del Punto Fijo en órdenes Parciales [9] y han sido usadas en Gravitación Cuántica [23]. Para amplia bibiliografía sobre gráficas de clanes puede consultar [24, 22, 11]. Algunos trabajos recientes se encuentran en [1, 17, 18, 4].

Una *triangulación de Whitney* de un espacio topológico X es una gráfica G tal que la realización geométrica del complejo de completas de G es homeomorfa a X. En el caso en que X es una superficie compacta (con o sin frontera), es fácil ver que las triangulaciones de Whitney son precisamente las gráficas en las que los vecinos de cada vértice inducen una subgráfica isomorfa a algún ciclo (de longitud al menos 4) o a alguna trayectoria (de longitud al menos 2). Las triangulaciones de Whitney de superficies cerradas (i.e. sin frontera) son aquellas en las que los vecinos de cada vértice inducen una subgráfica isomorfa a algún ciclo de longitud al menos 4.

Se sabe [14] que casi toda superficie cerrada admite una triangulación clan-convergente (las posibles excepciones son: la esfera, el plano proyectivo, el toro y la botella de Klein). También se probó previamente [16] que toda superficie cerrada admite una triangulación clan-divergente. En el trabajo reportado en este resumen, hemos generalizado estos estudios al caso de superficies compactas (i.e. que pueden tener frontera). Los resultados obtenidos se presentan en los siguientes dos teoremas.

Teorema 1. *Casi toda superficie compacta admite una triangulación de Whitney clan-divergente; la única posible excepción es el disco.*

Teorema 2. *Casi toda superficie compacta admite una triangulación de Whitney clan-convergente; las únicas posibles excepciones son: la esfera, el plano proyectivo, el toro y la botella de Klein.*

Para probar estos teoremas fue necesario usar la mayoría de las técnicas que han sido desarrolladas a la fecha para determinar el clan-comportamiento: retracciones [21, 20], cubrimientos [12], relojes [15], gráficas rango-divergentes [16] y gráficas con cuello local grande [13] además de los resultados ya conocidos sobre la clan-convergencia de triangulaciones de superficies cerradas [14].

Estas técnicas, sin embargo, no bastan para construir una triangulación clan-divergente del disco, ni bastan para construir triangulaciones clan-convergentes para la esfera, el plano proyectivo, el toro y la botella de Klein. Esto viene a reforzar las ideas de que ni el disco admite triangulaciones clan-divergentes, ni la esfera admite triangulaciones clan-convergentes, ideas que ya han sido presentadas como conjeturas en [13].

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Forbidden configurations for (k, l)-cographs

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1 Introduction

Many problems in graphs, such as coloring and covering problems, can be seen as partition problems. We consider the problem of verifying whether the vertex set of a given graph G can be partitioned into k independent sets and l cliques, known as the (k, l)-*partition problem*. Graphs that can be partitioned in this way are called (k, l)-*graphs*, and were introduced by Brandstädt in [1]. The (k, l)-partition problem generalizes the k-coloring problem, which corresponds to checking whether a given graph G is a $(k, 0)$-graph. Brandstädt proved that for $k \geq 3$ or $l \geq 3$ the problem of recognizing (k, l)-graphs is NP-complete. Since then, many papers have been devoted to the study of special families of (k, l)-graphs, such as chordal (k, l)-graphs. In [10], a characterization of chordal (k, l)-graphs and a polynomial-time recognition algorithm for this family are given. Perfect (k, l)-graphs are studied in [6]. A generalization of the (k, l)-partition problem named *M-partition problem* [8, 9], besides allowing *internal restrictions* in the definition of the parts (such as being an independent set or a clique), also introduces *external restrictions* involving the parts (such as being completely linked by edges, or completely isolated). A common way of characterizing the existence of partitions is via the absence of finitely describable obstructions. We provide such a characterization for the family of (k, l)-*cographs*, cographs that are (k, l)-graphs.

The problem of partitioning a cograph into independent sets and cliques has also been investigated in other aspects in [4, 5, 6, 7]. We extend the result presented in [4], which caracterizes $(2,1)$-cographs and $(2,2)$-cographs in terms of obstructions.

Given a simple graph $G = (V, E)$, we denote by $\overline{G}$ the complement of G. For $V' \subseteq V$, $G[V']$ denotes the subgraph of G induced by V'. A *clique* (*independent set*) is a subset of vertices inducing a complete (edgeless) subgraph, not necessarily maximal. G is a (k,l)*-graph* if V can be partitioned into k independent sets and l cliques. For a (k,l)-graph G, we write $V = S_1 \cup \ldots \cup S_k \cup C_1 \cup \ldots \cup C_l$, where each S_j is an independent set and each C_i is a clique. It is worth mentioning that in the above definition some sets may be empty. Such a partition is called a (k,l)-*partition* of G. The complete (resp. edgeless) graph on r vertices is denoted by K_r (resp. I_r). Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the graph $G_1 \cup G_2$ (called the *union* of G_1 and G_2) is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and the graph $G_1 + G_2$ (called the *join* of G_1 and G_2) is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{(x,y) \mid x \in V_1,\ y \in V_2\}$. A cograph G is a graph which contains no P_4 (a path with four vertices). Cographs have many properties which are useful to characterize and recognize (k,l)-cographs. One of them is the auto-complementarity: G is a (k,l)-cograph if and only if $\overline{G}$ is an (l,k)-cograph. Lerchs [11] showed how to associate a cograph G with a unique tree $T(G)$ called the *cotree* of G, defined as follows:

- If G is not trivial then every internal node of $T(G)$ has at least two children.

- Internal nodes are labelled 0 (0-nodes) or 1 (1-nodes) in such a way that 0-nodes and 1-nodes alternate along every path in $T(G)$ starting at the root.

- Leaves of $T(G)$ are precisely the vertices of G, such that vertices x and y are adjacent in G if and only if the lowest common ancestor of x and y in $T(G)$ is a 1-node.

The following definition will be useful in order to characterize (k,l)-cographs. Denote by $\mathcal{F}(a,b)$ $(a, b > 0)$ the family of cographs such that each member $G \in \mathcal{F}(a,b)$ satisfies the following properties:

- $|V(G)| = ab$;

- G contains a disjoint cliques, each of size b;

- G contains b disjoint independent sets, each of size a.

A simple way of representing a member $G \in \mathcal{F}(a,b)$ is by means of a matrix called (a,b)*-template*, defined as follows. Let M be an $a \times b$ matrix of points. Each row of M represents a clique of size b, and each column of M represents an independent set of size a. There are two types of (a,b)-templates: (a,b)*-union templates* (representing disconnected members of $\mathcal{F}(a,b)$) and (a,b)*-join templates* (representing connected members of $\mathcal{F}(a,b)$), recursively defined as follows:

- M is an (a,b)-union template if either $b = 1$ or $b > 1$ and there are $p-1$ horizontal lines dividing M into p sub-matrices $M_1, M_2, \ldots, M_p$ such that each M_i is an (a_i, b)-join template, $a_i > 0$.

- M is an (a,b)-join template if either $a = 1$ or $a > 1$ and there are $q-1$ vertical lines dividing M into q sub-matrices $M_1, M_2, \ldots, M_q$ such that each M_i is an (a, b_i)-join template, $b_i > 0$.

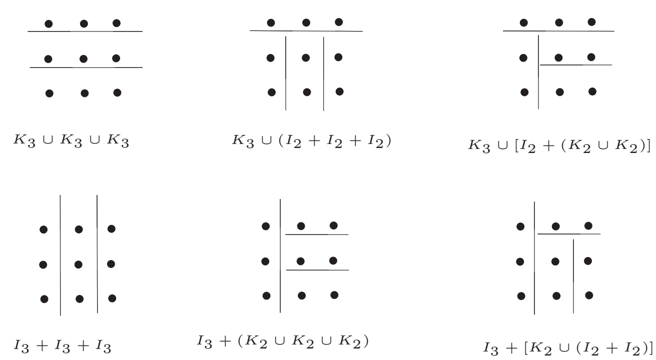

Figure 1: The family of $(3,3)$-templates and corresponding cographs.

Figure 1 shows the $(3,3)$-templates. Given an (a,b)-template M, the associated cograph G_M is easily obtained as follows. Each point represents a vertex of G_M. If M is a union template and $b = 1$ then the a points represent I_a, otherwise the sub-matrices $M_1, M_2, \ldots, M_p$ represent connected components of G_M. Similarly, if M is a join template and $a = 1$ then the b points represent K_b, otherwise the sub-matrices $M_1, M_2, \ldots, M_q$ represent connected components of $\overline{G_M}$. Figure 1 shows the cographs associated with the $(3,3)$-templates.

2 Main results

The following proposition relates $\mathcal{F}(a,b)$ and the (a,b)-templates.

Proposition 1. *A graph G is a member of $\mathcal{F}(a,b)$ if and only if there exists an (a,b)-template M such that $G = G_M$.*

Proof. Let $G \in \mathcal{F}(a,b)$. Let us show, by induction on the height $h(T(G))$ of $T(G)$, how to define an (a,b)-template M such that $G = G_M$. If $h(T(G)) = 0$ then choose M as the $(1,1)$-template. If $h(T(G)) = 1$ and the root of T_G is a 0-node, i.e., G is disconnected, then $G = I_a$ and M is chosen as the $(a,1)$-union template; otherwise, if the root of $T(G)$ is a 1-node, then $G = K_b$ and M is chosen as the $(1,b)$-join template. Assume now that $h(T(G)) \geq 2$ and the root of T_G is a 1-node. Let $G_1, G_2, \ldots, G_p$ be the children of the root. Since G contains b disjoint I_a's, each G_i, $1 \leq i \leq p$, contains b_i disjoint I_a's in such a way that $b_1 + b_2 + \cdots + b_p = b$. By induction, each G_i is associated with an (a,b_i)-union template M_i. Therefore, G is associated with the (a,b)-join template

$$M_1 \mid M_2 \mid \cdots \mid M_q.$$

We use a similar argument, if the root of $T(G)$ is a 0-node. Conversely, if $G = G_M$ for some (a,b)-template, then $|V(G)| = ab$ and G contains a disjoint cliques of size b and b disjoint independent sets of size a, i.e., $G \in \mathcal{F}(a,b)$. This completes the proof. □

Having described the family $\mathcal{F}(a,b)$, let us show how to use it in order to characterize (k,l)-cographs. We start with the following lemma.

Lemma 2. [2] *Let G be a cograph and S^* a maximum independent set of G. If $G[V \backslash S^*]$ contains K_r as a subgraph then G contains K_{r+1} as a subgraph.*

The following lemma, proved in [4], will be useful to prove the characterization of (k,l)-cographs.

Lemma 3. [4] *A graph G is a cograph if and only if for every $V' \subseteq V(G)$ the following property holds: if $G[V']$ is a (k,l)-graph with $k > 0$ and S' is a maximum independent set of $G[V']$ then $G[V' \backslash S']$ is a $(k-1,l)$-graph.*

Theorem 4. [2] *A cograph G is a (k,l)-graph if and only if it contains no member of $\mathcal{F}(l+1,k+1)$ as an induced subgraph.*

Proof. The proof is done by induction on the sum $p + k$. For the lack of space, the proof is omitted. □

3 Conclusion

From lemma 3 we have a linear time algorithm for recognizing (k,l)-cographs. We can extend the family of forbidden subgraphs for P_4-sparse graphs.

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Partial characterizations of balanced graphs

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A {0,1}-matrix is *balanced* if it has no square submatrix of odd order with exactly two 1's per row and per column. A graph is *balanced* if its clique-vertex incidence matrix is balanced. Balanced matrices are perfect and, accordingly, balanced graphs are perfect. Few years ago, perfect graphs were characterized by minimal forbidden induced subgraphs. There is a characterization of balanced graphs by forbidden induced subgraphs, but this characterization is not by *minimal* forbidden induced subgraphs. We present some characterizations of balanced graphs by minimal forbidden induced subgraphs restricted to graphs that belong to any of the following classes: P_4-sparse graphs, line graphs and their complements, and Helly circular-arc (HCA) graphs.

Keywords: balanced graphs, HCA graphs, line graphs, P_4-sparse graphs

1 Introduction

A *clique* is a maximal complete subgraph of a graph. Berge defined *balanced matrices* as those {0,1}-matrices not having a square submatrix of odd order with exactly two 1's per row and per column. Balanced matrices play a remarkable role on combinatorial optimization. Given an enumeration $Q_1, \ldots, Q_k$ of the cliques and $v_1, \ldots, v_n$ of the vertices of a graph G, the $k \times n$ {0,1}-matrix $A = (a_{ij})$, where $a_{ij} = 1$ iff $v_j \in Q_i$, is a clique-matrix of G. A graph G is *balanced* if its clique-matrix is balanced [4]. Balanced graphs constitute a subclass of the famous class of perfect graphs [3, 5].

We denote the chordless cycle (resp. path) on n vertices by C_n (resp. P_n). $\overline{G}$ denotes the complement of G. A *hole* is an induced C_k for some $k \geq 5$. Figure 1 shows some small graphs. Let G_1 and G_2 be two graphs and assume that $V(G_1) \cap V(G_2) = \emptyset$. The *join of* G_1 *and* G_2 is the graph $G_1 + G_2$ with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$.

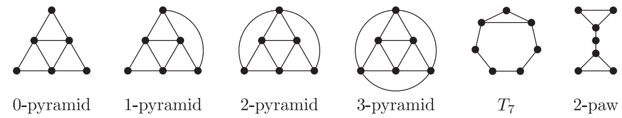

Figure 1: Some small graphs

Hoàng [7] defined *P_4-sparse* graphs as those graphs such that every set of five vertices induces at most one P_4. These graphs are a natural generalization of cographs and they are also perfect. A *spider* is a graph whose vertex set can be partitioned into three sets S, C and R, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j iff $i = j$ (a *thin spider*), or s_i is adjacent to c_j iff $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and nonadjacent to all the vertices in S. *If G is a P_4-sparse graph then G or $\overline{G}$ is disconnected, or G is a spider* [7].

Given a graph R, the *line graph* $L(R)$ of R is the graph whose vertices are the edges of R, two edges of R being adjacent iff they share an endpoint. The graph G is a *line graph* if it is isomorphic to $L(R)$ for some graph R.

A graph is a *circular-arc graph* if it is the intersection graph of a family of open arcs of a circle. Such a family of arcs is called a *circular-arc model* of the graph. A *Helly circular-arc (HCA)* graph is a circular-arc graph that has a circular-arc model satisfying the Helly property (i.e. any subfamily of pairwise intersecting arcs has a nonempty intersection).

Few years ago, perfect graphs were characterized by minimal forbidden induced subgraphs: they are exactly the graphs without odd holes and their complements [2]. In [1], a forbidden induced subgraphs characterization of balanced graphs was given, but it was not by *minimal* forbidden induced subgraphs. A graph is a minimal forbidden induced subgraph for the class of balanced (or simply, minimally not balanced) if it is not balanced but all its proper induced subgraphs are. In Section 2, we present minimal forbidden induced subgraphs characterizations of balanced graphs restricted to graphs that belong to any of the following graph classes: P_4-sparse graphs, line graphs and their complements, and HCA graphs.

2 Partial characterizations

The class of *trivially perfect graphs* was defined by M. Golumbic in [6]. He also showed that it coincides with the class of $\{C_4, P_4\}$-free graphs.

Lemma 1. *Let $G = G_1 + G_2$. Then G is balanced if and only if at least one of the following statements holds: (i) At least one of G_1 and G_2 is a complete, and the other is balanced; (ii) G_1 and G_2 are both trivially perfect.*

Sketch of the proof. It is easy to see that the join of a complete and a balanced graph is balanced. On the other hand, the join of $2K_1$ and P_4 (resp. C_4) is the 2-pyramid (resp. 3-pyramid), so if one of G_1 and G_2 is not a complete, the other one must be trivially perfect. The main part of the proof consists on proving that the join of two trivially perfect graphs is balanced. □

Now we are in position to characterize those P_4-sparse balanced graphs.

Theorem 2. *Let G be a P_4-sparse graph. Then G is balanced if and only if G contains no induced 0-, 2-, or 3-pyramid.*

Sketch of the proof. Let H be P_4-sparse graph that is minimally not balanced. If $\overline{H}$ is disconnected then, by Lemma 1, we can conclude that H is 2- or 3-pyramid. If $\overline{H}$ is connected then, since by minimality H is connected, H is a spider. If we assume that H is a thin spider, we reach a contradiction by inspection of its clique-matrix. So H is a thick spider. Therefore H contains an induced 0-pyramid and by minimality H is 0-pyramid, contradiction. □

The results obtained for line graphs and their complements and for HCA graphs are the following. We omit the proofs due to space limitations.

Theorem 3. *Let G be a line graph. Then G is balanced if and only if G has no odd holes and it contains no induced 0-pyramid, 1-pyramid, or 3-pyramid.*

Theorem 4. *Let G be the complement of a line graph. Then G is balanced if and only if G contains no induced 0-, 2-, 3-pyramid, C_5, $\overline{C_7}$, $\overline{T_7}$, or $\overline{\text{2-paw}}$.*

Let $t \geq 1$. For each p even such that $2 \leq p \leq 2t$, we define the graph V_p^{2t+1} whose vertex set is $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2\}$, $v_1 v_2 \ldots v_{2t+1}$ is a cycle whose only chord is $v_1 v_3$, $N(u_1) = \{v_1, v_2\}$ and $N(u_2) = \{v_2, v_3, \ldots, v_{p+1}\}$. We define the graph D^{2t+1} whose vertex set is $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2, u_3\}$ such that $v_1 v_2 \ldots v_{2t+1}$ is a cycle whose only chords are $v_{2t+1} v_2$ and $v_1 v_3$, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3\}$ and $N(u_3) = \{v_1, v_2\}$. For each p even such that $4 \leq p \leq 2t$, we define the graph X_p^{2t+1} whose vertex set is $\{v_1, v_2, \ldots, v_{2t+1}, u_1, u_2, u_3, u_4\}$ such that $v_1 v_2 \ldots v_{2t+1}$ is a cycle whose only chords are $v_{2t+1} v_2$ and $v_1 v_3$, $N(u_1) = \{v_{2t+1}, v_1\}$, $N(u_2) = \{v_2, v_3, u_4\}$, $N(u_3) = \{v_{2t+1}, v_1, v_2, u_4\}$, $N(u_4) = \{v_1, v_2, v_3, \ldots, v_p, u_2, u_3\}$.

Theorem 5. *Let G be a HCA graph. Then G is balanced if and only if G has no odd holes and contains no induced 0-, 1-, 2-pyramid, $\overline{C_7}$, V_p^{2t+1}, D^{2t+1}, or X_p^{2t+1} for any $t \geq 1$ and any valid p.*

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Interval Count and Maximal Cliques

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An *interval graph* is the intersection graph of a family of intervals of the real line, called an *interval model*. An order $P = (X, \prec)$ is an *interval order* if P can be associated to an interval model $\mathcal{R} = \{I_x \mid x \in X\}$ such that $x \prec y$ if and only if I_x is entirely to the left of I_y.

The interest in interval graphs and orders comes from both their central role in many applications and purely theoretical questions [2]. We consider the problem of computing, for some given graph G, the least amount of interval lengths for an interval model of G, named the *interval count problem* [3, 2]

Let $IC(\mathcal{R})$ denote the number of distinct lengths of the interval model $\mathcal{R}$. The *interval count* of an interval order P is the least number $IC(P)$ of interval lengths for an interval model of P, i.e., $IC(P) = \min\{IC(\mathcal{R}) \mid \mathcal{R}$ is an interval model of $P\}$. Similarly, the *interval count* of an interval graph G is the least number $IC(G)$ of interval lengths for an interval model of G, i.e., $IC(G) = \min\{IC(\mathcal{R}) \mid \mathcal{R}$ is an interval model of $G\}$. The problem of computing either $IC(P)$ or $IC(G)$ is called the *interval count problem*. It is not known currently whether the problem is NP-complete. In what follows, all graphs considered are interval graphs. Denote the left and right extreme points of an interval I by $\ell(I)$ and $r(I)$, respectively. For convenience, we may use the concepts of vertex and its corresponding interval interchangebly.

Deciding whether $IC(G) = 1$ for a given graph G is equivalent to recognizing whether G is a unit interval graph. The latter problem is well-studied and there are several efficient solutions to it, including linear-time algorithms [1]. In fact, $IC(G) = 1$ if and only if G is claw-free. However, given a graph G (order P) and some fixed integer $k \geq 2$, deciding efficiently whether $IC(G) = k$ ($IC(P) = k$) is an open problem.

We consider the constraint in which is assumed the intervals have their extreme points represented by integers and we ask whether the interval count of a graph (order) is affected. It is usual to assume such a property without loss of generality in most of the problems related to interval graphs. However, regarding to the interval count problem, this question should be analyzed carefully. A slightly change in any extreme point of some interval (which is the basic operation to transform interval models) affects the length of that interval, therefore potentially affecting the number of distinct lengths. We show that in fact such a property does not affect the interval count of a graph (order). This result is formally stated next.

Theorem 1. *Let P be an interval order. Then, there exists an interval model $\mathcal{R}$ of P with $IC(\mathcal{R}) = IC(P)$ in which all extreme points are distinct integers.*

Corollary 2. *Let G be an interval graph. Then, there exists an interval model $\mathcal{R}$ of G with $IC(\mathcal{R}) = IC(G)$ in which all extreme points are distinct integers.*

Note that the results stated in Theorem 1 and Corollary 2 do not provide any insight into the problem of deciding whether $IC(G) = k$ ($IC(P) = k$) running a combinatorial algorithm whose worst case time complexity depends only on the size of the input, namely G (P) and k. To address this issue, the next straightforward lemma is used as a basis to design such an algorithm.

Lemma 3. *Let $P = (X, \prec)$ be an interval order and $k \geq 1$ be an integer. Then, $IC(P) = k$ if and only if k is the least integer for which there exists a partition $S_1 \cup \cdots \cup S_k$ of X and an interval model $\{I_x \mid x \in X\}$ of P such that $|I_x| = |I_y| \iff \{x, y\} \subseteq S_i$ for some $1 \leq i \leq k$.*

Let $P = (X, \prec)$ be an interval order and $S = S_1 \cup \cdots \cup S_k$ be a partition of X. Let (G_S) be the linear programming instance defined below, where $\delta > 0$ and $\epsilon > 0$ are constants, ℓ_x, r_x, s_i are variables for all $x \in X$ and $1 \leq i \leq k$, and $F(P)$ is a linear objective function on such variables.

$$
\begin{array}{llll}
(G_S): & \min & F(P) & (2) \\
& s.t. & & \\
& r_x - \ell_x = s_i & \text{, for all } x \in S_i, 1 \leq i \leq k & (3) \\
& s_i + \delta \leq s_{i+1} & \text{, for all } 1 \leq i < k & (4) \\
& r_x + \epsilon \leq \ell_y & \text{, for all } x \prec y & (5) \\
& \ell_y + \epsilon \leq r_x & \text{, for all } x \parallel y & (6) \\
& \ell_x, r_x \geq \epsilon & \text{, for all } x \in X & (7) \\
& s_1 \geq \epsilon & & (8)
\end{array}
$$

Theorem 4. *Let $P = (X, \prec)$ be an interval order and $k \geq 1$. Then, it is possible to decide whether $IC(P) = k$ running a combinatorial algorithm whose worst case time complexity depends only on the sizes of P and k.*

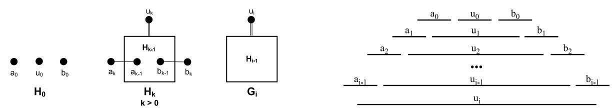
Proof. By Theorem 1, it is clear that G_S is feasible if and only if there exists an interval model $\mathcal{R} = \{I_x \mid x \in X\}$ of P such that $\ell_x = \ell(I_x)$, $r_x = r(I_x)$ for each $x \in X$, and $|I_x| = |I_y| \iff \{x, y\} \subseteq S_i$ for some $1 \leq i \leq k$. Therefore, by Lemma 3, $IC(P) = k$ if and only if k is the least for which G_S is feasible for some partition $S = S_1 \cup \cdots \cup S_k$ of X. □

It is straightforward to obtain the counterpart of Theorem 4 for graphs. These results emphasize the combinatorial aspects of the interval count problem, which may not be much clear at first glance, since interval extremes vary continuously on the real line.

We also address a very different question: an extremal problem concerning the interval count. Let $f(q)$ be the function which associates the maximum possible interval count for a graph with q maximal cliques, i.e., $f(q) = \max\{IC(G) \mid q(G) = q\}$, where $q(G)$ is the number of maximal cliques of G.

Theorem 5. *$f(q) = \lfloor (q+1)/2 \rfloor$, for all $q \geq 1$.*

Proof. (Sketch) The result is clear when $q \leq 2$, since such a graph is claw-free. Assume $q > 2$.
Let G_i, $i \geq 1$, be the graph defined schematically in the figure (left part). The fancy edges (u_i, H_{i-1}) mean that the vertex u_i is adjacent to all vertices of the induced subgraph H_{i-1}.



For each $i \geq 1$, the number of maximal cliques of G_i can be easily worked out as being $q(G_i) = 2i + 1$. Moreover, for any interval model $\{I_v \mid v \in V(G_i)\}$ of G_i, $|I_{u_k}| > |I_{u_{k-1}}|$ for every $1 \leq k \leq i$. Thus, $IC(G_i) \geq i + 1$. As a matter of fact, $IC(G_i) = i + 1$ as testified by the interval model of G_i depicted in the figure (right part).
Let G be either the graph $G_{(q-1)/2}$ if q is odd or the graph $G_{(q-2)/2}$ plus an isolated vertex, otherwise. Therefore, $f(q) \geq IC(G) = \lfloor (q+1)/2 \rfloor$.
On the other hand, let G be a graph with q maximal cliques. We show an upper bound for $f(q)$ by designing an algorithm which builds an interval model $\mathcal{R}$ of G such that $IC(\mathcal{R}) \leq \lfloor (q+1)/2 \rfloor$. Since G is a general graph with q maximal cliques, then $f(q) \leq \lfloor (q+1)/2 \rfloor$.
Let $C_1, \ldots, C_q$ be the maximal cliques read from left to the right of some interval model of G and let $m = \lfloor (q+1)/2 \rfloor$. Initially, assume all intervals in the cliques C_m or C_{m+1} are unit length. Then, on each step $i = 1, \ldots, m-1$, move every left extreme point of the intervals in $C_{m-i} \cap C_{m-i+1}$ to the left and move every right extreme point of the intervals in $C_{m+i} \cap C_{m+i+1}$ to the right in a manner that the following two conditions hold: (i) the modified intervals have the greatest length and (ii) there exist two points $p_1 < p_2$ such that the set of intervals which contain p_1 is $C_{m-i} \cap C_{m-i+1}$ and p_2 is $C_{m+i} \cap C_{m+i+1}$. For each $v \in C_{m-i} \setminus C_{m-i+1}$, add I_v such that $|I_v| = 1$ and $r(I_v) = p_1$. Symmetrically, for each $v \in C_{m+i+1} \setminus C_{m+i}$, add I_v such that $|I_v| = 1$ and $\ell(I_v) = p_2$. In each iteration clearly at most one new interval length is added. Therefore, $IC(G) \leq IC(\mathcal{R}) \leq (m-1) + 1 = \lfloor (q+1)/2 \rfloor$. □

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Surface triangulations and a large family of no dismantleable k-null graphs

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Abstract

We show a large class of graphs k-null that are not dismantleable and include a infinity family of Whitney triangulations of disk. *Keywords: clique graphs, Whitney triagulations, k-null, k-convegent.*

1 Introduction

Our graphs are simple, finite and connected. Making a noun out of an adjective we shall refer to complete subgraphs just as completes. We shall identify a induced subgraph (hences completes) and their vertices set. A *clique* of a graph is a maximal complete. The *clique graph* of a graph G is the intersection graph $k(G)$ of the set of all cliques of G. Iterated clique graphs $k^n(G)$ are defined by $k^0(G) = G$ and $k^{n+1}(G) = k(k^n(G))$. We say that G is *k-null* if $k^n(G)$ is the trivial (i.e. one-vertex) graph for some $n \geq 0$. More generally, if there are m and n with $m \neq n$ such that $k^m(G) \cong k^n(G)$, we say that G is *k-convergent*. It is easy to see that if G is not k-convergent then the sequence of orders $|k^n(G)|$ tends to infinity, in this case we say that G is *k-divergent*.

We will distinguish this three k-*behaviors*. The first examples of large families of this three distinct k-behavior are in [4], [1] and [7]. The open neighborhood of $x \in V(G)$ is $N_G(x) = \{y \in G | xy \in E(G)\}$. The closed neighborhood is $N[x] = N(x) \bigcup \{x\}$. Given $x, y \in V(G)$ if closed neighborhood of x is subset of closed neighborhood of y we say that x is *dominated* by y, and we say that x is dominated if x is dominated by some else, this concept was introduced by Escalante [1]. In [2] was proved that erase dominated vertex do not change the k-behavior. We say that graph G is *dismantleable* if we eliminate dominated vertex one by one we obtain the one-vertex graph, Prisner [8] proved that dismantleable graphs are k-null. I If $\mathfrak{T}$ is a triangulation (simplicial descomposition) of a compact surface $\mathbb{X}$ and G is the underlying graph (1-skeleton) of $\mathfrak{T}$. We call $\mathfrak{T}$ a *Whitney triangulation* if any face of $\mathfrak{T}$ is a triangule of G. We do not distinguish between $\mathfrak{T}$ and G. Larrión, Neumann-Lara and Pizaña in [5] and [6] was conjectured that every Whitney triangulation of the disk is k-null, we will call this, *the disk conjeture*; particulary in [5] has been proved that a disk, which interior vertices have at least six degree, always has a dominated vertex in the border then the disk is dismantliable and therefore it is k-null. In the literature there are not examples of a large family of k-null graphs that are not dismantleable. In this paper (section 2) we show a large family of k-null graphs no dismantleables that include the Whithney triagulation of a familie of disks with arbitrary radio. This is one more step to prove the disk conjetura. For more notation or basic definitions see [3].

2 Crowns

We write $G \overset{\#}{\to} H$, if there is H' subgraph of G such that $H \cong H'$ and for each $x \in V(G) - V(H')$ there is $y \in V(H')$ such that x is dominated by y. We write $G \overset{\#}{\leadsto} H$ if there are $\{G_i\}_{i=0}^{n}$ such that $G \cong G_0 \overset{\#}{\to} G_1 \overset{\#}{\to} G_2 \overset{\#}{\to} \ldots \overset{\#}{\to} G_n \cong H$. In [2] is proved the next theorem.

Theorem 1. *[2]. Let G and H be a graphs. If $G \overset{\#}{\to} H$ then $k(G) \overset{\#}{\to} k(H)$*

Definition 2. *We say that a graph H' is a coronation of a graph H and if there are completes q_1, q_2, ..., q_m in H such that:*

1. $m \geq 4$.
2. $q_i \cap q_{i+1}$ *have one and only one vertex.*

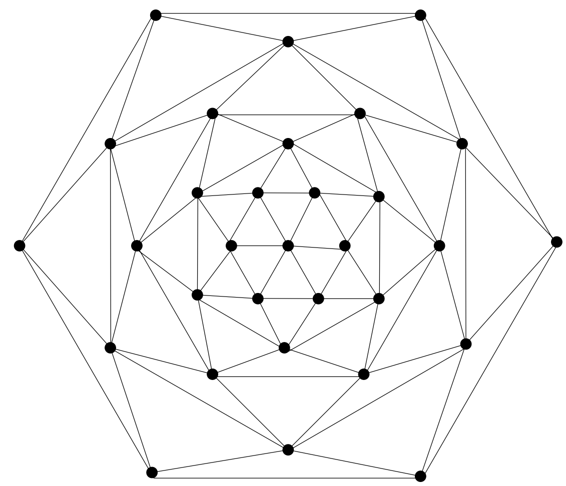

Figure 1:

3. $q_i \cap q_j = \emptyset$ *when* $j \notin \{i-1, i, i+1\}$.
4. $H' \cong H''$ *where* $V(H'') = V(H) \cup V(C_m)$, $E(H'') = E(H) \cup E(C_m) \cup \{\{x, v_i\} : x \in q_i, v_i \in C_m,$ *para* $i = 1, 2, ..., m\}$ *and where C_m is cycle of m vertices* $v_1, v_2, ..., v_m$.

We will name $C_m \cup \cup_{i=1}^{m} q_i$ *crown* of H. The vertices of C_m we will call them *exterior vertices of the crown.* The vertices of $\cup_{i=1}^{m} q_i$ we will call them *interior vertices of the crown.* If H' is a coronation of H we write $H' \overset{\mho}{\to} H$ and too we say that H' crown H.

Theorem 3. *Let H' and H graphs such that $H' \overset{\mho}{\to} H$ then $k(H') \overset{\#}{\leadsto} k(H)$ or there is graph G such that $k(H') \overset{\#}{\to} G \overset{\mho}{\to} k(H)$*

*proof:*If $k(H') \overset{\#}{\leadsto} k(H)$ we have finished. If not, let $r_i = \{Q \in k(H) : q_i \cap q_{i+1} \subseteq Q\}$ for $i = 1, 2, ..., m$. The r_1, r_2, ..., r_m play the role of $q_1, q_2, ..., q_m$ of definition 2. Let $G = \Omega(\{Q \in k(H)\} \cup \{\{v_i, v_{i+1}\} \cup (q_i \cap q_{i+1}) : i = 1, 2, ..., m\})$. Obviously $G \overset{\mho}{\to} k(H)$ or $G \overset{\#}{\leadsto} k(H)$. We can see that $k(H') \cong \Omega(\{Q \in k(H) : Q \neq q_i,\ i = 1, 2, ..., m\} \cup \{q_i \cup \{v_i\} : i = 1, 2, ..., m\} \cup \{\{v_i, v_{i+1}\} \cup (q_i \cap q_{i+1}) : i = 1, 2, ..., m\})$, when q_i is not a clique in H let Q_i a clique of H such that $q_i \subseteq Q_i$ then $q_i \cup \{v_i\}$ is dominated by Q_i, and so $k(H') \overset{\#}{\to} G$. ■

Theorem 4. *Let H' and H graphs such that $H' \stackrel{\mho}{\to} H$ and such that H is k-null then H' is k-null.*

proof: Let n such that $k^n(H) = K_1$ then by aplying the theorem 3 and theorem 1 n times, we have that $k^n(H') \stackrel{\#}{\rightsquigarrow} k^n(H) = K_1$ then by theorem of dismantlings of [8] we have the result. ∎

Example 5. *If we start with wheel. Then we crown many times; in each step we take the exterior vertices of the last step how new interior vertices, in such way, we have always a disk. And then we obtain a underlaying graph G that is Whitney triangulation of disk. Adicionally G is k-null by theorem 4 and G is not dismantliable. See figure 1.*

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Clique decomposition and the monophonic hull number of a graph[3]

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Keywords: Computational complexity, graph convexity, hull number, monophonic convexity.

1 Introduction

A *convexity* over a non-empty set X is a family $\mathcal{C}$ of subsets of X, called *convex sets*, such that $\emptyset, X \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections and nested unions [5, 6]. In graphs, the most studied convexities use particular kind of paths to define the convex sets. For example, given a set of paths $\mathcal{P}$ of a graph G, a set S is $\mathcal{P}$*-convex* if, for every pair of vertices $u, v \in S$, every path from u to v belonging to $\mathcal{P}$ contains only vertices of S.

In this work we deal with the *monophonic convexity*, where $\mathcal{P}$ is formed by all minimal paths, that is, all induced paths of the graph. Then, we say that a set S is *monophonically convex*, or simply *m-convex*, if all induced paths between two vertices of S are contained in S.

Given a set S, the minimum set containing S which is m-convex is called the *m-convex hull of* S, and is denoted by $J_h[S]$; and the set $J[S]$ containing all induced paths among the vertices of S is the *monophonic interval of* S. It is clear that $S \subseteq J[S] \subseteq J_h[S]$. It is also clear that one can compute the m-convex hull of a set S, by repeatedly applying the monophonic interval, until obtaining an m-covex set.

Given a graph G, the *m-convexity number of* G*,* $c_m(G)$*,* is the cardinality of a maximum m-convex set $S \subsetneq V(G)$; the *monophonic number of* G*,* $m(G)$*,* is the cardinality of a minimum set S such that $J[S] = V(G)$; and the *monophonic hull number of* G*,* $h_m(G)$*,* is the cardinality of a minimum set S such that $J_h[S] = V(G)$.

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Another widely studied convexity is the *geodesic convexity*, where $\mathcal{P}$ is formed by all shortest paths of the graph. The parameters in this convexity, corresponding to the monophonic convexity number, monophonic number and monophonic hull number are the geodesic convexity number, geodetic number and geodesic hull number. It is known that it is NP-hard to compute these parameters for the geodesic convexity [1, 2, 3, 7].
In the monophonic convexity, deciding whether the m-convexity number of a graph is greater than an integer k, and whether the monophonic number is less than an integer k are NP-complete problems for general graphs [4]. Furthermore, computing the monophonic interval of a set is also computationally hard [4]. From these results, it was expected that deciding whether the monophonic hull number of a general graph is less than an integer k was an NP-complete problem. However, we propose a polynomial-time algorithm for finding a minimum monophonic hull set of a graph G in time $O(nm)$, where n is the number of vertices and m the number of edges of G. The algorithm is based on clique decomposition.

2 The algorithm

Given a connected graph G, a set $C \subseteq V(G)$ is a *clique separator* of G if C is a complete set and $G - C$ is a disconnected graph. We say that G is an *atom* if G contains no clique separator. A *(clique) decomposition tree T of a connected graph G* is recursively defined as being a rooted tree such that:

1. G is an atom and T consists of a sole node associated to G, or

2. G has a clique separator C. In this case, the root of T is associated to C, and the connected components of $G - C$ are partitioned into $p > 1$ parts. Each of these parts, together with C, forms a connected induced subgraph G_i of G, and the subtrees T_i of the root of T are tree decompositions of the subgraphs G_i.

A clique separator (atom) associated to a non-leaf (leaf) of T is called a *clique separator* (*atom*) of T. Write $\mathcal{C}(T)$ as the union of the clique separators of T. A decomposition tree T of a graph G is an *s-decomposition tree of* G if, for every subtree T_i of T, any vertex of C_i has neighbors in at least two of the sets $V(H_1)\backslash C_i, \ldots, V(H_\ell)\backslash C_i$, where $H_i, \ldots, H_\ell$ are the induced subgraphs associated to the children of T_i.

Theorem 1. *If G is an atom that is not a complete graph, then every pair of non-adjacent vertices is an m-hull set of G.*

Let T be a decomposition tree for a graph G and $F_1, \ldots, F_k$ its atoms. We partition the set of atoms of T into the following four types. Let F_i be an atom.

- Type 0: $V(F_i) \cap \mathcal{C}(T)$ is not a complete set;
- Type 1: $V(F_i) \cap \mathcal{C}(T)$ is a complete set and there is a vertex $u \in V(F_i)$ not adjacent to some vertex of $V(F_i) \cap \mathcal{C}(T)$;
- Type 2: $V(F_i) \cap \mathcal{C}(T)$ is a complete set, F_i is not a complete graph and every vertex $u \in V(F_i)$ is adjacent to all vertices of $V(F_i) \cap \mathcal{C}(T)$;
- Type 3: F_i is a complete graph.

Next, we describe an algorithm for constructing a minimum m-hull set S of a graph G. Let T be an s-decomposition tree of G.

Algorithm 1 Computation of a minimum m-hull set

Define $S := \emptyset$. Examine each atom F of T. If F is of Type 1, include in S one vertex of F, not adjacent to any vertex of $V(F) \cap \mathcal{C}(T)$. If F is of Type 2, include in S one pair of non-adjacent vertices of F. If F is of Type 3, include in S all vertices of $V(F) \setminus \mathcal{C}(T)$. The construction is terminated.

Theorem 2. *The set obtained by the Algorithm* 1 *is a minimum m-hull set of the graph.*

Sketch of the proof. First we prove that the set S obtained by the algorithm is an m-hull set of G. This is made by induction on the height of the s-decomposition tree of the graph. Next, we prove that S is minimum by an analysis of the atoms of the s-decomposition tree. ■

Corollary 3. *Let G be a graph, T an s-decomposition tree of it. Denote by s the number of simplicial vertices of G, and by f_i the number of atoms of T of Type i, for $i = 1, 2$. Then $h_m(G) = s + f_1 + f_2$.*

In order to apply Algorithm 1, we need to employ a decomposition tree which is in fact an s-decomposition tree. Such a decomposition is obtained by Tarjan's algorithm [8] in $O(nm)$ time. To determine the type of each atom, we need no more than $O(m)$ time, and T has $O(n)$ atoms. Consequently, the complexity of the entire algorithm for computing $h_m(G)$ is $O(nm)$.

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